Democratising Economic Knowledge Acquisition by Products of Computational Intelligence and Informatics

Ladislav Andrášik

Institute of Management, Slovak University of Technology, Bratislava
ladislav.andrasik@stuba.sk

Abstract: There are a large set of complex economic tasks that caused inconveniences in knowledge acquisition on, because they need strenuous mathematical procedures. Unfortunately not only students but sometimes experienced economic scholars face up such difficulties, too. These facts are narrowing the group of people who are interesting about such problems to a small group of best mathematically prepared ones. On the other hand, thanks to progress of Computational Intelligence and Informatics there are several possibilities to solve those problems. First of all among them we have in mind possibilities for experimentations in virtual laboratories, created with their helps. We have rich experiences with creation such laboratories and with experimentation in them with a large set of economic tasks difficult for understanding not only per see, but after deep conventional studying too. On those bases we claim that experimentation in virtual economic laboratory widening the group of interested people possibilities to understand complex economic phenomena. In this paper we will show that already relatively simple devices, from the point of view of informatics, can effectively help in advancing economic knowledge acquisition. This reality is the reason why we argue with verb "democratisation". Our purpose is the exhibition some of well known economic theories and/or models based on mathematics but simples one. One of them is model of cyclical economic growth created by famous Hungarian economist Lord Nicolas Kaldor, the other is the world wide known duopoly model of Auguste A. Cournot, which is the precursor of theory of game created in 1928 by János (John) von Neumann also born in Budapest. For exhibition we are using several modification of duopoly. Another mode for visualising of democratisation methods is the model of monopolist's searching created by Finnish economist Tönu Puu and so on. For that visualising we are using comparatively simply programming and languages devices notably STELLA, **iDmc** *and, maybe surprisingly Excel too. The economic knowledge acquisition process realised by the aid of experimentation in virtual laboratories is easy, effective and bring excellent knowledge's with very long perpetuity. It is proved so in student's learning as in scientific research too. On the other hand those methods helping basically in exosomatic knowledge dissemination, printing on different media and spreading in various nets. In these senses virtual laboratories are self-proving as important tools of knowledge technology.*

Keywords: absorbing area, basins of attractors, chaos, Cournot duopoly, critical curves, cycles, cyclical economic growth, Kaldorian models of cycles, economic behaviour, learning from experiences attained on the base of experimentation in virtual laboratories, Lyapunov exponents, Neimark bifurcation, orbits, theory of games, manifolds

1 Introduction

Endosomatic knowledge acquisition in economics may be in several cases rather difficult but for some of them needed knowledge entirely not coming up to. This is right not only for university students in first-second semester and/or trimestri for the reason that such analysis needs strenuous mathematical procedures. That is to say there are a large set of complex economic tasks in macro/micro economics and also in other economic and managerial task that caused inconveniences in knowledge acquisition. So a large group of interested people, sometimes experienced economic scholars inclusive, must face up various difficulties, mainly for lacking of needed mathematical knowledge and analytical tools and experience. In our opinion these facts are narrowing the group of people who are interesting about such problems to a small group of best mathematically prepared ones. On the other hand, thanks to progress of Computational Intelligence and Informatics there are several possibilities to solve those problems. First of all among them we have in mind possibilities for experimentations in virtual laboratories, created with their helps. We have rich experiences with creation such laboratories and with experimentation in them with a large set of economic tasks difficult for understanding not only per see, but after deep conventional studying too. Based on these opinions we can argue that with experimentation in virtual laboratories the student has opportunity to achieve at least qualitative imagination about complex economic systems. On those bases we claim that experimentation in virtual economic laboratory widening the group of interested people possibilities to understand complex economic phenomena. In this paper we will show that already relatively simple devices, from the point of view of informatics, can effectively help in advancing economic knowledge acquisition. This reality is the reason why we argue with verb "democratisation". In this paper we will show that already relatively simple devices, from the point of view of informatics, can effectively help in advancing economic knowledge acquisition. Our intention is the exhibition some of well known economic theories and/or models based on mathematics but simples one. One of them is the model of cyclical economic growth created by famous Hungarian economist Lord Nicolas Kaldor [6], the other is the world wide known duopoly model of Auguste A. Cournot [3], which is the precursor of theory of game created in 1928 by John von Neumann[1] also born in Budapest. For exhibition we are using several modification of original duopoly model. Another mode for visualising of democratisation methods is the model of monopolist's adaptive searching created by Finnish economist Tönu Puu[2] and so on. For that visualising we are using comparatively simply programming and languages devices notably STELLA, iDmc and, maybe

[1] We have "in natura" only his book written with Oscar Morgenstern [3], but the original essay one can find in Internet.

[2] For a shortage of place we unfortunately cannot exhibit our large group of virtual experiments with that very interesting model [11, pp. 118-130].

surprisingly Excel too. The economic knowledge acquisition process realised by the aid of experimentation in virtual laboratories is easy, effective and bring excellent knowledge's with very long perpetuity. It is proved so in student's learning as in scientific research too. On the other hand those methods helping basically in exosomatic knowledge dissemination, printing on different media and spreading in the nets. In these senses virtual laboratories are self-proving as important tools of knowledge technology usable in helping deeper understanding of economic phenomena. It is our debt to say that these approaches are correct first of all in some causes of general economics, not in econometrics where we need another sophisticated simulation methods.

2 The Model of Cyclical Growth in Kaldor Tradition

The solving the theoretical and/or practical problems concerning business cycles and/or cyclical economic growth is typical among those one that for deeper understanding necessarily need sophisticated mathematical tools. Among other such tools as belong to theory of differential equations, topology, theory of groups and groupoids and so on. This is the reason why reaching deeper understanding of such phenomena was been restricted on relatively small group of interested people before wide availability of hardware, software devices and products of informatics. But this situation in a few former decades was dramatically changed. We have a rich offer of devices, tools, programs and sophisticated methods suitable for helping in difficult mathematical obstacles arising in study of business cycle and similar economic phenomena, process and tasks, for example the dynamics of the Cournot duopoly model [3] dating from 1838. The rationale of this part of paper is to exhibit or visualise some procedures that successfully proved someone in fulfilling such tasks in the cause of general theory of business cycle.

In traditional economic imaginations about business cycle and/or cyclical economic growth their graphs is drawing (and/or their qualitative representation) like as look near as being sinusoidal waves. On the other hand the graphs plotted on the base of statistical dates look as some disturbed saw. Because of such shape majority of explanation is arguments with its randomness'. But not in all cases this shapes is caused by randomness of business processes. Most of them are based on the holistic collective product of subjective and/or rational reasoning of business people (i. g. players) in their decision making. The individual decision making of consumer population members whether consume or save on the one hand and individual decision making of entrepreneurs whether invest or not, is not necessary cooperative in any of two different populations[3].

One of famous founder of upper pointed out interpretation of that reason is Lord N. Kaldor born in Budapest. He propose for founding of sequence of consumption-saving-investment decision making to use “S”-shaped curve (generated by cubic and/or arc tan function). But this goes to non trivial, somewhat strange behaviour of such type of growth model. This property of Kaldorian model is very appropriate for our purpose in these exhibitions of experimentation in virtual laboratories possibilities for endogenous knowledge acquisition. That is the reason for beginning our exposition of the possibilities of economic knowledge acquisition in the process of experimentation in a virtual economy with relatively simply model of cyclical growth. Namely we are operating with a upper mentioned nonlinear business cycle model of Kaldor-type but in discrete-time, to illustrate the difficulties in understanding such behaviour because of existence some particular global bifurcations in that process. Such behaviour determines the appearance or disappearance of attracting and repelling closed invariant curves. The consequences of such bifurcations sequences, which involve homoclinic tangencies and transversal intersection of the stable and unstable manifolds of saddle cycles, is that it may increase the complexity of the basins of attraction of multiple, coexisting attractors. Particularly, for the business cycle model going we examined by experimentation in virtual laboratory, such dynamic phenomena explain the co-existence of two stable steady states and an attracting closed curve, with an intricate basin structure, for wide ranges of the parameters. Encouragement for construction virtual Kaldor-type laboratory for that task is due to Italian economists A. Agliary, R. Dieci and L. Gardini (ADG model [2]) for first example. The second one[4] is the Kaldor-type model of Hermann [5]. Our purpose is not developing of new model but only exhibition of possibilities for making easer for student understand more difficult economic tasks.

3 The expression “population” we use in those judgment not for their assemblage as physical individuals but for differentiating them on the base of their behavioural (ethological) function in an economy. So the very same (lawfully, legitimately authentic) physical person may play several different economic roles, and he can be rank among several different populations simultaneously.

4 There are several other scholars developed similar models based on N. Kaldor, P. A. Samuelson, M. Kalecki, R. Hicks etc., but for shortening this paper we are used only that two among them. We are obliged to note that our virtual laboratory is based on Lorenz H-W., modification of Herrmann’s model [8].

2.1 Kaldor's like Model: "ADG-Model"

In the Kaldorian model there are used customary notations such as in textbook economics, that is: K… is capital stock; Y… is national income (or *GDP*); C… is consumption; I… is investment, which both depend only on the output Y and the capital stock K; δ… is rate of depreciation of capital; κ… is (Keynesian) multiplier; a… is accelerator (due to Aftalion), and so on. The authors of their simplification of original Kaldor model, upper abbreviated with us as "ADG model" begin their analysis with following statements and formulas.

$$\begin{aligned} Y_{t+1} &= Y_t + \alpha(I_t - S_t), \\ K_{t+1} &= (1+\delta)K_t + I_t, \end{aligned} \tag{1a}$$

where α… is the adjustment velocity[5]. In [2] it is also assumed that firms adjust supply Y to demand $C + I$ *through* a variation of production proportional to excess demand. These assumptions may be also formalized through the following well-known discrete-time scheme

$$\begin{aligned} Y_{t+1} &= Y_t + \alpha(C_t + I_t - Y_t) = Y_t + \alpha(C_t - S_t); \\ K_{t+1} &= (1-\delta)K_t + I_t. \end{aligned} \tag{1b}$$

In (1b) $S{=}Y - C$ represents the savings function, as another expression of $C{=}Y - S$, that is consumption function, because the formula of product distribution is $Y{=}\ C + S$. Kaldor [6, 1940] assumed that, for a given K, the total marginal propensity to spend ($\partial C/\partial Y + \partial I/\partial Y$) is greater than one for "normal" levels of production and less than one for high or low levels of production. A possible way to incorporate this idea into the general model (1b) is to use, for instance, a savings function $S = Y - C$ proportional to income ($S = \sigma Y$, where $\sigma = 1 - \partial C/\partial Y$ is the *constant propensity to save*) and to assume that *investment demand is an S-shaped function of income* for any given level of capital. Moreover, in Kaldor's original idea the investment and savings schedules are short period functions and are assumed to shift as capital stock shifts: in particular, a rise (fall) in capital stock will shift the investment function down (up). The latter feature can be formalized by assuming that the investment demand is a decreasing function of K, for any given Y. A simple mathematical formulation which satisfies the above requirements is the one

5 The adjustment speed parameter $\alpha, (\alpha > 0)$ in market economy is in fact velocity of adjustation of firms to difference between supply of production and demand $C + I$ to be proportional to excess demand. In macroeconomic environment it is the difference between demand for investment (I_t) and of savings (S_t). Some small chosen value of α mean *cautious reaction*, the situation can be referred to as high level of *risk aversion*, or as appropriate degree of monopoly. On the contrary high level of $\alpha, (\alpha > 0)$ are caused fierce reactions because of *risk bearing* or of the detriment of competitive compression. This reason is interesting because of possible evolving to coordination failure.

proposed in Herrmann [5] and in conception of several Italian mathematical economists as Rodano, Bischi et al., that was used for creation of ADG model [1], in which the saving and investment schedules are specified, respectively, as

$$\begin{aligned} S_t &= \sigma Y_t \\ I_t &= \sigma\alpha + \gamma\left(\frac{\sigma\alpha}{\delta} - K_t\right) + \arctan(Y_t - \alpha) \end{aligned} \tag{2}$$

where the coefficient $\sigma \equiv 1 - \partial C/\partial Y$, $0<\sigma<1$, represents the (constant) *propensity to save,* α represents the "normal" level of income (exogenously assumed in firms' expectations), while $\sigma\alpha/\delta$ is the "normal" level of capital stock. In particular, in the investment demand I_t two short run components are considered: the first one is proportional to the difference between normal capital stock and current stock, according to a coefficient γ ($\gamma > 0$), which captures the presence of adjustment costs; the second one is an increasing *S*-shaped function of the difference between current income and its "normal" level. By substituting the expressions of I_t and S_t into the dynamic model (1b), the authors of [2, pp. 327-8] obtained the following two-dimensional nonlinear discrete-time dynamic system:

$$M: \begin{cases} Y' = Y + \alpha\left[\delta\mu + \gamma\left(\dfrac{\sigma\mu}{\delta} - K\right) + \arctan(Y-\mu) - \sigma Y\right] \\ K' = (1-\delta)K + \sigma\mu + \gamma\left(\dfrac{\sigma\mu}{\delta} - K\right) + \arctan(Y-\mu). \end{cases} \tag{3}$$

where the symbol "'" denotes the unit time advancement operator. In topology such entity as (3) is called *map*. The authors, in order to make their explanation based on the analysis of the model (3), as simple as possible they introduced for new coordinates, so that the normal level of income and capital (α, $\sigma\alpha/\delta$) is represented by the origin (*0, 0*) of the plane; that is, we consider

$$\Phi: \begin{cases} x_t = K_t - \dfrac{\sigma\alpha}{\delta} \\ y_t = Y_t - \alpha \end{cases} \tag{4}$$

Thus, the model they and we focus on too, becomes

$$T: \begin{cases} x' = (1-\gamma-\delta)x + \arctan y \\ y' = -\alpha\gamma x + (1-\alpha\sigma)y + \alpha \arctan y\ . \end{cases} \tag{5}$$

Evidently the new map T is topologically conjugate to the map M describing the Kaldorian model, and the dynamics of the latter can be obtained from that of (6) by applying the inverse transformation of Φ.

The equilibrium points of the model described by map (5) are the fixed points of that one. Solution of map T become (the graph is in Fig. 1 right)

$$\begin{cases} x = \dfrac{\sigma}{\delta} y \\ \dfrac{(\gamma+\delta)\sigma}{\delta} y = \arctan y \end{cases} \tag{6}$$

and is obtained by reformulation of (5) with $x' = x$ and $y' = y$.

Besides the upper solution $E^*=(0,0)$, it can be observed that may exists further fixed points of following equation

$$\frac{(\gamma+\delta}{\delta} y = \arctan y \tag{7}$$

if (7) admits non-zero solutions, that is, if the straight line of equation $z = ((\gamma + \delta)\sigma/\delta)y$ and the sigmoid shaped graph of the function $z = \arctan(y)$ intersect in some points other than the origin. Since this occurs if and only if the slope of the straight line is lower than the slope of the function $f(y) = \arctan y$ (at the origin), the authors of ADG model state the following:

Proposition 1. *If $\sigma \geq (\delta/(\gamma + \delta))$ then $E^* = (0, 0)$ is the unique fixed point of the map T defined by* (5). *If $\sigma < (\delta/(\gamma + \delta))$ then two further fixed points, P^* and Q^*, exist, symmetric with respect to the fixed point E^*.*

Proposition 2. *The fixed point $E^* = (0, 0)$ is locally asymptotically stable if the parameters μ and σ belong to the region ABCD of the plane (μ,σ), with vertices $A = (0, (\delta/(\delta + \gamma))$, $B = (0, 1)$, $C = (((\delta + \gamma)/\gamma), 1)$, $D = (((\delta + \gamma)2/\gamma), (\delta/(\delta + \gamma)))$ where AB belongs to the line $\mu=0$, BC to the line $\sigma = 1$ and the side AD to the line*

$$\sigma = \sigma_p = \frac{\delta}{\delta+\gamma}, \tag{8}$$

while the arc connecting C and D belongs to the hyperbola of equation

$$\sigma = \sigma_h(\mu) = \frac{(1-\delta)\mu - \delta - \gamma}{(1-\delta-\gamma)\mu}. \tag{9}$$

Moreover, the arc CD of the stability region represents a Neimark-Sacker bifurcation curve, at which E is transformed from stable to unstable focus, while the segment AD represents a (supercritical) pitchfork bifurcation boundary, at which E* becomes a saddle and two new stable fixed points appear close to E*.*

For proving that it is need mathematical qualification and skill, which is not the equipment of all who interested in understanding this theory of trade cycle. Instead of proving we sketch in Excel proper graphs visualising those situation. Further, for better understanding them it is more appropriate to use experimentation in knowledgeable virtual laboratory.

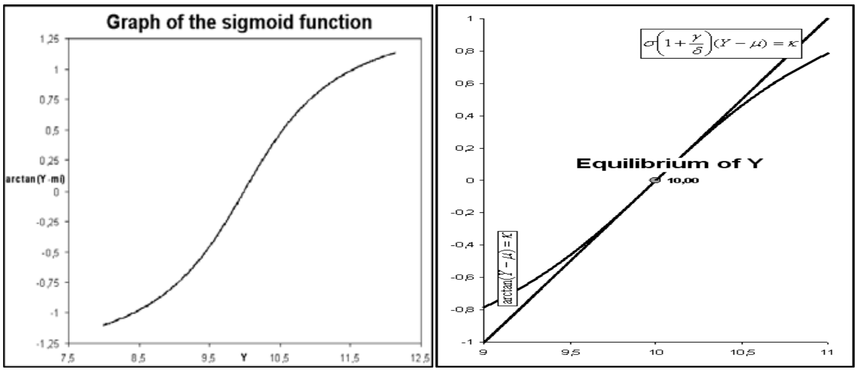


Figure 1
The simple insight to sigmoid graphs created in Excel

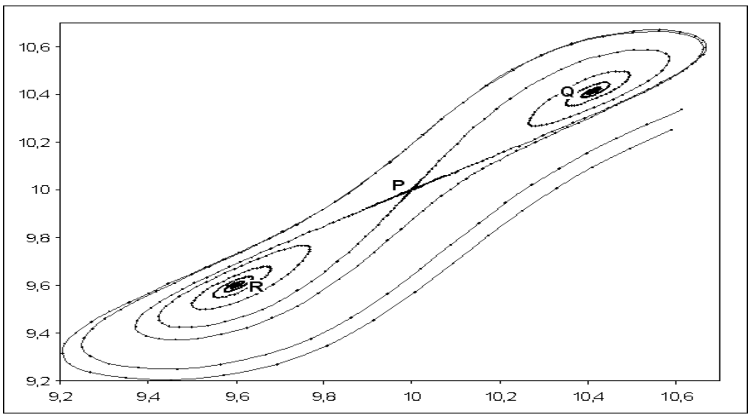


Figure 2
The quasi manifold received as a result of experiment in Excel made laboratory

We are using for exhibition of possibilities of virtual laboratory experimentation for democratization of endogenous knowledge acquisition more than a few algorithms from several offers of ***iDmc*** software, namely routines of: Trajectory, Basin of attraction, and further is Herrmann's model such routines as Absorbing area with possibility for generation a set of critical curves and generation of attractors, and algorithms for generation Lyapunov exponents.

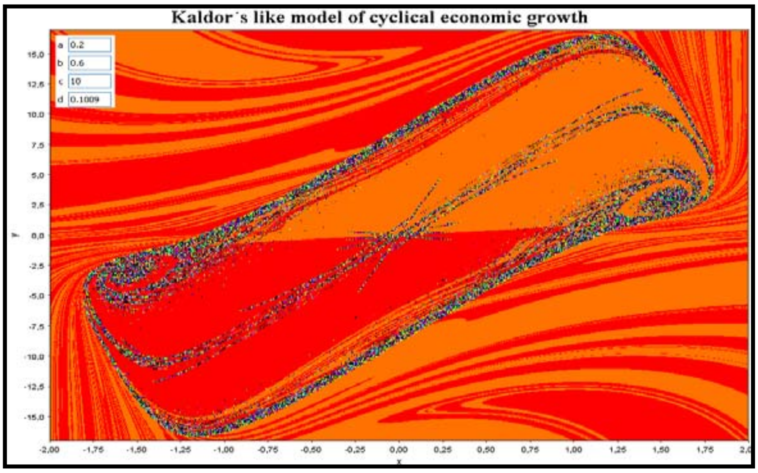


Figure 3
The great number of trajectories achieved by variation with starting values of variables *x* and *y* (the enormous set of discrete dots in front); in background there is a basin of attraction

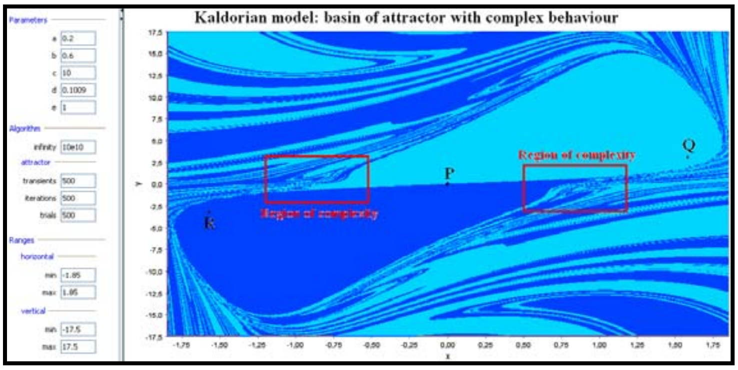


Figure 4
The basin of attraction of cyclical growth

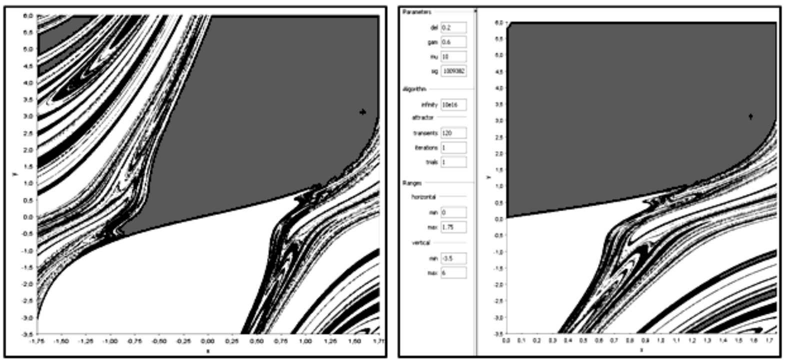


Figure 5
Two enlarging of the right region of complexity from Fig. 4

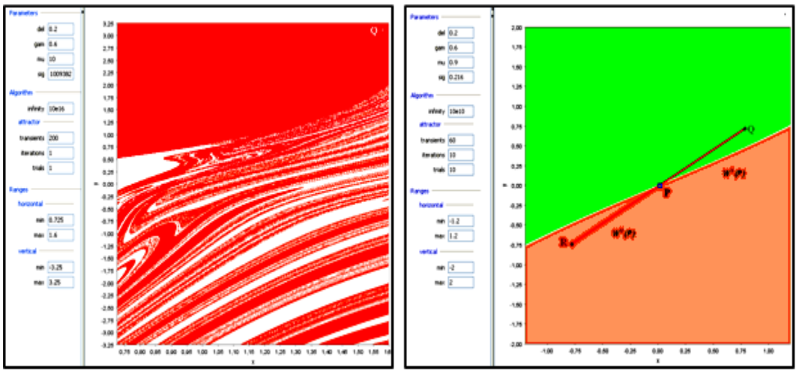


Figure 6

The third enlarging of complexity region (left) and fixed point ***P*** with stable nodes ***Q*** and ***R***

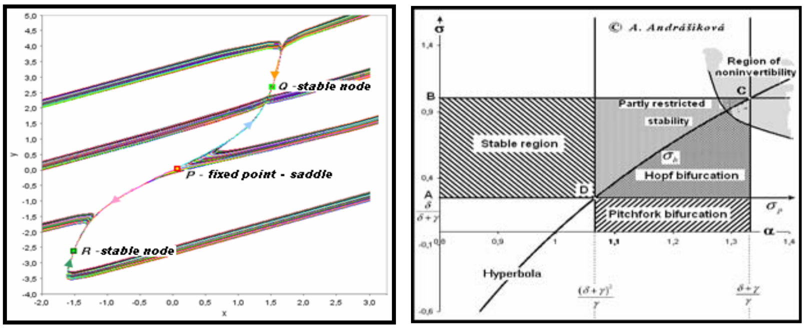


Figure 7

Stable nodes and set of trajectories achieved by variation with starting values of variables (left) and the situation belonging to **Proposition 2.** in α-σ space

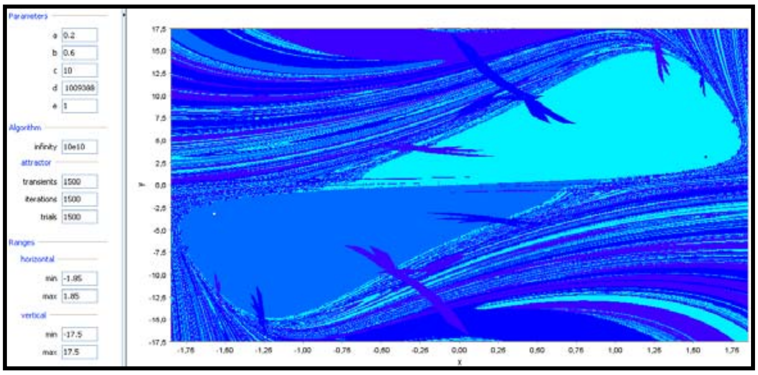


Figure 8

The basin of attraction of cyclical growth: higher level of parameter *d, (d=0.1009388)*

1.2 Kaldor's like Model: Herrmann's Approach

We are noted already, that in the field of mathematical analysis of the problem released by Kaldor there are several less or more different approaches. One of further model which we are using for our goals are created by R. Herrmann. Subsequent two-dimensional discrete-time business-cycle model we have directly used as a loaning from the monograph [8]. Formulas for creation our virtual laboratory in ***iDmc*** for Kaldor-Herman-Lorenz model is:

$$T: \begin{cases} Y' = \alpha[\beta(kY - K) + \delta K + \overset{Y}{C} - Y] + Y \\ K' = \beta(kY - K) + K \end{cases} \tag{10}$$

where the function $\overset{Y}{C}$ is suggested by Herrmann, that is his own construction of the *H-sigmoid* and there is in the form

$$\overset{Y}{C} = 20.0 + \frac{2}{\pi}10.0\arctan\left(\frac{0.85\pi}{20.0}(Y - Y^*)\right), \tag{11}$$

with $Y^*=22.22$, and α… is the speed of adjustment, β… is the propensity to investment, and $\beta>0$, δ… is the depreciation rate and $\delta>0$, and k… is the linear multiplication rate of accumulation for desired investment from output Y, that is $k = \frac{K_t^d}{Y_t}$, where K_t^d … is desired capital stock, and $k>0$. The investment function is

$$I(Y_t, K_t) = \beta(K_t^d - K_t + \delta K_t. \tag{12}$$

On that groundwork we are creating our virtual laboratory following right away.

Script the virtual laboratory in Java-LUA:

```
--@@
name = "Kaldor-Herrmann type model - Andrášik"
description = "See Model refs Andrášik guide"
type = "D"
parameters = {"alpha", "beta", "delta", "kappa", "C"}
variables = {"Y", "K"}
function f(alpha, beta, delta, kappa, C, Y, K)
   Y1 = Y + alpha*(beta*(kappa*Y - K) + delta*K + 20 + 20*math.atan((0.85*math.pi/20)*(Y - C))/math.pi - Y)
   K1 = K + beta*(kappa*Y - K)
   return Y1, K1
end
```

function Jf(alpha, beta, delta, kappa, C, Y, K)

return

*1+alpha*beta*kappa+(20*alpha/math.pi*((0.85*math.pi/20)*(Y-C)))-alpha, -alpha*beta*kappa+alpha*delta,*

*beta*kappa, 1+beta*kappa*

end (The end of the script)

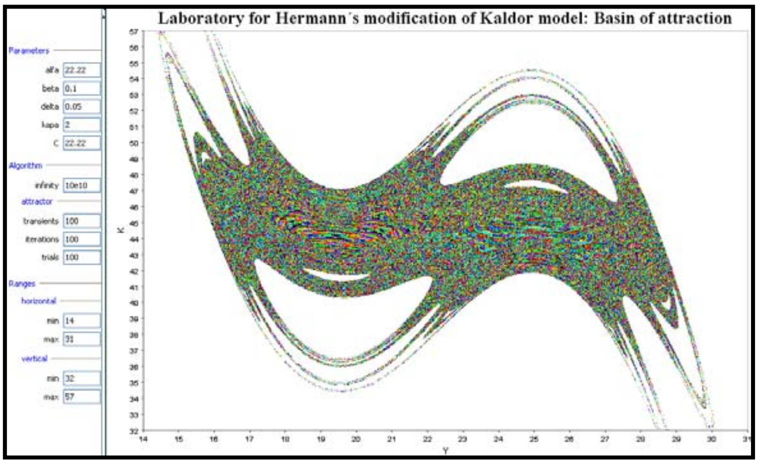


Figure 9
The basin of attraction of cyclical growth: Kaldor-Herrmann model

Using that virtual laboratory we can to obtain several very interesting and for understanding of cyclical economic growth important set of information, the first is in snapshot from the experiment fulfilled in routine “Basin of attraction”, Fig. 9. We can see a massive area of different coloured dots every of them is a loci of attraction arrived from a particular starting loci. The empty area is unreachable.

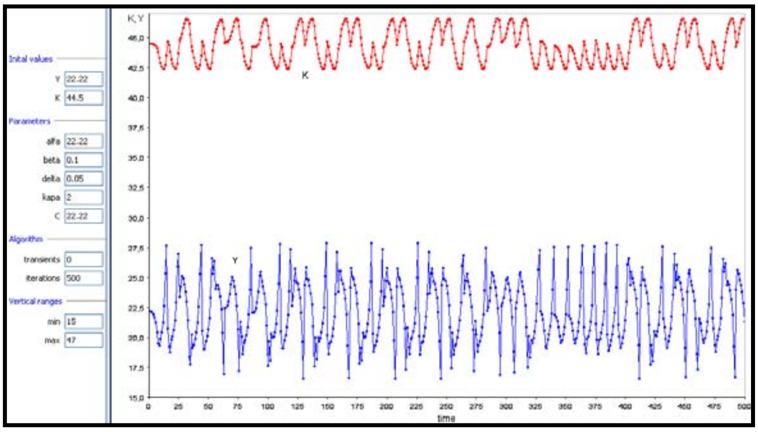


Figure 7
The variables K and Y time-step trajectories of cyclical growth: Kaldor-Herrmann model

For better understanding of the character of evolution we made another experiment in "Trajectory" algorithm but in time steps. In snapshot of Fig. 7 we see the evolution of both variables in interval *<0; 500>*. It is clear that so capital stock *K* and also income *Y* has two levels: upper and lower one. The artificial economy is evolving by jumping up and down but not in regular schedule between upper and lower level. In first sight it looks like random walk, but in reality the process is deterministic. Maybe it is such also some part and/or interval of evolution of that macroeconomic variables in objective reality.

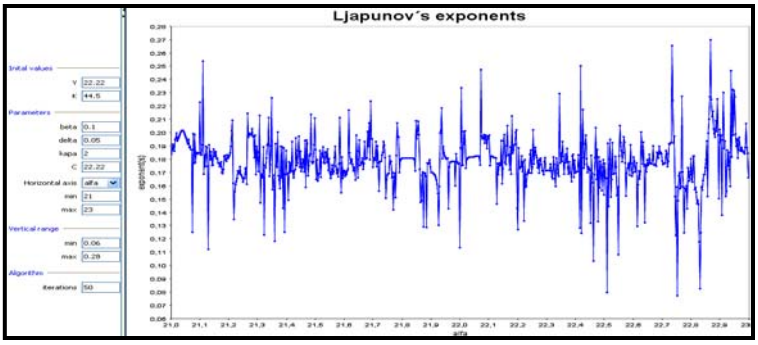


Figure 8
Model Kaldor-Herrmann of cyclical growth: space of parameter α against Lyapunov exponents

For deeper insight into stable-unstable character of the process we are used the algorithm for plotting Lyapunov's exponents exhibited in Fig. 8. It is clear that all exponents are in the positive part of coordinate surface, so the process is "Lyapunovian-unstable" (Ly-u). By the way to prove this with mathematical procedures isn't easy for every interested subject.

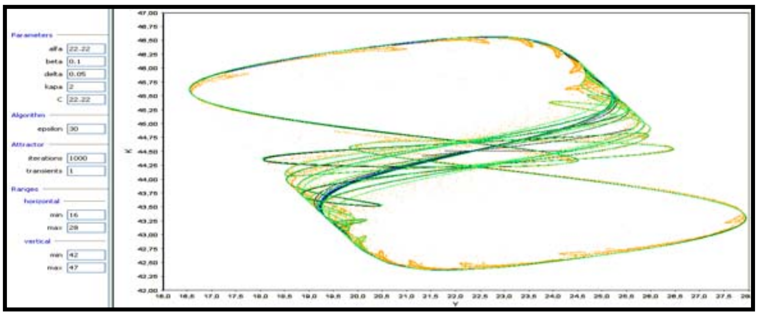


Figure 9
Absorbing area of the Kaldor-Herrmann type cyclical growth model: critical curves (front) and attractors (dark yellow dots - at the back)

There are further efficient algorithms in ***iDmc*** that is "Absorbing area" offering possibility for plotting wide set of critical curves and creating attractors, see Fig. 9. Deeper insight into absorbing are with attractor is in Fig. 10. In this few snapshots there are a lot of information showing the idiosyncrasy of artificial cyclical economic growth that serve important starting field for imagination on peculiarities and/or casualties of real economic growth and on the reason of such behaviours[6].

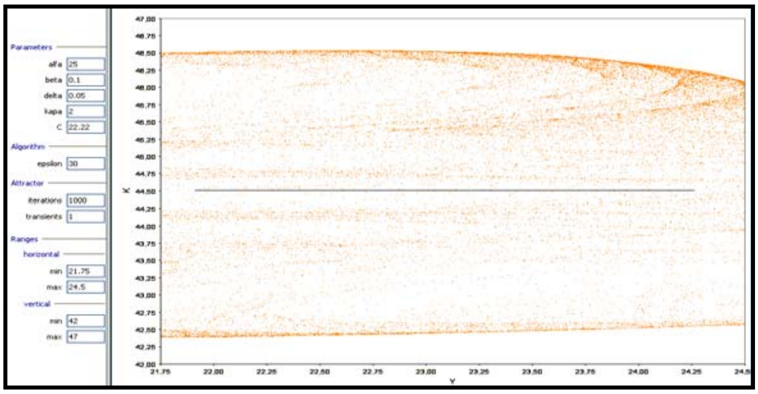


Figure 10
Deeper insight into an Absorbing area of the Kaldor-Herrmann type cyclical growth model: detail of attractors (dark yellow dots)

[6] Such imagination belongs to "Ethology of Investment Behaviour", or "Ethology of the Investors".

2 The Duopoly Models in Cournot Tradition

Let us consider of two players (or firms), denoted by X and Y in the market, as it is in the conventional duopoly model, which produce a homogenous good Q. We than may assume, following Puu´s formulation, in those tradition too, that

• the demand function of the good is isoelastic, i.e. $Q_t = \frac{a_0}{p_t}$, where Q is the total demand and p is the price of the good,

• the firms are producing the amounts of goods x and y,

• the costs are linear, $C_i(q) = c_i q$, where the marginal costs c_i ($i=1,2$) are positive. Linear cost functions provide a convenient heuristic setting to detect the analytical properties of the nonlinear Cournot output adjustment and make a formidable mathematical problem simpler and manageable. Goods are perfect substitutes so that, provided demand equals supply, the total demand equals the total supplies, $Q = x + y$ and consequently the price in the market will

$$p = \frac{1}{Q} = \frac{1}{x+y}. \tag{13}$$

This demand function is not unproblematic but remedy of that one is ease technical problem only, it is enough to add any positive constant to denominator of (?). If marginal costs are constant, the profit function of the two players become accordingly

$$\Pi_1 = \frac{x}{x+y} - ax, \tag{14}$$

$$\Pi_2 = \frac{y}{x+y} - bx. \tag{15}$$

$$\begin{aligned} &\max_x \Pi_1 \qquad \text{with } x, y \geq 0, \\ &\max_y \Pi_2 \qquad \text{with } x, y \geq 0. \end{aligned} \tag{16}$$

In () and () we use beside denotation of marginal cost c_i denotation a and b, for better manipulation with them in virtual laboratories because the lack of sub i. The reaction function become

$$\begin{aligned} x_{t+1} &= \sqrt{\frac{y_t}{a}} - y_t, \\ y_{t+1} &= \sqrt{\frac{x_t}{b}} - x_t. \end{aligned} \tag{17}$$

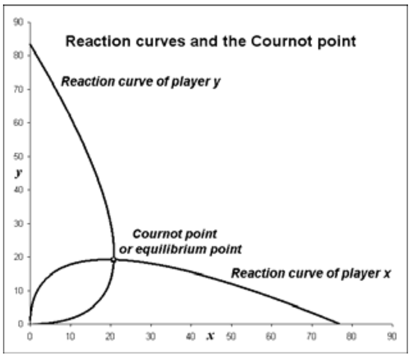


Figure 11
Reaction curves plotted in Excel

In this simple case of duopoly, we are assuming that adjustment of two player amount is *instantaneous*. We made, only for illustration of shapes of reaction functions, opposite graphs in Excel, as it is in Fig.11. The loci of Cournot point of output quantities in coordinates x and y is solved by simultaneous system of equations

$$x = \frac{b}{(a+b)^2}, \tag{18}$$

$$y = \frac{a}{(a+b)^2}, \tag{19}$$

and equilibrium profits are[7]

$$\Pi_1 = \frac{b^2}{(a+b)^2}, \quad \Pi_2 = \frac{a^2}{(a+b)^2}. \tag{20}$$

3.1 Simple Cournot Model of Duopoly in Virtual Laboratory

For the exhibitions of strange behaviour of simple Cournot duopoly so far, we created virtual laboratory again in iDmc. The script in Java/LUA is follow:

```
--@@
name = "Simple Cournot model of duopoly"
description = "See Model in Andrášik paper"
```

[7] These conventional Cournot formulas are so also in Puu's monographic book[11], pp. 136-7.

```
type = "D"
parameters = {"a", "b"}
variables ={"x", "y"}
function f(a, b, x, y)
        if (a*y <= 1) then
        x1= math.sqrt(y/a)-y
        else
        x1=0
        end
        if (b*x <= 1) then
        y1= math.sqrt(x/b)-x
        else
        y1=0
        end
return x1, y1
end
function Jf(a,b,x,y)
        if (a*y <= 1) then
        dxdy =math.sqrt(1/(4*a*y))-1
        else
        dxdy = 0
        end
        if (b*x <= 1) then
        dydx = math.sqrt(1/(4*b*x))-1
        else
        dydx = 0
        end
return
0,      dxdy,
dydx,   0
end
```
[8]

[8] This laboratory is essentially the same as L&M (M. Lines & A. Medio) model[], p. 33. Before availability of their iDmc software we made simulators in “C”, in Simulink of MATLAB, in STELLA and in Excel too, but iDmc is actually better for the “democratization”.

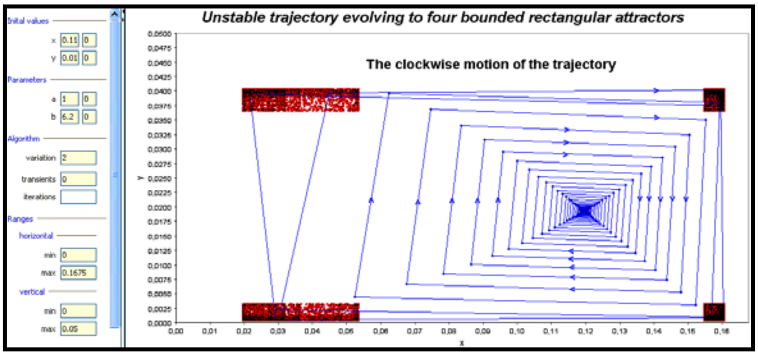


Figure 12
The snapshot of unstable evolution gets by "Trajectory" algorithm running

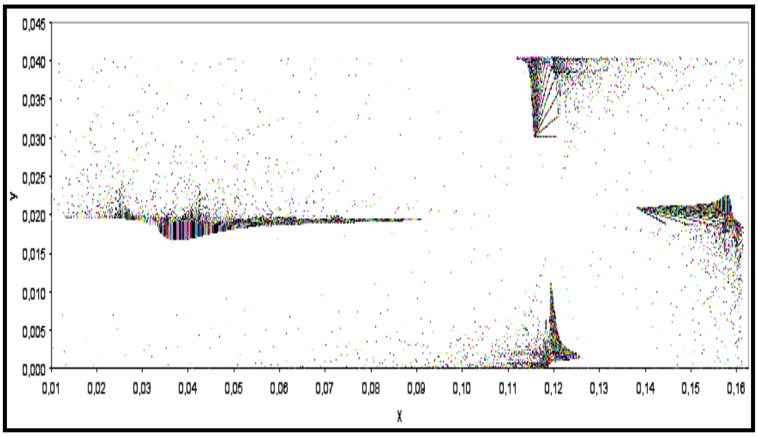


Figure 13
Quasi basin of attractors emerging on variation with parameter *a*: there are bifurcations from convergence to Cournot point to divergence to four attractors

We are using for exhibition of possibilities of virtual laboratory experimentation for democratization of endogenous knowledge acquisition more than a few algorithms from several offers of ***iDmc*** software, namely: Trajectory, trajectory with one parameter variation, basin of attraction and bifurcation map emerging as a results of growing iteration one of parameter selected from the set of them.

3.1.1 The Trajectories of the Simple Duopoly Evolution

In a subsequent snapshot of experiment from the virtual laboratory Fig. 12 we can clearly see four bounded areas visited in the process of clockwise motion of very high number of iterations.

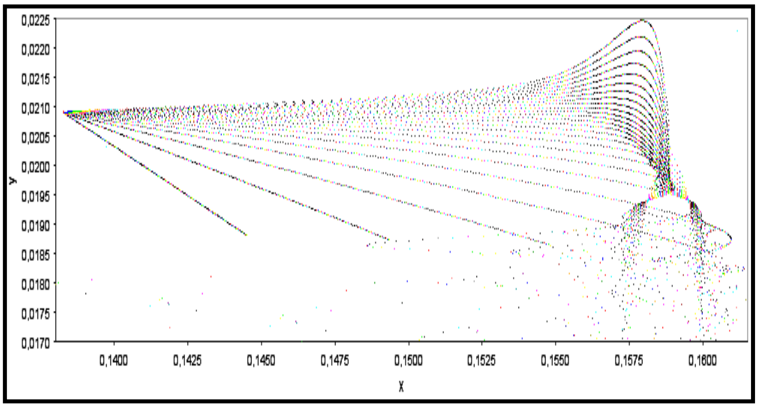


Figure 14
Enlarging of Fig. 13

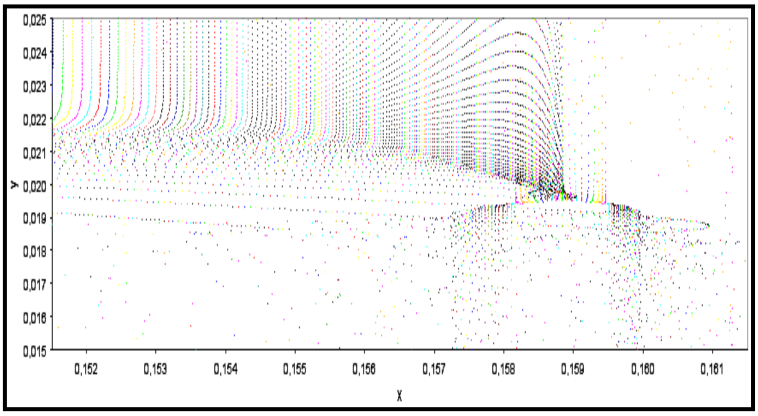


Figure 15
Second extending Fig. 13

3.1.2 Bifurcation Maps of the Simple Duopoly Model

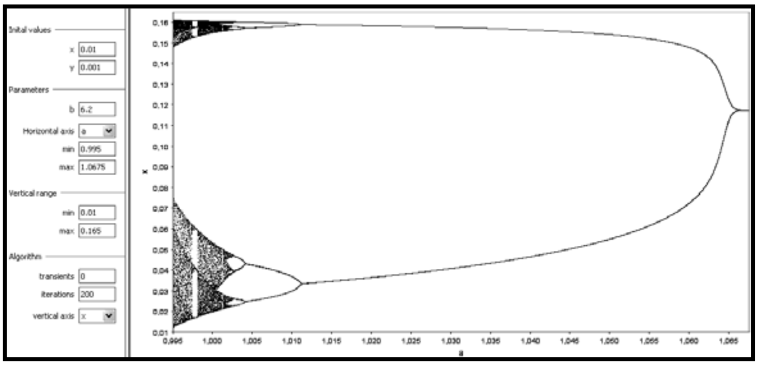


Figure 16
Bifurcation map for the growing parameter *a*

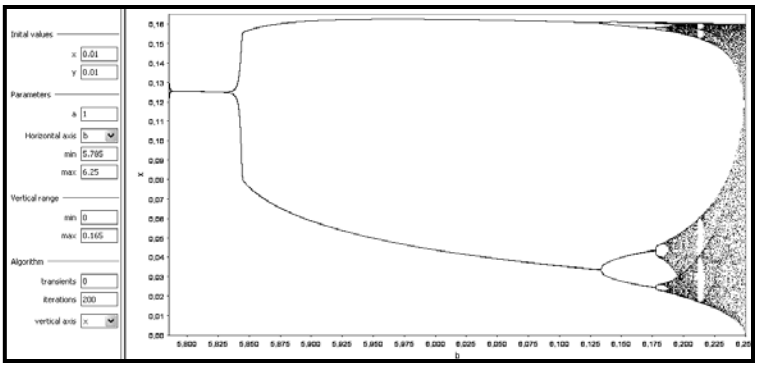


Figure 17
Bifurcation map for the growing parameter *b*

3.1.3 Absorbing Areas of the Simple Duopoly Model, Critical Curves and Attractors in It

The routine "Absorbing area" is very important for deeper understanding of behaviours in complex socio-economic systems because of their efficiency in revealing deep covered tangles in evolution. For such purposes is served generation of large set of critical curves and creation of dotted attractor ('s). In the subsequent snapshots we can see relatively clear situations as results of experimentation in Absorbing area. Specifically on Fig. 18 there are several critical curves in front and dark yellow dotted attractor on background of that space. We can clearly see process of bounding the fixed point from four sides in inner region made by critical curves and also dividing with them the space to for outer regions with rectangular attractive areas.

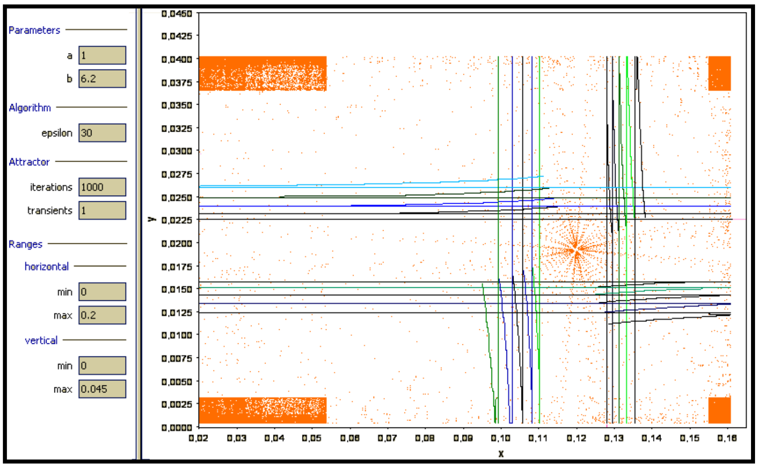


Figure 18

Absorbing area: Critical curves and four regions of attractors

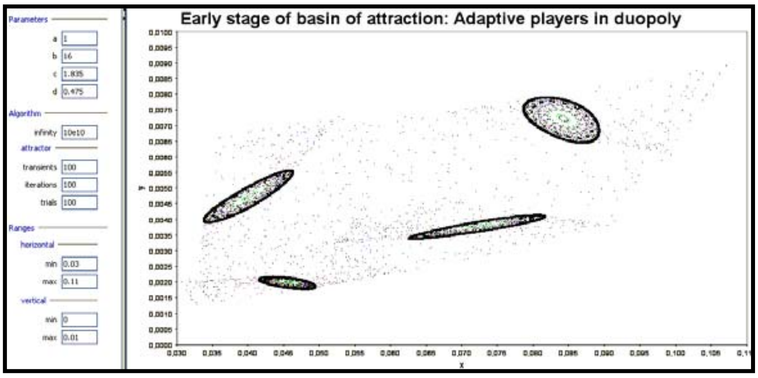
3.2 Cournot-like Duopoly with Adaptive Expectation of Players: T. Puu´s Modification if the Simple Model

In the simple model it was assumes that adjustment of players about amounts of goods was instantaneous. In this model it is, on the contrary, supposed, that players adjust their best replay to other player decision on the basis of previous decisions, in the direction of the optimal supply without necessarily reaching the optimal one immediately. So we can write new system of equations

$$x_{n+1} = \begin{cases} (1-c)x_n + c\left(\sqrt{\dfrac{y_n}{a}} - y_n\right) & \text{if } ay_n \le 1 \\ (1-c)x_n & \text{if } ay_n > 1 \end{cases} \tag{21}$$

$$y_{n+1} = \begin{cases} (1-d)y_n + d\left(\sqrt{\dfrac{x_n}{b}} - x_n\right) & \text{if } bx_n \le 1 \\ (1-d)y_n & \text{if } bx_n > 1 \end{cases} \tag{22}$$

Further routine that allows interested subject penetrate deeper to the very nature of play between two firms is the Basin of attraction. The snapshot of Fig. 19 shows four attractive regions of character similar to stable focus. But the picture we see it is only early phase of evolution of the game. In additional evolving of that game another pictures are emerge. The finished experiment is in Fig. 20.



Picture 19

Basin of attraction of two person game – evolution of duopoly model

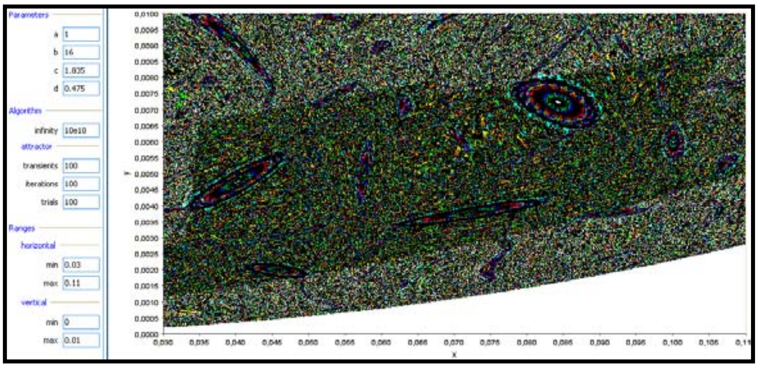


Figure 20

The basin of attraction: complete experiment

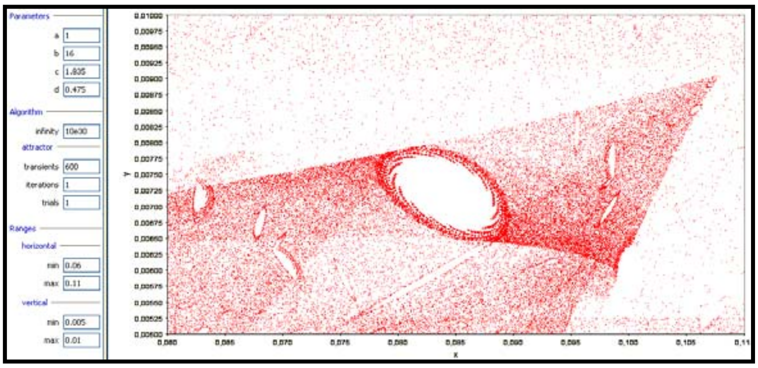


Figure 21

Focusing and orbiting in the space near to giant attractive region

If we are interesting how looks like do detailed picture of game evolution we have use “microscope” for enlarging the interested loci to realise this goal. To do the picture more lucid we have to use very small numbers of iterations and trials on the one hand and a high distance of transients on the other one. On this base we receive better image about the shape of behaviour in distinct areas of attractions, see Fig. 21 and Fig. 22.

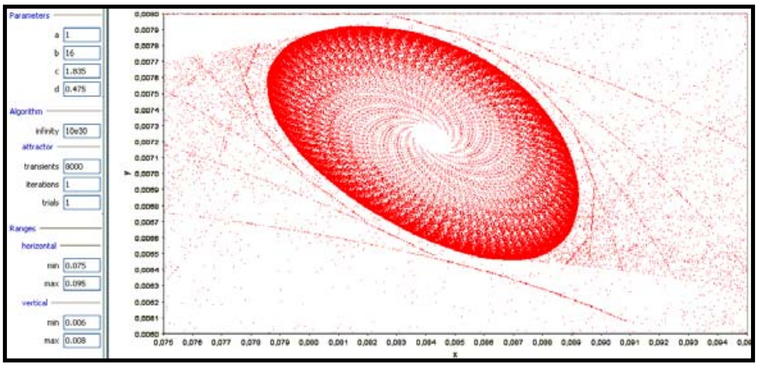


Figure 22

The giant attractive object are focusing: enlarged experiment

However we have further possibilities to deeper study the character of the gaming between two players. The algorithms allowing this is called “Bifurcation mapping” that uses for analysis only one parameter and “Double parameter bifurcation” using two parameters. The second one we suggest to call “Counterfeit basin of attraction” because by variation with pair of parameters we can gain important regions with great information filling and the space looks as genuine basin of core variables *x* and *y*.

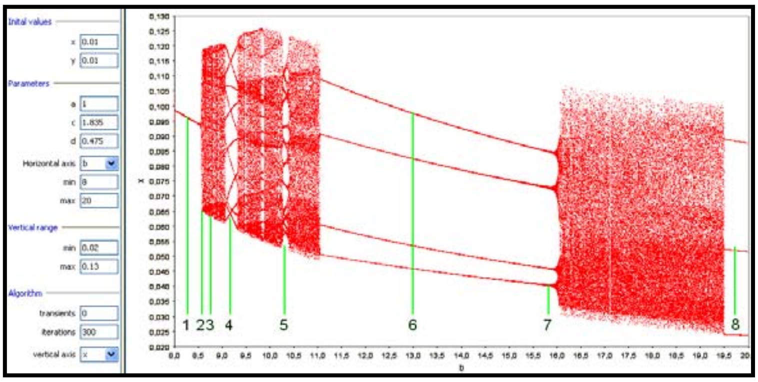


Figure 23

Bifurcation map for parameter *b* versus variables *x*

In Fig. 23 we can see very strange behaviour of variables *x* (product amount of ***Player 1***) cased by increasing parameter *b* (the marginal cost of ***Player 2***). In passageway denoted by figure “1” the behaviour is in equilibrium but diminished. The numeral “2” pick the rout to chaos. The number “3” shade light on very chaos. “4” and “5” pointed to existence of orbits with several rising and/or falling periodical trajectories and relatively long interval denoted by “6” is also orbit by diminishing four periods, “7” is denote four rout saw like twisting dispersed to long channel of massive chaos and “8” is again orbit but by three falling periods. If we get certain pair of parameters *a* and *b* and use it for creation trajectory we can exhibit the shape of evolution more instructively. Those approaches are visualised in Fig. 24 by different colours – the numbering is same as in Fig. 23.

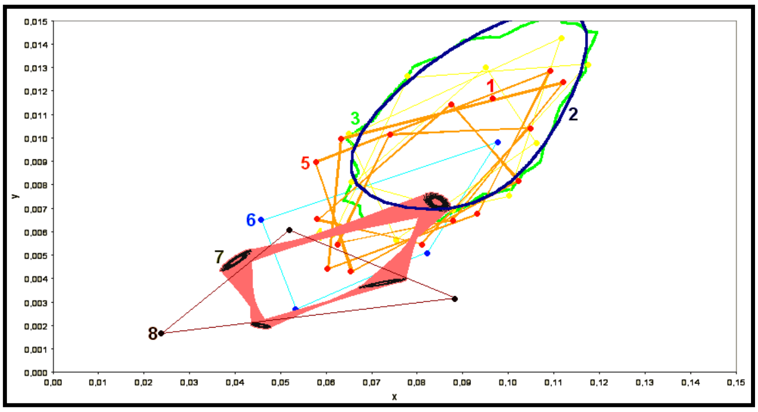


Figure 24

The trajectories of eight passages (regimes) realised on the base of Fig. 23

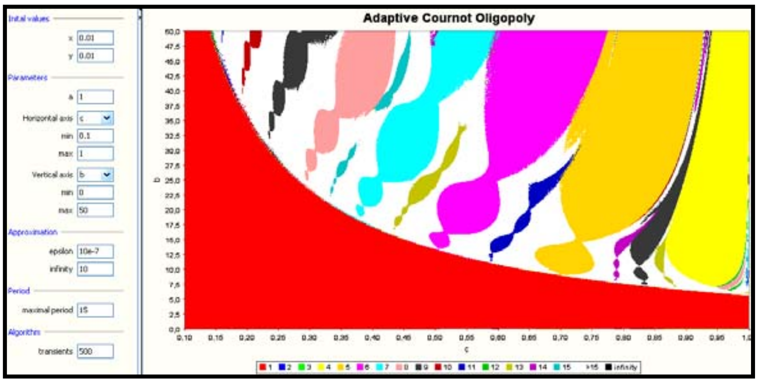


Figure 25

The Arnold’s tongues in *c* versus *b* parameters space

More precise pictures on the phenomena caused by parameters rising we can obtain by help of routine with two opposite parameters. In subsequent snapshot of Fig. 25 visualised that situations and the emergence of so called Arnold’s tongues.

The tongues that can be observed in Fig. 26, that is, the crosshair dots resulting from combinations of parameter *c* versus *b* values for which orbits converge to cycles less or equals to chosen numbers of period π, that is $1 \leq \pi \leq 15$. The curve bounding the red area represents the Neimark bifurcation curve on which the determinant of the Jacobian matrix calculated at the fixed point is equal to *1*.

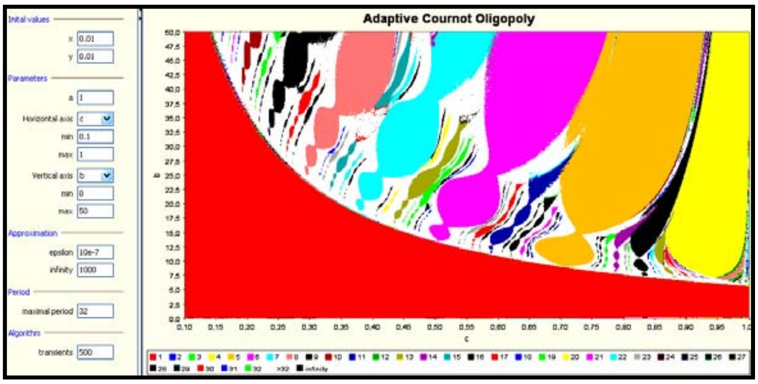


Figure 26

More comprehensive "Quasi basin of attraction" for parameter *c* versus *b* with π=32

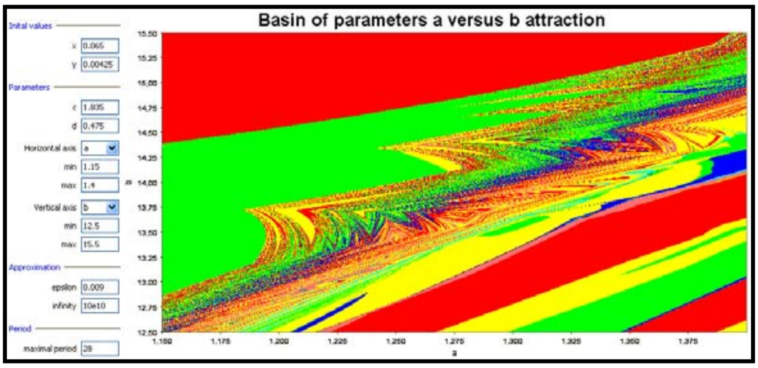


Figure 27

Much more huge, deep detail of "Quasi basin of attraction" for parameter *c* versus *b* with π=32

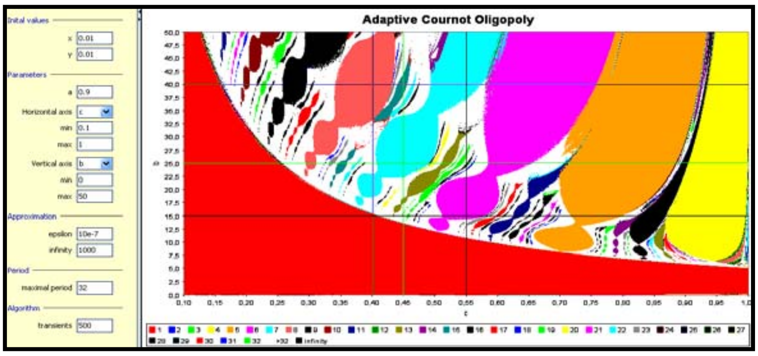


Figure 28
The three pair of coordinates of parameter *b* versus *c* is chosen

As we previously pointed above the double parameter bifurcation is very effective tool for revealing the number of period in this model it can be clearly demonstrating because of exact boundaries of Arnold's tongues. In Fig. 28 above and subsequent Fig. 29 we exhibiting that upper statement is true. In Fig. 28 there are chosen several inner point in some of Arnold's tongues and in Fig. 29 exhibits the realisation of real being in the right place trajectories. Those trajectories are more accurate than exhibited in Fig. 24 because of tongues are precisely bounded.

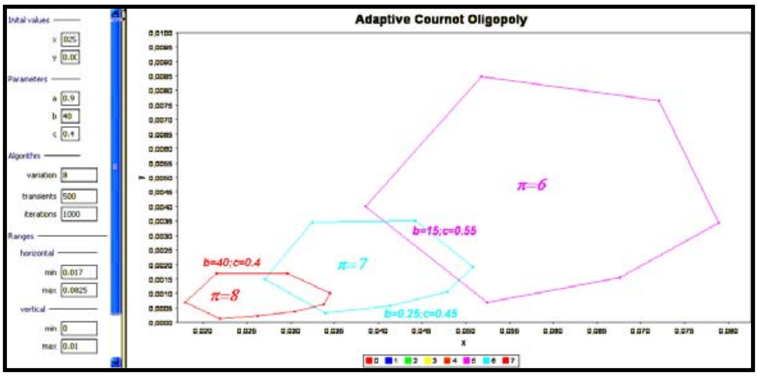
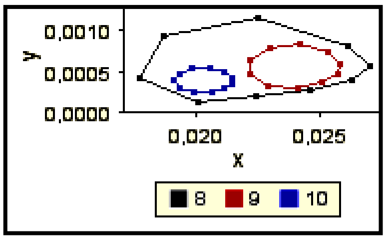


Figure 30
The realisation of chosen parameter pairs in trajectory algorithms



In the small snapshot left we can see the following three Arnold's tongues visualising three orbits with period π=9, π=10, π=11 (as we see in Fig. 19 the number of first red-colour is *0*, so to numbers of colour it is ought to add *+1*, to be the number of period).

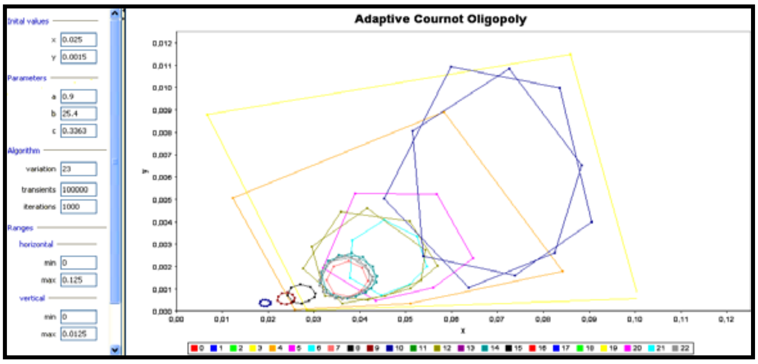


Figure 31
Single, doubled and tripled orbits

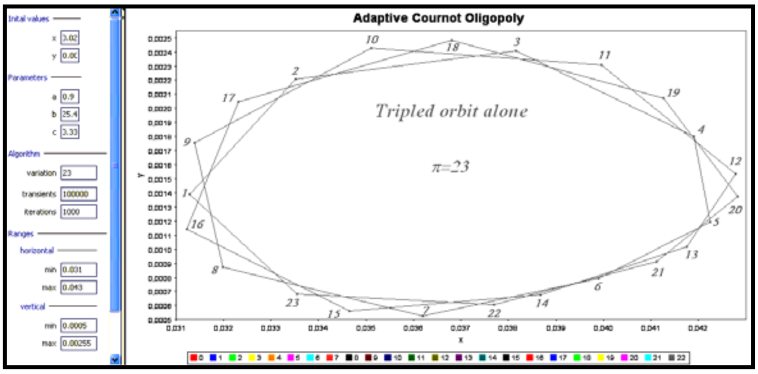


Figure 32
Tripled orbit alone with *23* periods

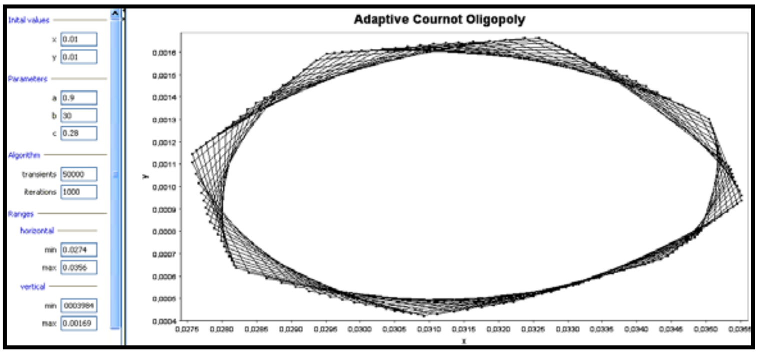


Figure 33
The orbit with a great number of periods: after 50000 transitions

In subsequent snapshots as the results of experimentations in virtual laboratory are exhibited further possibilities to study the strange behaviours in complex duopoly that is: Basin of attraction, trajectory in time steps created for two firms in competitive relation, critical curves and attractors in absorbing areas Fig. 34-37.

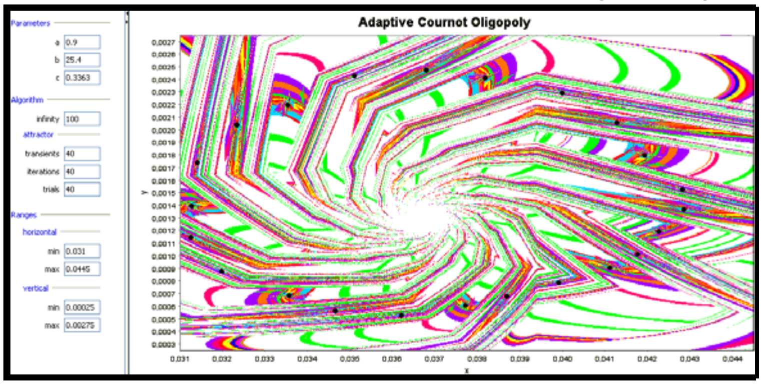


Figure 34
The basin of attraction with the *23* periods (black dots)

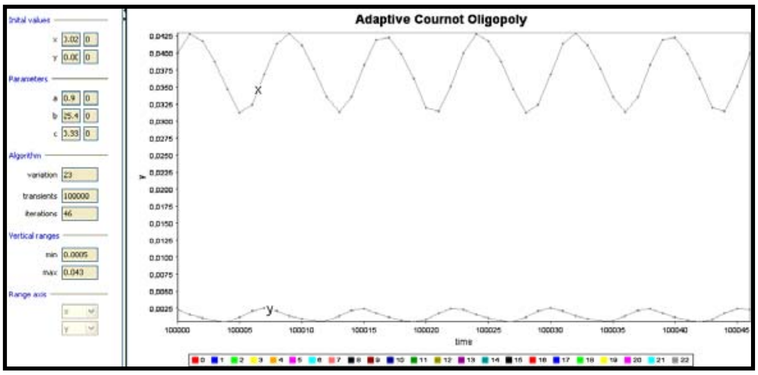


Figure 35
Evolution in time steps

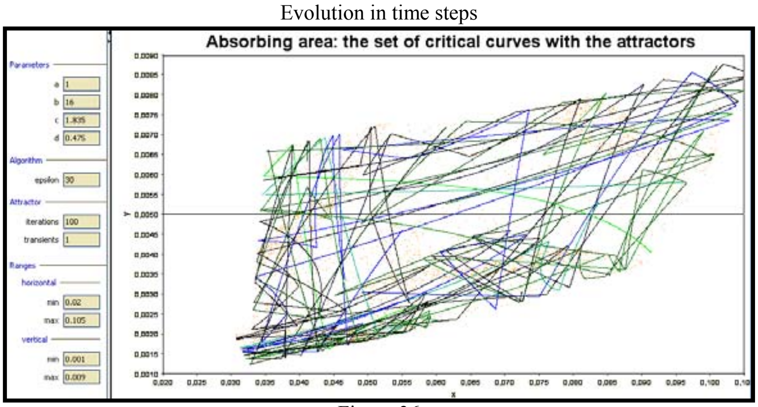


Figure 36
The set of critical curves in absorbing

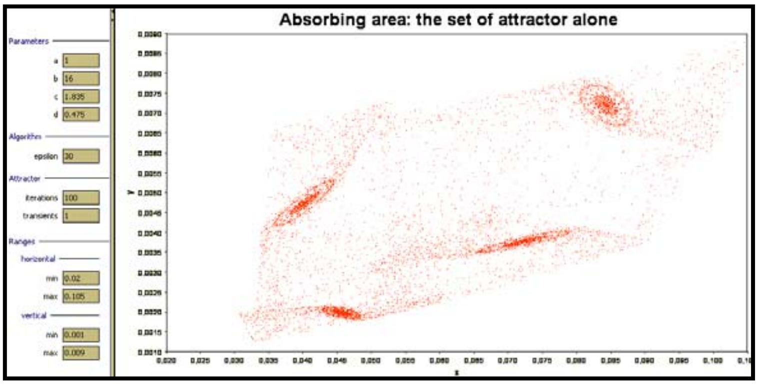


Figure 37
The attractors in absorbing area

3.3 Cournot-like Duopoly with Adaptive Expectation of Players – A. Agliari´s modification

In following sequences of snapshots from experimentations in virtual laboratories us created on ***iDmc*** for more sophisticated model of duopoly due to A. Agliari we won't only put visual basics for proving that in comparatively simple laboratories someone can gain very interesting material for creative imagination on complex economic phenomena.

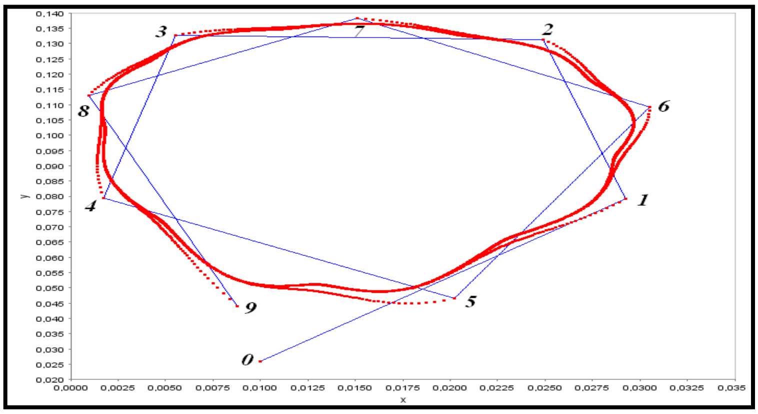


Figure 38
The nine periods cycles of Agliari model

We would like to note that behaviour is very similar to (three dimensional) biological-ecological system analyzed and numerically approximated in [4]. Numerical approximation of the invariant manifold ***M*** of the *8*- and *16*-periodic cycles for *M0 = 4.0*, *M1 = 1.0*, *M2 = 3.0*, *M3 = 4.0* and the unstable manifolds of

U_1^8 (blue), and U_2^8 (red) are two unstable *8*-periodic orbits and U_1^{16} (dark green) is one *16*-unstable periodic orbit, as it is exhibited in Fig. 39.

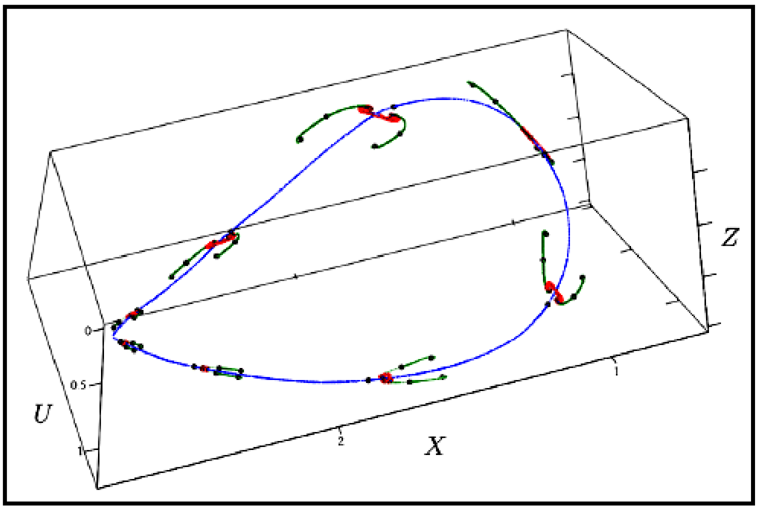


Figure 39
Complex behaviour in three dimensional biological-ecological system

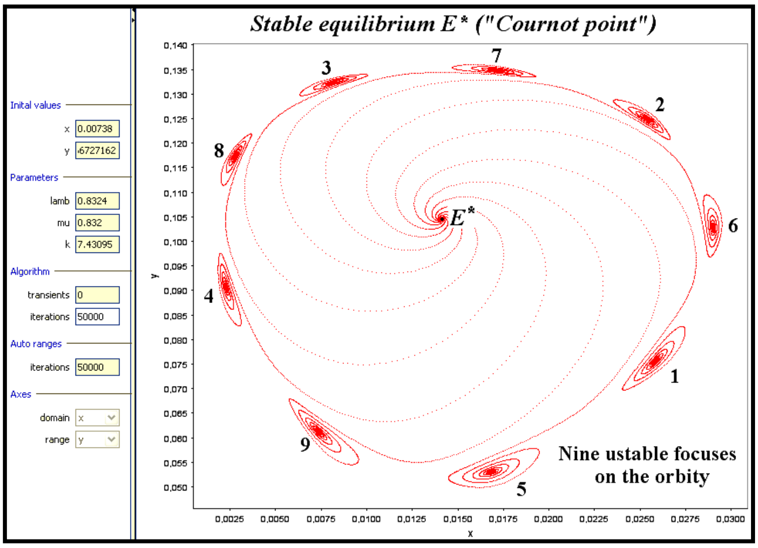


Figure 40
The trajectories approaching nine unstable focuses on the orbit

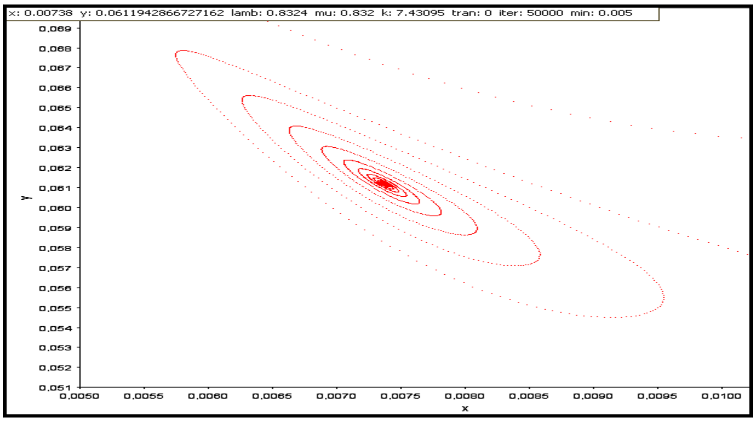


Figure 41

The snapshot detail from upper experiment result in virtual laboratory: unstable focus

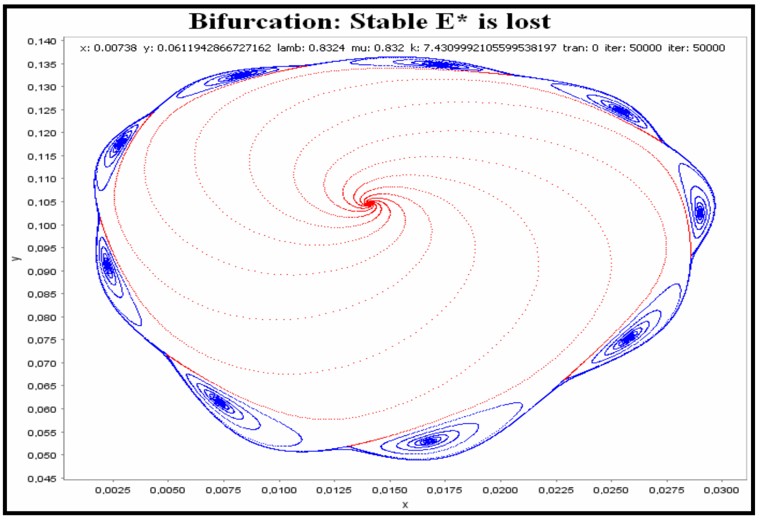


Figure 42

The edge of bifurcation

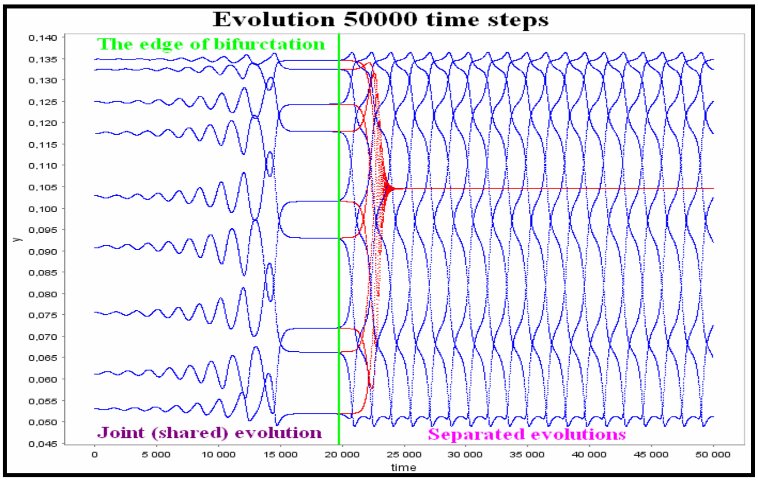


Figure 43

Joint and separated evolution: bifurcation disintegrate the separated and shared intervals

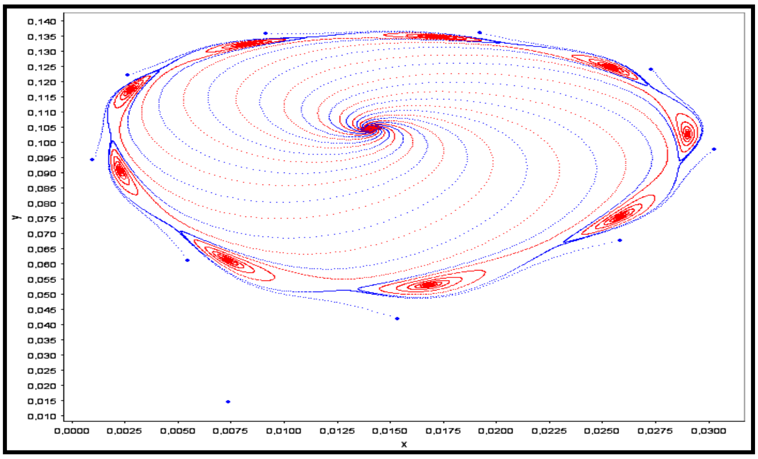


Figure 44

The snapshot from experiment result in virtual laboratory "Duopoly with players expectation"

Let us follow the approach of A. Agliari [1] in selecting the values of parameters. If we use her decision and fix the value $\lambda = 0.8323$, $\mu = 0.83235$ and start experiment in our virtual laboratory, constructed in ***iDmc,*** from $k = 7.4285$, we can obtain the phase space result represented in the snapshot on Fig. xy: the fixed point $\boldsymbol{E}^*$ is the unique attractor of the dynamical system and a period *9* repelling focus cycle $\boldsymbol{C}^*$ exists, as well as a period *9* saddle cycle $\boldsymbol{S}^*$. Both branches of the unstable set of the saddle (light grey curves in Fig. xy) tend to $\boldsymbol{E}^*$, one branch turns around the periodic points of the focus $\boldsymbol{C^*}$. The branch ω_1 of the stable set $\boldsymbol{W^s(S^*)}$ issues from the boundary of the set $\boldsymbol{F}$, whereas ω_2 issues from the repelling focus cycle.

In snapshot of Fig. 45 we can see a new experiment result appeared because of increasing the value of k up to level of $k=7.42905$. As a consequence of that we are seeing a global bifurcation occurrence, giving rise to a new attractor. A cyclical closed invariant curves $\tilde{\Gamma}_i$, ($i = 1, \ldots, 9$) surrounding the repelling focus cycle emerges. This bifurcation may be, as A. Agliari argued, explained via a homoclinic loop of the map M^9 due to the invariant manifolds of the saddle[9]. In fact, in each periodic point of the saddle (fixed point of the map M^9), the portion of the unstable branch a1 related to a periodic point merges with the portion of the stable branch x_2 associated with the same periodic point. After the bifurcation, *9* attracting closed curves appear (invariant for M^9, cyclical for M), each one surrounding a periodic point of $\boldsymbol{C^*}$. As shown in Fig. 46, now also the branch x_2 issues from the boundary of the set $\boldsymbol{F}$ and the whole stable set of the saddle separates the basin of attraction of $\boldsymbol{E^*}$ from the basin of the cyclical closed curves. The unstable branch a1 too changes its asymptotic behaviour: after the bifurcation it converges to the attracting closed curves.

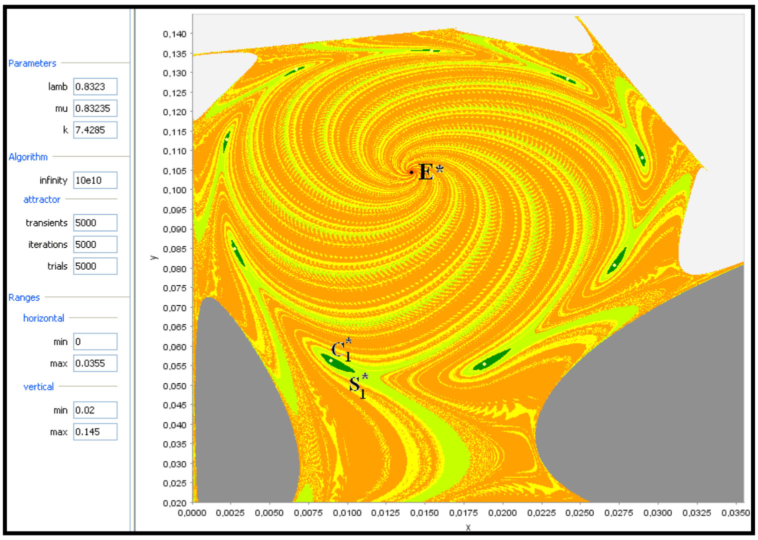


Figure 45

The basin of attractor with stable Cournot point, *9* repelling focuses and 9 saddles at k=7.4285

9 She have been noted in [1] that this situation is different from those described in her another essay at section: *4.1. Homoclinic connection and saddle-repelling focus connection*, p. 746

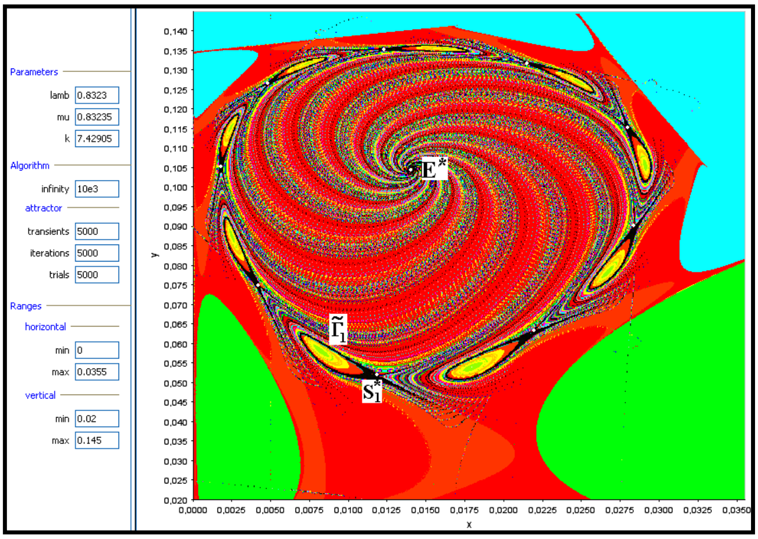


Figure 46

The basin of attractor with stable Cournot point *9* repelling focuses and 9 saddles *k=7.42905*

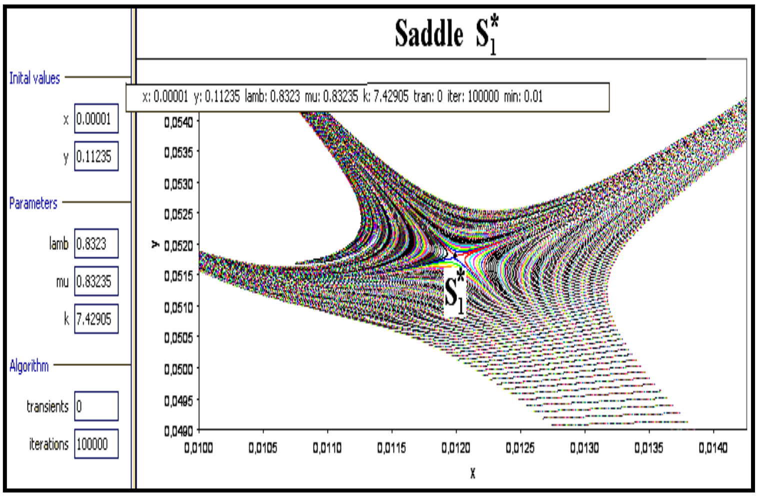


Figure 47

The basin of attractor: detail of S^{*}_{1} at level of k=*7.42905*

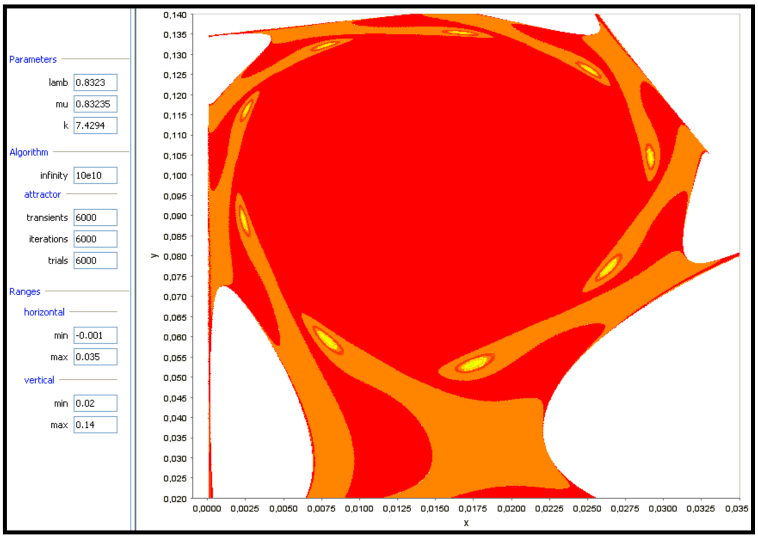


Figure 48

Basin of attraction and the borders of bifurcation regions – situation 1

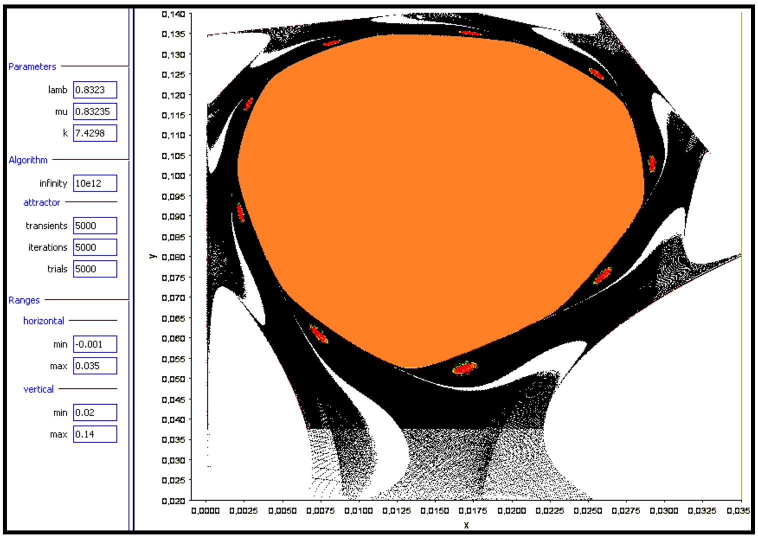


Figure 49

Basin of attraction and the borders of bifurcation regions – situation 2

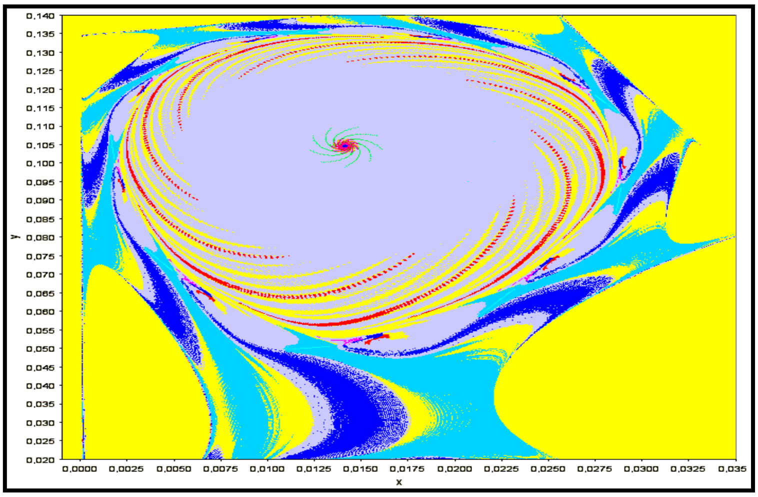


Figure 50
Basin of attraction and the borders of bifurcation regions – situation 3

Acknowledgement

In device named iDmc (Lines and Medio, 2005) there are several routines enabling qualitative arrangement of phases of dynamical systems, for example:

1. Absorbing area; 2. Basin of attraction; 3. Bifurcation (single and/or double parameter); 4. Cycles; 5. Lyapunov exponents (single parameter and/or parameter space); 6. Manifolds; 7. Trajectories (plotting in state space and/or plotting in time line); 8. One-dimensional maps (that is for single difference equation – shifted map and/or cobweb animation. However, some of such arrangement one can create in Excel, STELLA, VENSIM, MATLAB, MATHEMATICA, SWARM and other ones too. I am shoving some of them in upper parts of paper but mainly with aim to give preliminary visual impression and for obtaining general opinion on very complexity of those topics.

Conclusions

At the end of first decade of 21th Century there are wide offer of advanced ICT product, hardware, software and advanced result of research in informatics that can be suitable for using in improving the process of new endosomatic knowledge gaining. In several branches and classes of contemporary sciences such as biology, ecology, ethology, psychology, economics and other not technical branches there are great groups of strange, extraordinary complex problems very pretentious for understanding. To take hold of such problems need deep qualitative reasoning with powerful aid of sophisticated mathematical methods and tools. Not everybody who is interesting or is obliged in gaining endosomatic knowledge about such complex problems has proper mathematical skill for reaching such goals. Fortunately the advances in computational intelligence and informatics can very successfully help in solving such difficulties. Computational intelligence and informatics support of qualitative reasoning against technical computation support is differing. The results of various computations enable in social and cognitive sciences discerning of the shape of complex behaviour using sophisticated imaging. Visual information gaining from two and three dimensional complex graphics is rendering more convenient basics for qualitative reasoning about the complex behaviour than difficult mathematical analysis. In this impression there is putting smaller wage on exact figure than on shapes. On the other hand computational intelligence and informatics outstandingly is helping by its supporting exact figures gaining in technical branches and in physics (famous example is the research in CERN). But, we must keep in mind that social and cognitive sciences need exact number gaining too, for example such as sociology, econometrics, prognostics and other. Statistical analysis of enormous databases in those sciences is clear proof of utility of computational intelligence support for them. However we must have in mind that sometimes very careful statistical procedures are bringing ambiguous results. For example in economics, on the one hand econometric analysis revealed the trend of longitude evolutions of macroeconomic variables, but on the other hand that exact analysis covering up the reason of rugged objective evolution (and subjective motivation staying behind) which statistical routines are consequently smoothing, flatting, levelling and straightening. Such covering routines are protecting interested subject from understanding the core of analysed phenomena and/or process. The latter is to motivation for preparing this paper. Our purpose was show how however relatively simple devices, approaches and tools can very successfully help in revealing various wonders, strange phenomena, twisting and other irregularities in complex socio-economic evolution. Naturally, there are much more sophisticated and efficient set of approaches, methods and toolboxes offered by computational intelligence and informatics, but our opinion in current gaining of endogenous economic knowledge is that we can achieving quietly, without high level of stress from difficult mathematics good and log lasting learning results.

References

[1] Agliari, A., Homoclinic Connections and Subcritical Neimark Bifurcation in a Duopoly Model with Adaptively Adjusted Productions, Chaos, Solitons and Fractals, 29, 2006, pp. 739-755, http://www.elsevier.com/locate/chaos

[2] Agliari, A., Dieci, R., Gardini, L., Homoclinic Tangles in a Kaldor-like Business Cycle Model, Journal of Economic Behavior & Organization, Vol. 62, 2007, pp. 324-347, see also: http://www.elsevier.com /locate/econbase

[3] Cournot, A., Récherces sur les principes mathématiques de la théorie des richesses, Paris, 1838

[4] Fundinger, D., Lindström, T., Osipenko, G., Applied Mathematics and Computation, Vol. 184, 2007, pp. 429-444, see also: http://www.elsevier.com /locate/amc

[5] Herrmann, R., Stability and Chaos in a Kaldor-type Model, DP 22, Department of Economics, University of Göttingen, 1985

[6] Kaldor, N., A Model of the Trade Cycle, Economic Journal, Vol. 50, pp. 78-92

[7] Lines, M., Nonlinear Dynamics: Computer Exercises, in: Lines, M., Medio, A., Nonlinear Dynamics: A Primer, Cambridge University Press, 2001, see also: http://www.dss.uniud.it/nolinear

[8] Lorenz, H-W., Nonlinear Dynamical Economics and Chaotic Motions, Springer-Verlag, Berlin - Heidelberg - Budapest, 1993

[9] Matsumoto, A., Controlling the Cournot-Nash Chaos, Journal of optimization theory and applications, Vol. 128, No. 2, February 2006, pp. 379-392

[10] Puu, T., Attractors, Bifurcations and Chaos, Springer-Verlag, Berlin - Heidelberg - New York, 2000

[11] Puu, T., Nonlinear Economic Dynamics, Springer-Verlag, Berlin - Heidelberg - New York, 1997

[12] von Neumann, J., Morgenstern, O., Theory of Games and Economic Behavior, Princeton University Press, 1944