First order linear fuzzy differential equations under generalized differentiability

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Abstract. We provide solution of first order linear fuzzy differential equations by variation of constant formula. The differentiability concept used in this paper is the strongly generalized differentiability since a fuzzy differential equation under this differentiability can have decreasing legth of support function. Under some conditions we provide solution with decreasing support and so the behaviour of the solution better reflects the behaviour of real-world systems.

1 Introduction

Fuzzy differential equations are a natural way to model dynamical systems under uncertainty. First order linear fuzzy differential equations are one of the simplest fuzzy differential equations which appear in many applications. However the form of such an equation is very simple it raise many problems since under different fuzzy differential equation concepts, the behaviour of the solutions is different (it depends on the interpretation which is used).

The H-derivative of a fuzzy-number-valued function was introduced in [13] and it has its starting point in the Hukuhara derivative of multivalued functions. First approach to modelling uncertainty of dynamical systems uses the H-derivative or it's generalization, the Hukuhara derivative. Under this setting mainly existence and uniqueness theorems for the solution of a fuzzy differential equation are obtained (see e.g. [12], [14], [16], [15]). This approach has the disadvantage that it leads to solutions with increasing length of their support ([6]). This shortcoming is solved by interpreting a fuzzy differential equation as a system of differential inclusions (see e.g. [11], [6]). The main shortcoming of using differential inclusions is that we do not have a derivative of a fuzzy-number-valued function.

The concept of strongly generalized differentiability introduced in [2] and studied in [3] allows us to solve the above mentioned shortcomings and so we use this differential in the present paper.

First order linear fuzzy differential equations or systems are studied under different interpretations by several papers (see [9], [7], [4], [5]). The solutions provided in these papers have the disadvantages mentioned above. We propose to solve this problem under strongly generalized differentiability and to show some advantages of our method.

After a preliminary section we study differentiability of the H-difference and of the product of two functions. Then we solve first order linear fuzzy differential equations by variation of constants formula. Here we provide also some examples. At the end of the paper we present some conclusions and further research topics.

2 Preliminaries

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy subsets of the real axis (i.e. $u : \mathbb{R} \to [0,1]$) satisfying the following properties:

(i) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is normal, i.e. $\exists x_0 \in \mathbb{R}$ with $u(x_0) = 1$;

(ii) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is convex fuzzy set (i.e. $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall t \in [0,1]$, $x, y \in \mathbb{R}$);

(iii)$\forall u \in \mathbb{R}_{\mathcal{F}}$, u is upper semicontinuous on $\mathbb{R}$;

(iv) $\overline{\{x \in \mathbb{R};\ u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers (see e.g. [8]). Obviously $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. Here $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$ is understood as $\mathbb{R} = \left\{\chi_{\{x\}}; x \text{ is usual real number}\right\}$. For $0 < r \leq 1$, denote $[u]^r = \{x \in \mathbb{R}; u(x) \geq r\}$ and $[u]^0 = \overline{\{x \in \mathbb{R};\ u(x) > 0\}}$. Then it is well-known that for each $r \in [0,1]$, $[u]^r$ is a bounded closed interval. For $u, v \in \mathbb{R}_{\mathcal{F}}$, and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product $\lambda \cdot u$ are defined by $[u+v]^r = [u]^r + [v]^r$, $[\lambda \cdot u]^r = \lambda[u]^r$, $\forall r \in [0,1]$, where $[u]^r + [v]^r$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see e.g. [8], [17]).

Let $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$, $D(u,v) = \sup_{r \in [0,1]} \max\{|u_-^r - v_-^r|, |u_+^r - v_+^r|\}$, be the Hausdorff distance between fuzzy numbers, where $[u]^r = [u_-^r, u_+^r]$, $[v]^r = [v_-^r, v_+^r]$. The following properties are well-known (see e.g. [10] or [17]):

$D(u + w, v + w) = D(u, v)$, $\forall u, v, w \in \mathbb{R}_{\mathcal{F}}$,

$D(k \cdot u, k \cdot v) = |k| D(u, v)$, $\forall k \in \mathbb{R}$, $u, v \in \mathbb{R}_{\mathcal{F}}$,

$D(u + v, w + e) \leq D(u, w) + D(v, e)$, $\forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}$

and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

Also are known the following results and concepts.

Theorem 1 *(see e.g. [1]). (i) If we denote $\widetilde{0} = \chi_{\{0\}}$ then $\widetilde{0} \in \mathbb{R}_{\mathcal{F}}$ is neutral element with respect to $+$, i.e. $u + \widetilde{0} = \widetilde{0} + u = u$, for all $u \in \mathbb{R}_{\mathcal{F}}$.*

(ii) With respect to $\widetilde{0}$, none of $u \in \mathbb{R}_{\mathcal{F}} \setminus \mathbb{R}$, has opposite in $\mathbb{R}_{\mathcal{F}}$.

(iii) For any $a, b \in \mathbb{R}$ with $a, b \geq 0$ or $a, b \leq 0$ and any $u \in \mathbb{R}_{\mathcal{F}}$, we have $(a+b) \cdot u = a \cdot u + b \cdot u$; For general $a, b \in \mathbb{R}$, the above property does not hold.

(iv) For any $\lambda \in \mathbb{R}$ and any $u, v \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda \cdot (u+v) = \lambda \cdot u + \lambda \cdot v$;

(v) For any $\lambda, \mu \in \mathbb{R}$ and any $u \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda \cdot (\mu \cdot u) = (\lambda \cdot \mu) \cdot u$;

Definition 2 *(see e.g. [13]). Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then z is called the H-difference of x and y and it is denoted by $x - y$.*

In this paper the "$-$" sign stands allways for H-difference and let us remark that $x - y \neq x + (-1)y$.

Let us recall the definition of strongly generalized differentiability introduced in [2] and [3].

Definition 3 *Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a, b)$. We say that f is strongly generalized differentiable at x_0, if there exists an element $f'(x_0) \in \mathbb{R}_{\mathcal{F}}$, such that*

(i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) - f(x_0)$, $f(x_0) - f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) - f(x_0-h)}{h} = f'(x_0),$$

or

(ii) for all $h > 0$ sufficiently small, $\exists f(x_0) - f(x_0 + h)$, $f(x_0 - h) - f(x_0)$ and the limits

$$\lim_{h \searrow 0} \frac{f(x_0) - f(x_0+h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0-h) - f(x_0)}{(-h)} = f'(x_0),$$

or

(iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) - f(x_0)$, $f(x_0 - h) - f(x_0)$ and the limits

$$\lim_{h \searrow 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0-h) - f(x_0)}{(-h)} = f'(x_0),$$

or

(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) - f(x_0 + h)$, $f(x_0) - f(x_0 - h)$ and the limits

$$\lim_{h \searrow 0} \frac{f(x_0) - f(x_0+h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0) - f(x_0-h)}{h} = f'(x_0).$$

(h and $(-h)$ at denominators mean $\frac{1}{h}\cdot$ and $-\frac{1}{h}\cdot$, respectively).

Let us remind the following theorem which allows us to consider case (i) or (ii) of the previous definition almost everywhere in the domain of the functions under discussion.

Theorem 4 *Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ be strongly generalized differentiable on each point $x \in (a, b)$ in the sense of Definition 3, (iii) or (iv). Then $f'(x) \in \mathbb{R}$ for all $x \in (a, b)$.*

Another result concernes the derivation of a fuzzy constant multiplied by a crisp function (see [3]).

Theorem 5 *If $g : (a, b) \to \mathbb{R}$ is differentiable on (a, b) such that g' has at most a finite number of roots in (a, b) and $c \in \mathbb{R}_{\mathcal{F}}$, then $f(x) = g(x) \cdot c$ is strongly generalized differentiable on (a, b) and $f'(x) = g'(x) \cdot c$, $\forall x \in (a, b)$.*

The following theorems concern the existence of solutions of a fuzzy initial value problem under generalized differentiability (see [3]).

Theorem 6 *Let us suppose that the following conditions hold: (a) Let $R_0 = [x_0, x_0 + p] \times \overline{B}(y_0, q)$, $p, q > 0$, $y_0 \in \mathbb{R}_{\mathcal{F}}$, where $\overline{B}(y_0, q) = \{y \in \mathbb{R}_{\mathcal{F}} : D(y, y_0) \leq q\}$ denote a closed ball in $\mathbb{R}_{\mathcal{F}}$ and let $f : R_0 \to \mathbb{R}_{\mathcal{F}}$ be a continuous function such that $D\left(\widetilde{0}, f(x, y)\right) = \|f(x, y)\|_{\mathcal{F}} \leq M$ for all $(x, y) \in R_0$. (b) Let $g : [x_0, x_0 + p] \times [0, q] \to \mathbb{R}$, such that $g(x, 0) \equiv 0$ and $0 \leq g(x, u) \leq M_1$, $\forall x \in [x_0, x_0 + p]$, $0 \leq u \leq q$, such that $g(x, u)$ is nondecreasing in u and g is such that the initial value problem $u'(x) = g(x, u(x))$, $u(x_0) = 0$ has only the solution $u(x) \equiv 0$ on $[x_0, x_0 + p]$. (c) We have $D(f(x, y), f(x, z)) \leq g(x, D(y, z))$, $\forall (x, y), (x, z) \in R_0$ and $D(y, z) \leq q$. (d) There exists $d > 0$ such that for $x \in [x_0, x_0 + d]$ the sequence $\overline{y}_n : [x_0, x_0 + d] \to \mathbb{R}_{\mathcal{F}}$ given by $\overline{y}_0(x) = y_0$, $\overline{y}_{n+1}(x) = y_0 - (-1) \cdot \int_{x_0}^{x} f(t, \overline{y}_n(t))dt$ is defined for any $n \in \mathbb{N}$. Then the fuzzy initial value problem*

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

has two solutions (one differentiable as in Definition 3, (i) and the other one differentiable as in Definition 3, (ii)) $y, \overline{y} : [x_0, x_0 + r] \to B(y_0, q)$ where $r = \min\left\{p, \frac{q}{M}, \frac{q}{M_1}, d\right\}$ and the successive iterations

$$y_0(x) = y_0$$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt, \qquad (1)$$

and

$$\overline{y}_0(x) = y_0$$

$$\overline{y}_{n+1}(x) = y_0 - (-1) \cdot \int_{x_0}^{x} f(t, \overline{y}_n(t))dt \qquad (2)$$

converge to these two solutions respectively.

The following theorem deals with fuzzy differential equations with input data triangular fuzzy-numer-valued functions. We recall that for $a < b < c$, $a, b, c \in \mathbb{R}$, the triangular fuzzy number $u = (a, b, c)$ determined by a, b, c is given such that $u_-^r = a + (b - a)r$ and $u_+^r = c - (c - b)r$ are the endpoints of the $r-$level sets, for all $r \in [0, 1]$. Here $u_-^1 = u_+^1 = b$ and it is denoted by u^1. The set of triangular fuzzy numbers will be denoted by $\mathbb{R}_{\mathcal{T}}$. The following lemma gives a sufficient condition for the existence of the H-difference of two triangular fuzzy numbers.

Lemma 7 *Let $u, v \in \mathbb{R}_{\mathcal{T}}$ be such that $u^1 - u_-^0 > 0$, $u_+^0 - u^1 > 0$ and $len(v) = (v_+^0 - v_-^0) \leq \min\{u^1 - u_-^0, u_+^0 - u^1\}$. Then the H-difference $u - v$ exists.*

Corollary 8 *Let $f : R_0^{\mathcal{T}} \to \mathbb{R}_{\mathcal{T}}$, where $R_0^{\mathcal{T}} = [x_0, x_0 + p] \times (\overline{B}(y_0, q) \cap \mathbb{R}_{\mathcal{T}})$, and $y_0 \in \mathbb{R}_{\mathcal{T}}$ such that $(y_0)^1 - (y_0)_-^0 > 0$ and $(y_0)_+^0 - (y_0)^1 > 0$. Let $m = \min\{(y_0)^1 - (y_0)_-^0, (y_0)_+^0 - (y_0)^1\}$. Under the assumptions (a), (b) and (c) of the preceding Theorem 6 the fuzzy initial value problem*

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

has two solutions $y, \overline{y} : [x_0, x_0 + r] \to B(y_0, q)$ where $r = \min\left\{p, \frac{q}{M}, \frac{q}{M_1}, \frac{m}{2M}\right\}$ and the successive iterations in relations (1), (2) converge to the two solutions.

3 Variation of constants formula for fuzzy differential equations

In [7] it is proved that variation of constants formula provides solution of first order linear fuzzy differential equations, using the approach given in [11], which interprets a fuzzy differential equation as differential inclusions. Similar to the cited results we will prove variation of constants formula for fuzzy differential equations under strongly generalized differentiability. The solutions provided in the present paper and in [7] are in the general case different.

Firstly let us compute the generalized differential of the H-difference of two functions. The H-difference of functions is understood pointwise.

Theorem 9 *Let $f, g : (a, b) \to \mathbb{R}_{\mathcal{F}}$ be strongly generalized differentiable such that f is (i)-differentiable and g is (ii)-differentiable or f is (ii)-differentiable and g is (i)-differentiable on an interval (α, β). If the H-difference $f(x) - g(x)$ exists for $x \in (\alpha, \beta)$ then $f - g$ is strongly generalized differentiable and*

$$(f - g)'(x) = f'(x) + (-1) \cdot g'(x),$$

for all $x \in (\alpha, \beta)$.

Proof. Since f is (i)-differentiable it follows that $f(x + h) - f(x)$ exists i.e. there exists $u_1(x, h)$ such that

$$f(x + h) = f(x) + u_1(x, h).$$

Analogously since g is (ii)-differentiable there exists $v(x, h)$ such that

$$g(x) = g(x + h) + v_1(x, h)$$

and we get

$$f(x + h) + g(x) = f(x) + g(x + h) + u_1(x, h) + v_1(x, h).$$

Since the H-differences $f(x) - g(x)$ and $f(x+h) - g(x+h)$ exist we get

$$f(x+h) - g(x+h) = f(x) - g(x) + u_1(x,h) + v_1(x,h),$$

that is the H-difference $(f(x+h) - g(x+h)) - (f(x) - g(x))$ exists and

$$(f(x+h) - g(x+h)) - (f(x) - g(x)) = u_1(x,h) + v_1(x,h). \tag{3}$$

By similar reasoning we get that there exist $u_2(x,h)$ and $v_2(x,h)$ such that

$$f(x) = f(x-h) + u_2(x,h),$$

$$g(x-h) = g(x) + v_2(x,h)$$

and so

$$(f(x) - g(x)) - (f(x-h) - g(x-h)) = u_2(x,h) + v_2(x,h). \tag{4}$$

We observe that

$$\lim_{h \searrow 0} \frac{u_1(x,h)}{h} = \lim_{h \searrow 0} \frac{u_2(x,h)}{h} = f'(x)$$

and

$$\lim_{h \searrow 0} \frac{v_1(x,h)}{h} = \lim_{h \searrow 0} \frac{v_2(x,h)}{h} = (-1)g'(x).$$

Finally, by multiplying (3) and (4) with $\frac{1}{h}$ and passing to limit with $h \searrow 0$ we get that $f - g$ is (i)-differentiable and

$$(f-g)'(x) = f'(x) + (-1) \cdot g'(x),$$

The case when f is (ii)-differentiable and g is (i)-differentiable is similar, the difference being that $f - g$ is (ii)-differentiable. ■

Firstly we will extend Theorem 5 to the case of the product of a crisp function and a fuzzy-number-valued function, in some cases.

Theorem 10 *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be two differentiable functions (g is strogly generalized differentiable as in Definition 3, (i) or (ii)).*

a) If $f(x) \cdot f'(x) > 0$ and g is (i)-differentiable or

b) If $f(x) \cdot f'(x) < 0$ and g is (ii)-differentiable then $f \cdot g$ is strongly generalized differentiable and we have

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Proof. Case a) can be found in [**?**]. For the case b) we have the following subcases:

1) If $f(x) < 0$, $f'(x) > 0$ and g is (ii)-differentiable then the H-difference $g(x) - g(x+h)$ exists, i.e. there exists $u(x,h) \in \mathbb{R}_{\mathcal{F}}$, such that

$$g(x) = g(x+h) + u(x,h).$$

Also, we have

$$f(x) = f(x+h) + v(x,h),$$

where $v(x,h) = f(x) - f(x+h) < 0$. By Theorem 1, (iii) and (iv) we get

$$\begin{aligned} f(x) \cdot g(x) = {} & f(x+h) \cdot g(x+h) + f(x+h) \cdot u(x,h) \\ & + v(x,h) \cdot g(x+h) + v(x,h) \cdot u(x,h), \end{aligned}$$

that is the H-difference $f(x) \cdot g(x) \ominus f(x+h) \cdot g(x+h)$ exists and we have

$$\begin{aligned} f(x) \cdot g(x) \ominus f(x+h) \cdot g(x+h) = {} & f(x+h) \cdot u(x,h) \\ & + v(x,h) \cdot g(x+h) + v(x,h) \cdot u(x,h). \end{aligned}$$

By multiplying with $-\frac{1}{h}$ and passing to limit with $h \searrow 0$, we get

$$\begin{aligned} \lim_{h \searrow 0} \frac{f(x) \cdot g(x) \ominus f(x+h) \cdot g(x+h)}{-h} = {} & \lim_{h \searrow 0} f(x+h) \cdot \frac{u(x,h)}{-h} \\ & + \lim_{h \searrow 0} \frac{v(x,h)}{-h} \cdot g(x+h) + \lim_{h \searrow 0} \frac{v(x,h)}{-h} \cdot u(x,h). \end{aligned}$$

Since g is continuous, the last term is 0.

Analogously we get

$$\begin{aligned} \lim_{h \searrow 0} \frac{f(x-h) \cdot g(x-h) \ominus f(x) \cdot g(x)}{-h} = {} & \lim_{h \searrow 0} f(x-h) \cdot \frac{u'(x,h)}{-h} \\ & + \lim_{h \searrow 0} \frac{v'(x,h)}{-h} \cdot g(x-h) + \lim_{h \searrow 0} \frac{v'(x,h)}{-h} \cdot u(x,h), \end{aligned}$$

and finally by Definition 3 it follows that

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

2) If $f(x) > 0$, $f'(x) < 0$ and g is (ii)-differentiable then the H-difference $g(x) \ominus g(x+h)$ exists, i.e. there exists $u(x,h) \in \mathbb{R}_{\mathcal{F}}$, such that

$$g(x) = g(x+h) + u(x,h).$$

Also, we have

$$f(x) = f(x+h) + v(x,h),$$

where $v(x,h) = f(x) - f(x+h) > 0$.

Similar to the first case we obtain

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

■

Now we consider the fuzzy initial-value problem

$$\begin{cases} y'(x) = a \cdot y(x) + b(x) \\ y(x_0) = y_0 \end{cases}, \tag{5}$$

where $a \in \mathbb{R}$, $y_0 \in \mathbb{R}_{\mathcal{F}}$ and $b : (x_0, x_1) \to \mathbb{R}_{\mathcal{F}}$.

The following theorem is variation of constants formula for fuzzy differential equations.

Theorem 11 *(i) If $a > 0$ then*

$$y(x) = e^{a(x-x_0)} \left(y_0 + \int_{x_0}^{x} b(t) \cdot e^{-a(t-x_0)} dt \right)$$

is (i)-differentiable and it is a solution of the problem (5).

(ii) If $a < 0$ and if the H-difference $y_0 - \int_{x_0}^{x} (-b(t)) \cdot e^{-a(t-t_0)} dt$ exists then

$$y(x) = e^{a(x-x_0)} \left(y_0 - \int_{x_0}^{x} (-b(t)) \cdot e^{-a(t-x_0)} dt \right)$$

is (ii)-differentiable and it is a solution of the problem (5).

Proof. The case (i) follows by Theorem 10 and by [17], Theorem 3.6.

For the case (ii) we observe that the H-difference $y_0 - \int_{x_0}^{x} (-b(t)) \cdot e^{-a(t-x_0)} dt$ is differentiable by Theorem 9 and by [17], Theorem 3.6 we get

$$\left(y_0 - \int_{x_0}^{x} (-b(t)) \cdot e^{-a(t-x_0)} dt \right)' = b(x) e^{-a(x-x_0)}.$$

Since $a < 0$ $e^{a(x-x_0)}$ and $\left(e^{a(x-x_0)}\right)'$ have opposite sign and then by Theorem 10 we obtain that y is a solution of the problem (5). ■

Remark 12 *The solution provided by the preceeding Theorem 11, (ii) has decreasing legth of the support of the level sets, and so in this case, if for example $y_0 - \int_{x_0}^{x} (-b(t)) \cdot e^{-a(t-x_0)} dt$ exists for any $x \in (x_0, \infty)$ and if it is bounded then $\lim_{x\to\infty} y(x) = 0$, i.e. we get assimptotical stability. This solution do not allways exist globally, however, by Corollary 8, this solution allways exists locally. The global existence of such solution is ensured if for example the function b is real-valued, since in that case the H-difference allways exists $y_0 - \int_{x_0}^{x} (-b(t)) \cdot e^{-a(t-x_0)} dt$ exists for any $x \in (x_0, \infty)$.*

In what folows we provide two examples.

Example 13 *(see [3]) In [7] it is solved for example the fuzzy differential equation $y'(x) = -2y$, $y(0) = (0, \frac{1}{2}, 1)$. Since in the crisp case this equation has a solution that decreases asymptotically to 0, we expect solution with decreasing support, i.e. (ii)-differentiable. In [7] it is obtained that the solution to this problem (i.e. a function which satisfy some differential inclusions) has the level sets $[y(x)]^\beta = \left[\frac{\beta}{2} e^{-2x}, \left(1 - \frac{\beta}{2}\right) e^{-2x}\right]$, $\beta \in [0,1]$, so $y(x) = y(0) \cdot e^{-2x}$. It is easy to see that $y(x) - y(x+h) = y(0) \cdot (e^{-2x} - e^{-2(x+h)})$ exists and by Definition 3, (ii) we have $y'(x) = \lim_{h \searrow 0} \frac{y(x)-y(x+h)}{-h} = (-2) \cdot y(x)$, and so $y(x) = y(0) \cdot e^{-2x}$ is a solution of the initial value problem under strongly generalized differentiability.*

Example 14 *Let us solve the fuzzy initial-value problem*

$$\begin{cases} y'(x) = (-1) \cdot y(x) + x \\ y(0) = (1, 2, 3) \end{cases}, \tag{6}$$

Then since the H-difference $y_0 - \int_{x_0}^{x}(-t) \cdot e^{-t}dt$ exists for any $x \in (x_0, \infty)$, by Theorem 11, (ii) we get that

$$y(x) = e^{-x}\left((1,2,3) - \int_0^x (-t) \cdot e^t dt\right)$$
$$= x - 1 + \left(2e^{-x}, 3e^{-x}, 4e^{-x}\right).$$

We observe that in this case we have $D(y(x), x-1) \leq 4e^{-x}$, and it follows that $\lim_{x\to\infty} D(y(x), x-1) = 0$ behaviour which is similar to the assimptotic behaviour of solutions for crisp linear differential equations. Let us remark that under the H-differentiability concept since solutions of a fuzzy differential equation have allways increasing support, such an assimptotic behaviour was not possible.

4 Concluding remarks

By variation of constants formula we provided solutions to fuzzy initial value problems for first order linear fuzzy differential equations. These solutions may have decreasing length of their support. The examples provided in this paper show us that we can have in this case assimptotic behaviour of the solutions similarly to the crisp case, and also, we may have reversible processes which was not the case under H-differentiability.

The disadvantage of strongly generalized differentiability of a function with respect to H-differentiability and Hukuhara differentiability seems to be that a fuzzy differential equation has not a unique solution. The advantage is that the solution better reflects the behaviour of real-world system.

Generalized differentiability has also advantages with respect to differential inclusions. Firstly, it is more practical for numerical computation. Secondly, one can use the (partial) derivative of a fuzzy-number-valued function, which is not the case when interpreting a fuzzy differential equation as a system of differential inclusions, since this last one interprets directly the notion of fuzzy differential equation, without a derivative.

For further research we will study two-point boundary value problems for fuzzy differential equations and partial differential equations in fuzzy setting.

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