An Examination of the relationship between the Convex Hull and the Feasibility of LMI based control design

Kyriakos Ákos Matszangosz, Péter Baranyi, Szabolcs Nagy

matszangosz.akos@gmail.com

Abstract

This paper presents an examination which relates the sensitivity of the Linear Matrix Inequality based control design to the type of created convex hull defined by the Polytopic Model. We examine the aforementioned through the control design of the Parallel type Double Inverted Pendulum. We use the TP model transformation to derive various convex TP models of the Parallel type Double Inverted Pendulum. Then we base the design on the feasibility of Linear Matrix Inequalities derived under the Parallel Distributed Compensation based control design framework. Finally we use numerical simulation to compare the resulting control performances.

1 Introduction

In the paper we examine the relationship between the type of convex hull and the feasibility of the Linear Matrix Inequalities (LMIs) derived under the Parallel Distributed Compensation (PDC) based control design framework. Based on comparisons, the paper concludes that the feasibility of the Linear Matrix Inequalities (LMIs) and the resulting control performance is very sensitive to the type of convex hull derived. We examine this relationship via the control of a Parallel type Double Inverted Pendulum (PDIP) system. The system consists of two pendulums, with different lengths and weights, and a cart, on which a force acts as the actuator (see Figure 1). The control goal is to stabilize the pendulums in an upright position.

1.1 Proposed control design

We propose a mathematically non-heuristic, tractable control design. This solution is based on numerical steps that can be executed automatically with minimal human interaction in reasonable time.

First we derive the differential equations of motion via the physical considerations of the system and derive its quasi LPV (qLPV) state-space model. In the second step we execute the Tensor Product model (TP model) transformation to have such a polytopic representation of the qLPV model whereupon Parallel Distributed Compensation (PDC) design can immediately be executed. We derive four different convex type TP polytopic models with minimal complexity (e.g. with minimal number of LTI vertex components), where the LTI vertex components determine a CNO (Convex NOrmalised) type tight convex hull of the system. In this regard, first we show that the PDIP can exactly be given by a TP type polytopic model with minimal (144) number of LTI vertex systems.

In the second step we derive a controller including decay rate control design (finding largest Lyapunov exponent) that guarantees asymptotic stability and constraint on the control value. We base the design on the feasibility of Linear Matrix Inequalities (LMIs) derived under the PDC control design framework proposed by Tanaka and Wang [1]. We examine several CNO type tight convex hulls, and examine the feasibility of the corresponding LMIs.

2 Notations and General Principles

The purpose of this section is devoted to introduce the basic principles used in the current control design, and also, the notations utilised in the article.

2.1 Nomenclature

- $a, b, \ldots$: scalar values;
- $\mathbf{a}, \mathbf{b}, \ldots$: vectors;
- $\mathbf{A}, \mathbf{B}, \ldots$: matrices;
- $\mathcal{A}, \mathcal{B}, \ldots$: tensors;
- $\mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$: vector space of real valued $(I_1 \times I_2 \times \ldots \times I_N)$-tensors;
- $\mathbf{A}^+$: the pseudo inverse of matrix $\mathbf{A}$
- $\mathbf{A}_{(n)}$: n-mode matrix of tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$
- $\mathcal{A} \times_n \mathbf{U}$: n-mode matrix-tensor product;
- $rank_n(\mathcal{A})$: n-mode rank of tensor $\mathcal{A}$, that is $rank_n(\mathcal{A}) = rank(\mathbf{A}_{(n)})$;
- $\mathcal{A} \boxtimes_{n=1}^{N} \mathbf{U}_n$: multiple product as $\mathcal{A} \times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2 \times_3 \ldots \times_N \mathbf{U}_N$;
- Subscript defines lower order: for example, an element of matrix $\mathbf{A}$ at row-column number i, j is symbolized as $(\mathbf{A})_{i,j} = a_{i,j}$. Systematically, the ith column vector of $\mathbf{A}$ is denoted as $\mathbf{a}_i$, i.e. $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots \end{bmatrix}$.
- $(\cdot)_{i,j,n}, \ldots$: are indices;
- $(\cdot)_{I,J,N}, \ldots$: index upper bound: for example: $i = 1..I$, $j = 1..J$, $n = 1..N$ or $i_n = 1..I_n$.

2.2 Definitions

Consider the following linear parameter-varying (LPV) state-space model:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{D}(\mathbf{p}(t))\mathbf{u}(t), \end{aligned} \tag{1}$$

with input $\mathbf{u}(t) \in \mathbb{R}^k$, output $\mathbf{y}(t) \in \mathbb{R}^l$ and state vector $\mathbf{x}(t) \in \mathbb{R}^m$. The system matrix

$$\mathbf{S}(\mathbf{p}(t)) = \begin{pmatrix} \mathbf{A}(\mathbf{p}(t)) & \mathbf{B}(\mathbf{p}(t)) \\ \mathbf{C}(\mathbf{p}(t)) & \mathbf{D}(\mathbf{p}(t)) \end{pmatrix} \in \mathbb{R}^{(m+k)\times(m+l)} \tag{2}$$

is a parameter-varying object, where $\mathbf{p}(t) \in \Omega$ is a time-varying N-dimensional parameter vector, and is an element of the closed hypercube $\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N] \subset \mathbb{R}^N$. Parameter $\mathbf{p}(t)$ can also include some elements of $\mathbf{x}(t)$, in this case it is termed as quasi LPV (qLPV) model. Therefore this type of model is considered to belong to the class of non-linear models.

Definition 1 (Finite Element polytopic model) *:*

$$\mathbf{S}(\mathbf{p}(t)) = \sum_{r=1}^{R} w_r(\mathbf{p}(t))\mathbf{S}_r, \tag{3}$$

$\mathbf{S}(\mathbf{p}(t))$ *in (2) is given for any parameter vector* $\boldsymbol{p}(t)$ *as the parameter-varying combination of linear time-invariant (LTI) system matrices* $\mathbf{S}_r \in \mathcal{R}^{(m+k)\times(m+l)}$ *also called vertex systems. The combination is defined by the multi-variable weighting functions* $w_r(\mathbf{p}(t)) \in [0, 1]$. *Finite element means that R is bounded* $(R < \infty)$.

Definition 2 (Finite element TP type polytopic model) *: We say TP model for brevity.* $\mathbf{S}(\mathbf{p}(t))$ *in (2) is given for any parameter* $\mathbf{p}(t)$ *as the parameter varying combination of linear time-invariant (LTI) system matrices* $\mathbf{S}_{i_1 i_2 \ldots i_N} \in \mathcal{R}^{(m+k)\times(m+l)}$*:*

$$\mathbf{S}(\mathbf{p}(t)) = \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \ldots \sum_{i_N=1}^{I_N} \prod_{n=1}^{N} w_{n,i_n}(p_n(t))\mathbf{S}_{i_1,i_2,\ldots,i_N}, \tag{4}$$

that is with compact tensor notation:

$$\mathbf{S}(\mathbf{p}(t)) = \mathcal{S} \boxtimes_{n=1}^{N} \mathbf{w}_n(p_n(t)), \tag{5}$$

where the (N+2)-dimensional coefficient tensor $\mathcal{S} \in \mathcal{R}^{I_1\times I_2\times \ldots I_N\times(m+k)\times(m+l)}$ *is constructed from LTI vertex systems* $\mathbf{S}_{i_1 i_2 \ldots i_N}$ *and row vector* $\mathbf{w}_n(p_n(t)) \in [0, 1], (i_n = 1 \ldots I_n)$ *contains one variable and continuous weighting functions* $w_{n,i_n}(p_n(t)) \in [0, 1], (i_n = 1 \ldots I_n)$. *The function* $w_{n,i_n}(p_n(t))$ *is the* i_n*-th weighting function defined on the n-th dimension of* Ω*, and* $p_n(t)$ *is the n-th element of vector* $\mathbf{p}(t)$. *Note that the dimensions of* Ω *are respectively assigned to the elements of the parameter vector* $\mathbf{p}(t)$.

Remark 1 *TP model (5) is a special class of polytopic models (3), where the weighting functions are decomposed to the Tensor Product of one variable functions.*

Definition 3 (Convex type TP model) *: The TP model (5) is convex if the weighting functions satisfy*

$$\forall n, i, p_n(t) : w_{n,i}(p_n(t)) \in [0, 1], \tag{6}$$
$$\forall n, p_n(t) : \sum_{i=1}^{I_n} w_{n,i}(p_n(t)) = 1.$$

Definition 4 (NO/CNO type TP model) *The convex TP model is a normal (NO) type model, if its* $\mathbf{w}(p)$ *weighting functions are Normal (NO), that is, if it satisfies (6) and the largest value of all weighting functions is 1. Also, it is Close to NOrmal if it satisfies (6) and the largest value of all weighting functions is 1 or close to 1.*

Remark 2 *Let us introduce a geometric interpretation to the concepts defined above. We have a parameter-space in which each point can be interpreted as an LTI vertex system. With the introduction of the previous concepts, we can imagine the vertex systems of the Convex TP model as the convex hull of the system matrix* $\mathbf{S}(\mathbf{p}(t))$*. This means that we can represent each system matrix* $\mathbf{S}(\mathbf{p}(t))$ *in the parameter-space as a linear combination of these vertices. Furthermore, if this hull is NO/CNO type, it follows that in those points, that the weighting function is 1 (or is close to 1), the system matrix takes on the value of the specific vertex system that is related to that weighting function (or is very close to it in the* $\mathcal{L}_2$*-norm).*

Remark 3 *Various further types of convex TP models are defined in papers [2,3].*

3 Parallel type Double Inverted Pendulum system

This section is divided into 3 parts. The first part deals with the presentation of the PDIP system. In the second part, we derive the dynamic equations of the system, and in the final part we present the equations of motion in a qLPV form.

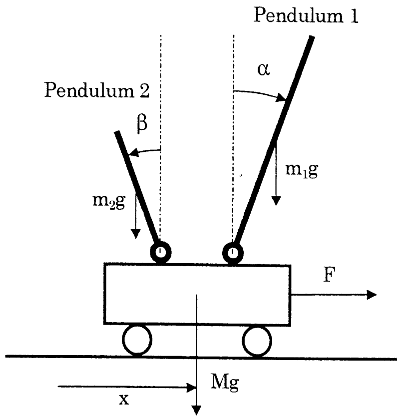

Figure 1: PDIP (Parallel type Double Inverted Pendulum) system

3.1 Description of the PDIP

The PDIP system is shown in Figure 1. The system consists of a straight line rail, a cart attached to it, a longer (1) and a shorter (2) pendulum. The cart is driven by an actuator force (F), and is able to move sideways. The force moves the cart solely in 1-Dimension. We assume homogenous weight distribution in each component. We also disregard friction. The length and mass of the pendulums are different, so they have different dynamics which enables control of the system to a certain degree. The used $\mathbf{p}(t) \in \Omega$ and notations of PDIP are given in Table 1.

3.2 qLPV model of the PDIP

We express the required system matrix:

$$\mathbf{S}(\mathbf{p}(t)) = \begin{pmatrix} \mathbf{E}^{-1}(\mathbf{p}(t))\tilde{\mathbf{A}}(\mathbf{p}(t)) & \mathbf{E}^{-1}(\mathbf{p}(t))\tilde{\mathbf{B}}(\mathbf{p}(t)) \end{pmatrix} = \begin{pmatrix} \mathbf{A}(\mathbf{p}(t)) & \mathbf{B}(\mathbf{p}(t)) \end{pmatrix} \tag{7}$$

Table 1: Parameters of the PDIP model

Symbols	Values	Unit	Description
m_c	1	kg	mass of the cart
x		m	horizontal position of the cart
m_1	0.3	kg	mass of the 1st pendulum
m_2	0.1	kg	mass of the 2nd pendulum
l_1	0.6	m	half length of the 1st pendulum
l_2	0.2	m	half length of the 2nd pendulum
α_1		rad	angular position of the 1st pendulum
α_2		rad	angular position of the 2nd pendulum
F		N	actuator force on the cart
g	9.81	$\frac{m}{s^2}$	standard gravity

$$\mathbf{A}(\mathbf{p}(t))_{1,1} = \mathbf{A}(\mathbf{p}(t))_{1,2} = \mathbf{A}(\mathbf{p}(t))_{1,3} = \mathbf{A}(\mathbf{p}(t))_{1,5} = \mathbf{A}(\mathbf{p}(t))_{1,6} = 0$$
$$\mathbf{A}(\mathbf{p}(t))_{2,1} = \mathbf{A}(\mathbf{p}(t))_{2,2} = \mathbf{A}(\mathbf{p}(t))_{2,3} = \mathbf{A}(\mathbf{p}(t))_{2,4} = \mathbf{A}(\mathbf{p}(t))_{2,6} = 0$$
$$\mathbf{A}(\mathbf{p}(t))_{3,1} = \mathbf{A}(\mathbf{p}(t))_{3,2} = \mathbf{A}(\mathbf{p}(t))_{3,3} = \mathbf{A}(\mathbf{p}(t))_{3,4} = \mathbf{A}(\mathbf{p}(t))_{3,5} = 0$$
$$\mathbf{A}(\mathbf{p}(t))_{4,4} = \mathbf{A}(\mathbf{p}(t))_{5,4} = \mathbf{A}(\mathbf{p}(t))_{6,4} = 0$$

$$\mathbf{A}(\mathbf{p}(t))_{1,4} = \mathbf{A}(\mathbf{p}(t))_{2,5} = \mathbf{A}(\mathbf{p}(t))_{3,6} = 1$$

$$\mathbf{A}(\mathbf{p}(t))_{4,2} = -\frac{m_1 g \sin(\alpha_1(t)) \cos(\alpha_1(t))}{\alpha_1(t) A_3(t)}$$
$$\mathbf{A}(\mathbf{p}(t))_{4,3} = -\frac{m_2 g \sin(\alpha_2(t)) \cos(\alpha_2(t))}{\alpha_2(t) A_3(t)}$$
$$\mathbf{A}(\mathbf{p}(t))_{4,5} = \frac{4}{3} \frac{m_1 l_1 \dot{\alpha_1}(t) \sin(\alpha_1(t))}{A_3(t)}$$
$$\mathbf{A}(\mathbf{p}(t))_{4,6} = \frac{4}{3} \frac{m_2 l_2 \dot{\alpha_2}(t) \sin(\alpha_2(t))}{A_3(t)}$$
$$\mathbf{A}(\mathbf{p}(t))_{5,2} = \frac{(M + m_1 + A_2(t))\, g \sin(\alpha_1(t))}{l_1 \alpha_1(t) A_3(t)}$$
$$\mathbf{A}(\mathbf{p}(t))_{6,3} = \frac{(M + m_2 + A_1(t))\, g \sin(\alpha_2(t))}{l_2 \alpha_2(t) A_3(t)}$$
$$\mathbf{A}(\mathbf{p}(t))_{5,3} = \frac{3}{4} \frac{m_2 g \sin(\alpha_2(t)) \cos(\alpha_2(t)) \cos(\alpha_1(t))}{l_1 \alpha_2(t) A_3(t)}$$
$$\mathbf{A}(\mathbf{p}(t))_{6,2} = \frac{3}{4} \frac{m_1 g \sin(\alpha_1(t)) \cos(\alpha_1(t)) \cos(\alpha_2(t))}{l_2 \alpha_1(t) A_3(t)}$$
$$\mathbf{A}(\mathbf{p}(t))_{5,5} = -\frac{m_1 \dot{\alpha_1}(t) \sin(\alpha_1(t)) \cos(\alpha_1(t))}{A_3(t)}$$
$$\mathbf{A}(\mathbf{p}(t))_{6,6} = -\frac{m_2 \dot{\alpha_2}(t) \sin(\alpha_2(t)) \cos(\alpha_2(t))}{A_3(t)}$$

$$\mathbf{A}(\mathbf{p}(t))_{5,6} = -\frac{m_2 l_2 \dot{\alpha_2}(t)\cos(\alpha_1(t))\sin(\alpha_2(t))}{l_1 A_3(t)}$$

$$\mathbf{A}(\mathbf{p}(t))_{6,5} = -\frac{m_1 l_1 \dot{\alpha_1}(t)\sin(\alpha_1(t))\cos(\alpha_2(t))}{l_2 A_3(t)}$$

$$\mathbf{B}(\mathbf{p}(t)) = \begin{bmatrix} 0 & 0 & 0 & \frac{4}{3}\frac{1}{A_3(t)} & -\frac{\cos(\alpha_1(t))}{l_1 A_3(t)} & -\frac{\cos(\alpha_2(t))}{l_2 A_3(t)} \end{bmatrix}^T, \tag{8}$$

where:

$$\begin{aligned} A_1(t) &:= \left(1 - \frac{3}{4}\cos^2(\alpha_1(t))\right) m_1, \\ A_2(t) &:= \left(1 - \frac{3}{4}\cos^2(\alpha_2(t))\right) m_2, \\ A_3(t) &:= \frac{4}{3}\left(M + A_1(t) + A_2(t)\right). \end{aligned} \tag{9}$$

4 Determination of several CNO type TP models

In this section we execute the TP model transformation on the qLPV model of the PDIP system (7-9). We derive 4 different CNO type TP models, and we examine the feasibility of the corresponding LMIs.

4.1 Determination of the CNO type TP model

In the following we detail the steps of the TP model transformation we apply to the PDIP system. Note that these steps can automatically be executed by computer via numerical algorithms (`http://tptool.sztaki.hu/`).

4.1.1 The parameter space Ω

First we determine the parameter space of interest where the PDIP system varies nonlinearly. Namely we define the intervals of each parameter values such as:

$$\Omega = \begin{pmatrix} -\pi/12 & \pi/12 \\ -\pi/12 & \pi/12 \\ -\pi & \pi \\ -\pi & \pi \end{pmatrix}.$$

Note that these intervals can be arbitrarily defined from the viewpoint of numerical execution of the TP model transformation.

4.1.2 Determination of the TP model with minimal components

According to the algorithm of the TP model transformation, first we discretize the system matrix $\mathbf{S}(\mathbf{p}(t))$ of the PDIP system over an equidistant rectangular grid in Ω. Then we execute Compact Higher Order Singular Value Decomposition (CHOSVD) to find the minimal number of weighting functions, hence the vertex systems of the TP model. Let the discretization grid be $M \times M \times M \times M$ where $M = 50$ in Ω. Then we compute the system matrix of PDIP over each grid point, and we store the resulting matrices $\mathbf{S}^d_{i,j,k,l}$, $(i, j, k, l = 1 \dots M)$ in tensor $\mathcal{S}^d$, where superscript 'd' means discretized.

This step yields tensor $\mathcal{S}^d \in \mathcal{R}^{50\times50\times50\times50\times6\times7}$(Note that the size of the parameter dependent system matrix $\mathbf{S}(\mathbf{p}(t))$ is 6×7). The rank of $\mathcal{S}^d$ on the first four dimensions are 6,6,2 and 2 respectively. From this we conclude that the qLPV model of the PDIP can be represented by a TP polytopic model where the minimal number of the vertex systems are $6\times6\times2\times2 = 144$. We execute CHOSVD [4] on the first four dimensions of $\mathcal{S}^d$:

$$\mathcal{S}^d = \mathcal{S} \boxtimes_{n=1}^{4} U_n.$$

Here tensor $\mathcal{S} \in \mathcal{R}^{6\times6\times2\times2\times6\times7}$ contains vertex systems $\mathbf{S}_{i,j,k,l}(i = 1\ldots6, j = 1\ldots6, k = 1\ldots2, l = 1\ldots2)$.

4.1.3 CNO type TP model

According to the concept of the TP model transformation, the columns of the matrices U_n determine the discretized weighting functions, namely, the values of the weighting functions over the discretization grid. According to Definition 3 of the convex TP model, the sum of the weighting functions must be 1 over any parameter values, and their values must not be negative. This means that the sum of the rows of matrices U_n must be equal to 1 and their elements must not be negative. The TP model transformation is capable of transforming

$$\mathcal{S} \boxtimes_{n=1}^{4} U_n$$

to

$$\mathcal{S} \boxtimes_{n=1}^{4} U_n = \mathcal{S}^{CO} \boxtimes_{n=1}^{4} U_n^{CO},$$

in such a way, that the sum of the rows of matrices U_n^{CO} equal 1 and their values are not negative. Superscript 'CO' denotes 'COnvexity'. (See Definition 3). Since this transformation is not unique, we can define various matrices U_n^{CO}. We select the CNO-type (Close to NOrmalised) transformation here from [2], that guarantees, that as many columns of matrices U_n^{CO} as possible contain element 1 and the maximum of the rest of the columns get close to 1. This actually guarantees that when $\mathbf{w}_n$ is generated from $\mathbf{U}_n$, later as many weighting functions as possible achieve 1, and the rest of them get close to 1. Since the weighting functions define the convex hull, the CNO type weighting functions determine a tight convex hull of the LTI systems where as many of the vertex systems $\mathbf{S}_{i,j,k,l}^{CNO}$ as possible will be equal to the parameter dependent system matrix $\mathbf{S}(\mathbf{p}(t))$ of the PDIP over certain parameter vector $\mathbf{p}(t)$ and the rest of the $\mathbf{S}_{i,j,k,l}^{CNO}$ get close to the system matrix of the PDIP in the sense of $\mathcal{L}_2$-norm. For further details about CNO transformation and geometrical discussion we refer to papers [2, 3]. The type of convex hull defined by the polytopic model considerably influences the feasibility of the LMI based control design and the resulting control performance, see later. In conclusion the resulting decomposition with the CNO transformation is:

$$\mathcal{S}^d = \mathcal{S}^{CNO} \boxtimes_{n=1}^{4} U_n^{CNO}.$$

4.1.4 Determination of the weighting functions

According to the concept of the TP model transformation we numerically reconstruct the weighting functions from matrix U_n^{CNO}. The weighting functions can be determined over any points by the help of the given $\mathbf{S}(\mathbf{p}(t))$.

The resulting weighting functions are depicted in Figure 2-4.

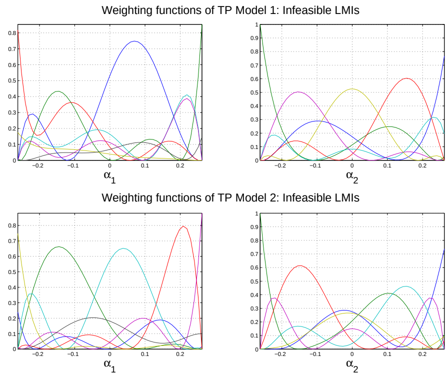

Figure 2: Weighting functions resulting Infeasible LMIs (CNO type TP Model 1 and 2)

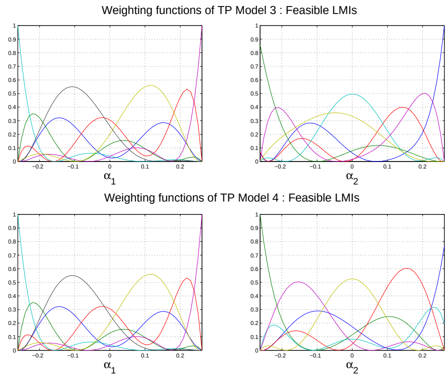

Figure 3: Weighting functions resulting Feasible LMIs (CNO type TP Model 3 and 4)

4.2 The resulting CNO type TP models

The resulting TP model of the PDIP system is:

$$\begin{pmatrix} \dot{\mathbf{x}}(t) \\ \mathbf{y}(t) \end{pmatrix} = \mathcal{S}^{CNO} \mathop{\boxtimes}_{n=1}^{4} \mathbf{w}_n^{CNO}(p_n(t)) \begin{pmatrix} \mathbf{x}(t) \\ u(t) \end{pmatrix}. \tag{10}$$

Without tensor operations it takes the form of:

$$\begin{pmatrix} \dot{\mathbf{x}}(t) \\ \mathbf{y}(t) \end{pmatrix} = \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{k=1}^{2} \sum_{l=1}^{2} w_{1,i}^{CNO}(x_2(t)) w_{2,j}^{CNO}(x_3(t)) w_{3,k}^{CNO}(x_5(t)) w_{4,l}^{CNO}(x_6(t)) (\mathbf{A}_{i,j,k,l}\mathbf{x}(t) + \mathbf{B}_{i,j,k,l}u(t)). \tag{11}$$

First, we generate three different weighting function systems (see Figures 2-3). Subsequently, to generate Model 4, we arbitrarily choose 2 seperate weighting function bundles from different models (Model 1 and Model 3) on seperate dimensions, and we utilise them in the same design. For this new model we have to recalculate the corresponding core tensor.

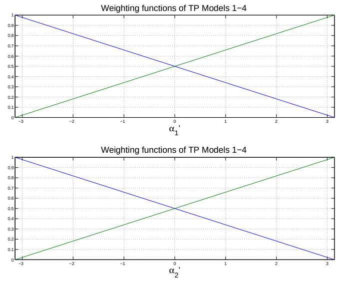

Figure 4: Weighting functions of $\dot{\alpha_1}$ and $\dot{\alpha_2}$ for CNO type TP Models 1-4

In order to conclude this section, we should emphasize here, that all the above steps can be readily executed via numerical steps automatically (`http://tptool.sztaki.hu/`). The result is a non-linear TP polytopic model, that defines a tight (CNO type) convex hull (in order to decrease the conservativeness of the control design) and hence, it is ready for LMI control design.

5 Feasibility of the LMI based design

First of all we specify the desired control performance. We use a robust control design strategy. We design asymptotic stability with decay rate control (finding the largest Lyapunov exponent) to have a fast controller and we have constraints on the control value($\mu = 40N$) according to physical considerations.

Our control design is based on LMIs developed under the PDC framework. The key idea of the PDC framework is that the non-linear controller has the same polytopic structure as the model has. In our case it means that the controller has the same CNO type TP form, namely, the same weighting functions as the model has. Thus, we search the feedback gains $\mathbf{F}_{i,j,k,l}$ over the weighting function system of the TP model as:

$$u(t) = -(\sum_{i=1}^{6}\sum_{j=1}^{6}\sum_{k=1}^{2}\sum_{l=1}^{2} w_{1,i}^{CNO}(x)w_{2,j}^{CNO}(x)w_{3,k}^{CNO}(x)w_{4,l}^{CNO}(x)\mathbf{F}_{i,j,k,l})\mathbf{x}(t), \qquad (12)$$

that is with compact tensor notation:

$$u(t) = -\mathcal{F}^{CNO} \underset{n=1}{\overset{4}{\boxtimes}} \mathbf{w}_n^{CNO}(p_n(t))\mathbf{x}(t). \qquad (13)$$

Thus, we readily substitute the vertex components of these models into the following LMIs.

Theorem 1 (Decay rate control) *Assuming convex type TP Model (10) and controller (13), solve:*

$$\underset{X,M_1,\ldots,M_g}{maximize}\ \alpha \text{ subject to}$$

$$\begin{aligned} \mathbf{X} &> \mathbf{0} \\ -\mathbf{X}\mathbf{A}_r^T - \mathbf{A}_r\mathbf{X} + \mathbf{M}_r^T\mathbf{B}_r^T + \mathbf{B}_r\mathbf{M}_r - 2\alpha\mathbf{X} &> \mathbf{0} \qquad (14) \\ -\mathbf{X}\mathbf{A}_r^T - \mathbf{A}_r\mathbf{X} - \mathbf{X}\mathbf{A}_s^T - \mathbf{A}_s\mathbf{X} + \mathbf{M}_s^T\mathbf{B}_r^T + \mathbf{B}_r\mathbf{M}_s + \mathbf{M}_r^T\mathbf{B}_s^T + \mathbf{B}_s\mathbf{M}_r - 4\alpha\mathbf{X} &\geq \mathbf{0} \end{aligned}$$

for $r < s \leq R$, *except the pairs* (r, s) *such that* $\forall \mathbf{p}(t) : w_r(\mathbf{p}(t))w_s(\mathbf{p}(t)) = 0$, *and where the feedback gains are determined from the solutions* $\mathbf{X}$ *and* $\mathbf{M}_r$ *by*

$$\mathbf{F}_r = \mathbf{M}_r\mathbf{X}^{-1}. \tag{15}$$

Theorem 2 (Constraint on the control value) *Assume that* $||\mathbf{x}(0)||_2 \leq \phi$, *where* $\mathbf{x}(0)$ *is unknown, but the upper bound* ϕ *is known. The constraint* $||\mathbf{u}(t)||_2 \leq \mu$ *is enforced at all times* $t \geq 0$ *if the LMIs*

$$\phi^2\mathbf{I} \leq \mathbf{X} \tag{16}$$

$$\begin{pmatrix} \mathbf{X} & \mathbf{M}_r^T \\ \mathbf{M}_r & \mu^2\mathbf{I} \end{pmatrix} \geq \mathbf{0} \tag{17}$$

hold for $r = 1, \ldots, R$.

The Decay rate control LMI-s (15) were used for this design because the upper bound of the input force we set $\mu = 40N$, the upper bound to the state vector 0.1. The implemented decay rate was chosen as 0.8. By using the LMI solver of MATLAB Robust Control Toolbox we solve all the previous LMIs simultaneously, we find that the in the case of Model 1 and 2, the LMIs are infeasible, while in the case of Model 3 and 4 the resulting LMIs are feasible, and we obtain the feedback gains of Controllers 3 and 4 respectively. In the following the controller generated with Model 3 shall be named Controller 3, the customly created controller from Model 4 shall be called Controller 4.

6 Resulting control performance

The stabilization properties are found on Figure 5. Both overshoot and stabilization time are smaller with Controller 4, briefly Controller 4 produced better results. The stabilization time domain can be found on Figure 6. We also marked the parameter domain Ω, and the upper bound on the initial state value Φ. Here we may also notice the difference between the 2 controllers, with Controller 4 having a greater stabilization domain.

7 Conclusion

We conclude the paper with the following results:

1. The type of convex hull considerably influences the feasibility of the corresponding LMIs. Finding the proper convex hull vastly improved the control performances.

2. The TP model transformation lets us creatively manipulate the convex hull via manipulation of the weighting function system, which resulted a better control performance.

3. Besides the 2 mentioned statements, we remark here, that the best control performance has been achieved as a combination of a convex hull with infeasible corresponding LMIs, and a convex hull with feasible corresponding LMIs.

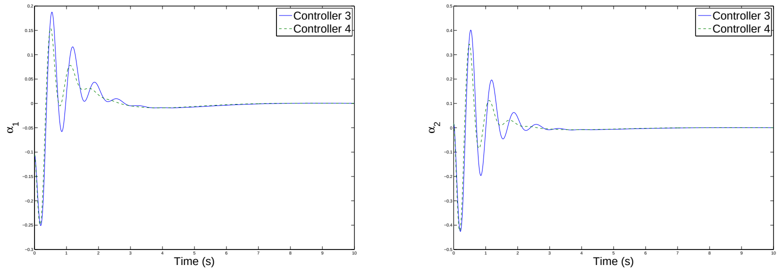

Figure 5: α_1 & α_2 parameters for Controllers 3 and 4

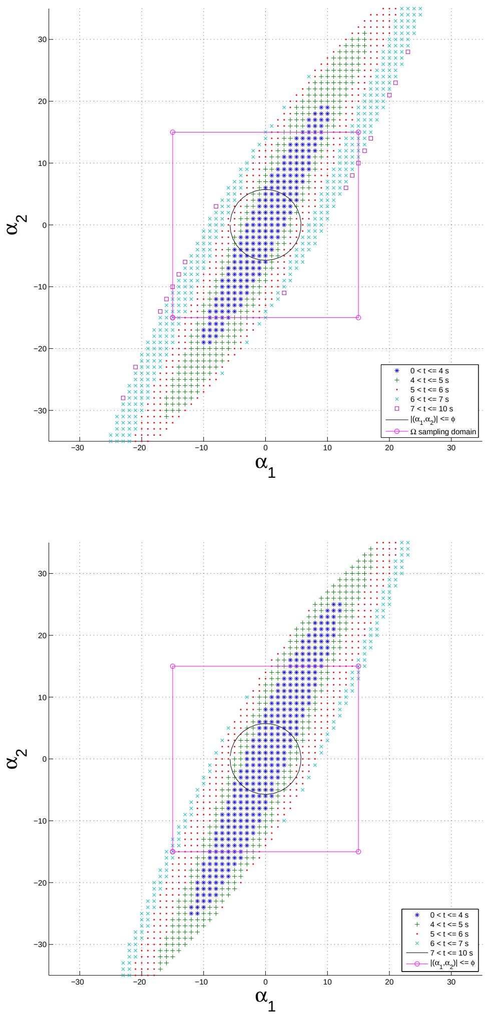

Figure 6: Stabilization times of Controller 3 and 4

References

[1] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*. John Wiley & Sons, Inc., 2001.

[2] P. Baranyi, Z. Petres, P. L. Várkonyi, P. Korondi, and Y. Yam, “Determination of different polytopic models of the prototypical aeroelastic wing section by TP model transformation,” *Journal of Advanced Computational Intelligence*, vol. 10, no. 4, pp. 486–493, 2006.

[3] Z. Petres, P. Baranyi, P. Gáspár, P. Várlaki, and P. Michelberger, “Different polytopic decomposition of the model of heavy vehicles by TP model transformation,” in *11th International Conference on Intelligent Engineering Systems (INES 2007)*, Budapest, Hungary, 29 June–1 July 2007 2007, pp. 265–270.

[4] L. D. Lathauwer, B. D. Moor, and J. Vandewalle, “A multlinear singular value decomposition,” *SIAM Journal on Matrix Analysis and Applications*, April 2000.