Boundary Element Method for Analyzing Fluid Movements in Network Profiles

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Abstract: Based on the results of [6], [7] and [9], in this paper we present a calculus algorithm for the study of the compressible fluid's stationary movement through profile grids, on an axial–symmetric flow–surface, in variable thickness of stratum. Using the D. Pompeiu integral-formula for the complex velocity in a multiply connected domain, the fundamental relation for the complex velocity and complex potential are obtained. The solving possibility of this problem by CVBEM method, using the established fundamental equations is also given [9]. We show the applicability of the boundary element methods (BEM) with real values, and the possibility of solving the integral equation of the velocity potential by using the successive approximation method w.r.t. the parameters ς *(fluid's density) and h (thickness variation of fluid stratum), and using the Lagrangian interpolation formula through five points for the calculation of the derivatives of the velocity potential.*

Keywords: boundary element method, hydrodynamic networks, fluid's velocity potential, Fredholme integral equation, Lagrange interpolation

1 Introduction

During the last years, the Boundary Element Method (BEM) proved to be a very efficient method for solving many boundary value problems appeared in engineering sciences. In most of the implied problems in engineering analysis, the real domain of the boundary value problems has irregular boundaries, with complex properties from one zone to the other of the domain, which are excluding any possibility to find analytical solutions for the fundamental equations. In this case, the modern numerical methods represent the only way to obtain the suitable solutions. They were used with a division of the complex domain by a grid, such as the method of finite elements (MFE), or by the division of the domain in finite elements, which will be then assembled.

BEM is an alternative of numerical study, where only the boundary of the analyzed domain is devised in finite elements thus obtaining fewer elements that in BEM.

The main point of the real-BEM [1], is the determination of the fundamental integral equation of the solution in a domain, with the aid of the values of the solutions on the boundary and of the 'flux values'. By aid of this formula the integral equation on the boundary domain is written, and by discretization of the integral equation on boundary, the algebraically equation system which result in the discretized solution a boundary is obtained. The solution in the inner points of the domain is obtained by using once more the fundamental integral equation. Depending on the physical nature of the mathematical solution (velocity, potential, stream function, etc.) there are three variants of the method: direct-BEM, indirect-BEM and semidirect-BEM, [1].

Lately, using the real-BEM ideas, Hromodka II T.V, [4], has presented a variant of the BEM-method in complex variables (CVBEM). This method is specialized for the solving of the boundary value problem for the bidimensional Laplace equation.

After that, D. Homentcovschi, [13], showed the possibility of applying the CVBEM in some boundary value problems of the Poisson equation.

In all these cases, the main request is that the wanted solution should be an analytical complex function.

In the study of the compressible fluid movement by profile grids using the integral equation method, the complex potential is a p-analytical function, which does not allow the use of the Cauchy integral formula for analytical functions, [8], [9].

In the present paper some developments of BEM and CVBEM for this actual problem of hydrodynamics are presented, establishing the fundamental equations and obtaining the solution of the integral equations by real-BEM using the iteration method related to the fluid density. Resembling solutions for the problem are found in [2], but with an essentially different numerical method.

In this paper we present practical aspects of the usage of the calculus algorithm for the study of the compressible fluid's stationary movement through profile grids, on an axial–symmetric flow–surface, in variable thickness of stratum. More precisely, we show the applicability of the boundary element methods (BEM) with real values, and the possibility of solving the integral equation of the velocity potential by using the successive approximation method w.r.t. the parameters ς (fluid's density) and h (thickness variation of fluid stratum), and using the Lagrangian interpolation formula through five points for the calculation of the derivatives of the velocity potential.

The rest of the paper is organized as follows: in Section 2 we state the problem and some theoretical considerations that are needed. In Section 3 we present the calculus algorithm for the study of the compressible fluid's stationary movement, together with its practical aspects.

2 Presenting the Problem

The fundamental equations (from the CVBEM method) in the problem of the compressible fluid's movement on a axial-symmetric flow–surface, in variable thickness of stratum, could be, [5], [6], [7]:

$$\begin{aligned} \overline{w}(z) &= \overline{V}_m + \int_{L_0} H(z,\zeta)\overline{w}(\zeta)d\zeta + i \iint_{D_{0*}^-} H(z,\zeta)\hat{q}(\zeta)d\xi d\eta \\ F(z) &= \overline{V}_m \cdot z + \Gamma \cdot G(z,\zeta_A) + \int_{L_0} H(z,\zeta)F(\zeta)d\zeta + \\ &\quad + i \iint_{D_{0*}^-} G(z,\zeta)\hat{q}(\zeta)d\xi d\eta \end{aligned} \tag{1}$$

where:

- $\overline{V}_m = \frac{\overline{V}_{1\infty} + \overline{V}_{2\infty}}{2}$ - asymptotic mean velocity;
- A – is a fixed point on the base profile L_0;
- t – is the grid step;
- Γ – is the circulation around L_0.

$$\begin{aligned} H(z,\zeta) &= \frac{1}{2ti} ctg \frac{\pi}{t}(z-\zeta) \\ G(z,\zeta) &= \frac{1}{2\pi i} ln\, sin \frac{\pi}{t}(z-\zeta) \\ \hat{q}(\zeta) &= 2\frac{\partial \overline{w}}{\partial \overline{\zeta}} = -\left[v_x \frac{\partial\, ln\, p^*}{\partial \xi} - v_y \frac{\partial\, ln\, p^*}{\partial \eta} \right], p^* = \frac{\varsigma \cdot h}{\varsigma_0} \end{aligned} \tag{2}$$

where:

- ς – is the fluid's density,
- h – is a function that represents the thickness' variation of the fluid stratum.
- D_{0*}^- – bounded simple convex domain, defined as:

$$D_{0*}^- : \left[-\frac{t}{2} \langle \xi \langle \frac{t}{2}, -\left(t+\frac{l}{2}\right) \langle \eta \langle \left(t+\frac{l}{2}\right) \right] \tag{3}$$

where l is the projection of L_0 profile's frame on the Oy axis.

Our purpose is to solve the fundamental equations (1) (obtained from the CVBEM method) using BEM in real variables. For doing so, we consider the fundamental integral–equation of the complex potential F(z) = φ' + iψ and transform it into an integral equation with real variables, i.e. we build the integral equation of the velocity potential φ'(s) (ψ (s) is the flow rate function).

Theorem 2.1 *[7], [11]. In the subsonic motion of the compressible fluid through the profile grid, on an axial–symmetric flow–surface, in variable thickness of stratum, the velocity potential* $\varphi(s)$, $s \in L_0$ *is the solution of the integral equation (4):*

$$\varphi(s) + \int_{L_0} \varphi(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma = b(s) + \iint_{D_{0^*}^-} \hat{q}(\sigma) N(s,\sigma) d\xi d\eta \tag{4}$$

where:

$s(x_0, y_0)$ *and* $\sigma(\xi, \eta)$ *are the curvilinear coordinates of the fixed point A on the* L_0 *base profile;*

$$b(s) = 2\left(x_0 V_{mx} + y_0 V_{my}\right) + \Gamma M(s, \sigma_A) + \int_{L_0} [\psi(s) - \psi(\sigma)] \frac{dN}{d\sigma} d\sigma$$

$$M(z_0, \zeta) = \frac{1}{\pi} arctg \frac{th \frac{\pi}{t}(\eta - y_0)}{tg \frac{\pi}{t}(\xi - x_0)} \tag{5}$$

$$N(z_0, \zeta) = \frac{1}{\pi} ln \sqrt{\frac{1}{2}\left[ch \frac{2\pi}{t}(\eta - y_0) - cos \frac{2\pi}{t}(\xi - x_0)\right]}$$

V_{mx}, V_{my} *are the components of the asymptotic mean velocity* V_m.

Proposition 2.1 *[10], [11]. In the case of an axial-subsonic movement of a perfect and compressible fluid through profile grids, the flow rate function is determined from the boundary condition (6):*

$$\psi(s) = u_0 \cdot \int_0^s p^*(s) \left(\frac{R}{R_0}\right) ds, \quad u_0 = \omega R_0 \tag{6}$$

where:

- ω *is the angular rotation velocity of the profile grid;*
- R0 *defines the origin of the axis system related to the turbine's axis.*

Equation (4) is an integro–differential equation. In this section, we will show a possibility of solving this equation applying the *method of successive* *approximation* (the iteration method), using also the result from [8] about the order of the term containing the double integral expression:

$$\varphi_{\hat{q}}(s) = \iint_{D_{0*}^{-}} \hat{q}(\sigma) N(s,\sigma) d\xi d\eta \qquad (7)$$

Proposition 2.2 *[10], [9]. In the case of the subsonic movement of the compressible fluid through the profile grid on an axial–symmetric flow–surface, in variable thickness of stratum, the integral equation of the velocity potential* $\varphi : D_{0*}^{-} \to R^2$ *is solvable by applying the method of successive approximations w.r.t. the parameter* $p^* = \frac{\varsigma \cdot h}{\varsigma_0}$.

Proof. For isentropic processes, by the Bernoulli–equation, we obtain:

$$\varsigma = \varsigma_0 \left(1 - \frac{\gamma - 1}{2}\frac{v^2}{c_0^2}\right)^{\frac{1}{\gamma-1}}, v^2 = v_\tau^2 + v_n^2, \qquad v\tau = \frac{d\varphi}{ds}, v_n = \frac{1}{p^*}\frac{d\psi}{ds} \qquad (8)$$

where:

- γ is the adiabatic constant;
- c_0 is the sound velocity in the zero velocity point;
- v_τ and v_n are, respectively, the tangential and normal velocities on L_0.

In the first approximation it is assumed that $\varsigma = \varsigma_0 =$ constant and $p^* = p^{*(0)} =$ constant. Thus, from (2), it results that $q^{(0)}(\sigma) = 0$. Hence, in the integral equation (4) the double integral (7) is neglected and results the following Fredholme integral equation of second type, with continuous nucleus:

$$\varphi^I(s) + \int_{L_0} \varphi^I(s) \frac{dM(s,\sigma)}{d\sigma} d\sigma = b^I(s) \qquad (9)$$

From solving equation (9) we obtain $\varphi^I(s)$ and furthermore from (6), (8), (12) ψ^I, ς^I are obtained. Finally, using the relation:

$$p^* = \frac{\varsigma \cdot h}{\varsigma_0}, \qquad \hat{q}(\sigma) = -grad\varphi \cdot grad\ ln\ p^* \qquad (10)$$

a p^{*I} and $\hat{q}^I(\sigma)$ are determined. In the second iteration $p^* = p^{*I}$ is assumed and for the determination of $\varphi^{II}(s)$ the following Fredholme integral equation of second type, with continuous nucleus, will be solved:

$$\varphi^{II}(s)+\int_{L_0}\varphi^{II}(\sigma)\frac{dM(s,\sigma)}{d\sigma}d\sigma=b^{II}(s)+\iint_{D_{0^*}^-}q^I(\sigma)N(s,\sigma)d\xi d\eta \tag{11}$$

where a ψ^I and $b^{II}(s)$ are previously calculated from (6) and (5), respectively.

From solving equation (11), we obtain φ^{II}. Furthermore, from (6), (8), (12) and (10) $\psi^{II}, \varsigma^{II}\ p^{*II},$ and $\hat{q}^{II}(\sigma)$ are obtained, respectively. Next, the third approximation might be done by assuming $p^* = p^{II}$, and so on.

Proposition 2.3 *[10]. Having given the values of the velocity potential on each element of the* L$_0$ *profile's division, the tangential velocity* v_τ *may be calculated in each division element of the* L$_0$ *basic profile's boundary by the formula, given by the Lagrange interpolation method through five points:*

$$\begin{aligned} v_{\tau i} &= \frac{d\varphi}{ds}(s_i)=\frac{2}{3\Delta s_i}(\varphi_{i+2}-\varphi_{i-2})-\frac{1}{12\Delta s_i}(\varphi_{i+4}-\varphi_{i-4}) \\ h &= \Delta s_i = s_{i+1}-s_{i-1}, i=1,3,5,\ldots,2n-1, \end{aligned} \tag{12}$$

where n *denotes the number of division elements and by* s_i *we refer to the* i^{th} *element of the division of* L$_0$.

To ensure the practical functionality of proposition 2.2, i.e. to indicate the solving method of the Fredholme integral equation of second type obtained in each approximation step (equation (6), (11)), let us reformulate and prove two more propositions.

Proposition 2.4 *[10], [11]. In the first approximation step, solving the velocity potential's Fredholme integral equation of second type is reduced to the solving of four systems of linear algebraic equations.*

Proof. Using the superposition rule of potential streams, we seek the solution of the Fredholme integral equation of second type (9) to be of the form:

$$\varphi^I = \varphi_1^I V_{mx} + \varphi_2^I V_{my} + \varphi_3^I \Gamma + \varphi_4^I u_0, \qquad u_0 = \omega R_o \tag{13}$$

where:

φ_k^I, k = 1 ÷ 4 are the solutions of the system (14) of integral equations:

$$\varphi_1^I(s) + \int_{L_0} \varphi_1^I(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma = 2x_0$$
$$\varphi_2^I(s) + \int_{L_0} \varphi_2^I(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma = 2y_0$$
$$\varphi_3^I(s) + \int_{L_0} \varphi_3^I(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma = M(s,\sigma_A) \quad (14)$$
$$\varphi_4^I(s) + \int_{L_0} \varphi_4^I(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma = b_4(s)$$

where:

$$b_4(s) = \int_{L_0} \left[\psi^I(s) - \psi(\sigma)\right] \frac{dN}{d\sigma} d\sigma \quad (15)$$

The integral equations (14) could be solved using the Bogoliubov-Krilov method, conform to which, solving each integral equation reduces to solving a system of linear algebraic equations.

Conform to the method, using an arbitrary division, we partition the boundary of L_0 in n subintervals $\Delta s = \Delta\sigma$. Note, that the chosen division might be not uniform, for instance at the trailing or the leading edge, where the variation of the function φ_k^I k is stronger from point–to–point, the length of subintervals might be shorter. In each subinterval, the function φ_k^I is assumed to be constant and equal to φ_{kj}^I where j represents the number of the middle–points of the considered subintervals. If the first division–points are debited by even numbers, and the division–points of the middle of the subintervals by odd numbers, then, conform to the approximation method, the integral equations (14) can be approximated by the following systems of linear algebraic equations:

$$\varphi_{ki}^I + \sum_{j=1}^{2n-1} \varphi_{kj}^I \Delta M_{ij} = b_{ki}, \qquad i = 1,3,5,.....,2n-1 \quad (16)$$
$$k = 1,2,3,4$$

where:

$$b_{1i}^{Ii} = 2x_i, \qquad b_{2i}^I = 2y_i, \qquad b_{3i}^I = M_{i,A}$$
$$b_{4i}^I = \sum_{j=1}^{2n-1} \Delta\psi_{i,j}^I \left(\frac{dN}{d\sigma}\right)_{i,j} \Delta\sigma_j \quad (17)$$
$$\Delta\psi_{i,j}^I = \psi_i^I - \psi_j^I, \qquad \Delta\sigma = \sigma_{j+1} - \sigma_{j-1}$$

Solving the algebraic system (16), we obtain φ_{ki}^{I} in n distinct point from the boundary of L_0. Finally, from equations (13), φ_i^I is determined in each point of the boundary's division.

Proposition 2.5 *[10], [11]. In the second approximation step, the Fredholme integral equation (11) of the velocity potential is reduced to solving four systems of linear algebraic equations.*

Proof. From (8) and (10), a ς^I and $\hat{q}^I(s)$ is determined, respectively. Consequently, using the superposition rule of potential streams, we seek the solution of the Fredholme integral equation of second type (11) to be of the form:

$$\varphi^{II} = \varphi_1^{II} v_{mx} + \varphi_2^{II} v_{my} + \varphi_3^{II} \Gamma + \varphi_4^{II} u_0, \qquad u_0 = \omega R_0 \tag{18}$$

where:

φ_k^{II}, k = 1 ÷ 4 are the solutions of the system (19) of integral equations:

$$\begin{aligned}
&\varphi_1^{II}(s) + \int_{L_0} \varphi_1^{II}(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma = 2x_0 + \iint_{D_{0^*}^-} \hat{q}^I(\sigma) N(s,\sigma) d\xi d\eta \\
&\varphi_2^{II}(s) + \int_{L_0} \varphi_2^{II}(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma = 2y_0 + \iint_{D_{0^*}^-} \hat{q}^I(\sigma) N(s,\sigma) d\xi d\eta \\
&\varphi_3^{II}(s) + \int_{L_0} \varphi_3^{II}(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma = M(s,\sigma_A) + \iint_{D_{0^*}^-} \hat{q}^I(\sigma) N(s,\sigma) d\xi d\eta \\
&\varphi_4^{II}(s) + \int_{L_0} \varphi_4^{II}(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma = b_4^{II}
\end{aligned} \tag{19}$$

where:

$$\begin{aligned}
&b_4^{II}(s) = \frac{1}{u_0} \int_{L_0} \left[\psi^{II}(s) - \psi^{II}(\sigma)\right] \frac{dN}{d\sigma} d\sigma + \iint_{D_{0^*}^-} q^I(\sigma) N(s,\sigma) d\xi d\eta \\
&\psi^{II}(s) = u_0 \int_0^s \left(\frac{R}{R_0}\right)^2 p^{*I}(s) ds, \qquad p^{*I} = \frac{\varsigma^I \cdot h^I}{\varsigma_0}
\end{aligned} \tag{20}$$

Using the numeric method presented in proposition 2.4, by applying the Bogoliubov-Krilov method, solution (19) is reduced to solving systems of linear algebraic equations.

These systems of linear algebraic equations will have the form:

$$\varphi_{ki}^{II} + \sum_{j=1}^{2n-1} \varphi_{kj}^{II} \Delta M_{ij} = b_{ki}, \qquad i = 1,3,5,.....,2n-1 \qquad k = 1,2,3,4 \tag{21}$$

where $b_{1i}^{II}, b_{2i}^{II}, b_{3i}^{II}$ and b_{4i}^{II} are obtained by using the Simpson formula for handling the double integral.

Solving the algebraic system (21), we obtain φ_{kj}^{II} k_j in n distinct point from the boundary of L_0. Finally, from equations (18), $\varphi_i^{II}\left(i = \overline{1,n}\right)$ is determined in each point of the boundary's division.

3 The Calculus Algorithm of the Fluid's Velocity Potentials through Profile Grids

1 Given are: the entering values into the profile grids of $p_1, v_{1\infty}, \alpha_1$ and the asymptotic mean velocity $\vec{V}_m$; the installation angle λ; the number of profiles n; the density ς_0 and the sound velocity c0 corresponding to the null–velocity point. The functions $h(\sigma)$ and $\frac{R}{R_0}(\sigma)$ are given by their table of values;

2 Conform to the chosen division, the coordinates $\sigma_i(\xi_i, \eta_i), i = 1,3,5,...,2n-1$ are determined. The circulation Γ is determined from the Jukovschi-Ciaplighin condition [10], [11];

3 From equation (5), the values of $\Delta M_{i,j}, \left(\frac{dN}{d\sigma}\right)_{i,j}, i, j = 1,3,5,...,2n+1$ are calculated;

4 Using the trapezoid method, ψ_i^I is calculated from the integral equation (6), and, using (17), $b_{ki}^I(k = 1 \div 4)$ re determined;

5 The linear algebraic system (16) is solved, and, thus, φ_{ki}^I is obtained. Furthermore, from (13), φ_i^I is also obtained;

6 Using the Lagrange interpolation formula through five points (12), $v_{\tau i}^I$ is calculated. Next, from (8), v_i^I is determined, and, furthermore, ς_i^I is also obtained;

7 Using ς_i^I and h = const, from the integral (20), by the trapezoid method, a $\psi_i^{II}(\sigma)$ is determined. Using equations (19) and (20), $b_{ki}^{II}(k = 1 \div 4)$ are obtained;

8 The integral equations (19) are solved, transforming them first into a linear algebraic system. Furthermore, φ_{ki}^{II} is obtained, and, from (18), φ_i^{II} is determined;

9 Using the Lagrange interpolation formula through five points (12), $v_{\tau i}^{II}$ is calculated. Next, from (8), v_i^{II} is determined, and, furthermore, ς_i^{II} is also obtained. Furthermore, using ς_i^{II}, the next iteration $\varsigma_i = \varsigma_i^{II}$ can be calculated, h = variable, and the algorithm continues.

Conclusions

We have shown some practical aspects of the usage of the calculus algorithm for the study of the compressible fluid's stationary movement through profile grids, on an axial–symmetric flow–surface, in variable thickness of stratum, namely:

- the usage of the boundary element method with real values;
- the applicability of the successive approximation method w.r.t. the parameters ς (fluid's density) and h (thickness variation of fluid stratum) for solving the integral equation of the velocity potential;
- the usage of the Lagrangian interpolation formula through five points for calculating the derivatives of the velocity potential.

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