Some Properties of Fuzzy Conjunction Useful to Fuzzy Rules

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Abstract: The fuzzy logical operators have a very important role in the fuzzy expert systems and fuzzy controllers. The premises of a rule in a fuzzy expert system are made of one or more logical fuzzy operations. In this paper we will consider the logical fuzzy conjunction defined by a t-norm, and we will determine its properties. From this properties the theorem of structure for the fuzzy disjunction of n arity, proved that the fuzzy Archimedean operator can be generated by a function $f \ :[0,1]\to[0,\infty]$, *continuous and strictly decreasing, with* $f(1)=0$. *Other results from this paper to refer to relation between numbers of AND arguments and her logical value.*

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1 Introduction

In many papers in the domain, as [2], [3], [7], [10], the logical operation of two fuzzy sets are defined on $[0,1]\times[0,1]\to[0,1]$. In this paper we will consider the logical fuzzy operator AND (fuzzy conjunction) defined by a t-norm, and we will extent this operator to an operator of n arity.

The fuzzy logical operations of n arity are used in fuzzy systems, fuzzy controllers and their properties proved in this paper will help us in tunning the mechanisms of fuzzy inference [10], [11], [1].

Let Ω be universal space.

Definition 1: If we have two classical sets of objects $X,Y\subseteq\Omega$ then the membership function of the conjunction of two fuzzy sets $\tilde{A}$ in X and $\tilde{B}$ in Y is given by:

$$AND: F\times F\to F\ ,\ AND\left(\mu_{\tilde{A}},\mu_{\tilde{B}}\right): X\times Y\to[0,1]$$

$$AND\left(\mu_{\tilde{A}}, \mu_{\tilde{B}}\right)(x, y) \overset{d}{=} T\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\right) \quad (1)$$

where F represents the set of the membership functions and T is a t-norm also named the generator of fuzzy conjunction. The extension of function AND to a function of n arity is made inductively by the use of associatively:

$$\begin{aligned} & AND(\mu_1, \mu_2, \ldots \mu_n) \overset{d}{=} AND(\mu_1, AND(\mu_2, \ldots \mu_n)) = \\ & = AND(AND(\mu_1, \mu_2, \ldots \mu_{n-1}), \mu_n) \forall n \in IN, \mu_n \in F \end{aligned} \quad (2)$$

2 The Properties of the Logical Fuzzy Conjunction of N Arity

In 1965 C. H. Ling [5] proved the theorem of structure for continuous and Arhimediean t-norms. In [4] J. Fodor and M. Roubens define the AND operator by t-norm. We will further prove the theorem of structure for the fuzzy conjunction of n arity:

Theorem 1: If the generator of fuzzy conjunction is continuous and Archimedean then there exists a function $f : [0,1] \to [0, \infty]$, continuous and strictly decreasing, with $f(1) = 0$, so that,

$$AND(\mu_1, \ldots, \mu_n)(x_1, \ldots, x_n) = f^{(-1)}\left(\sum_{i=1}^{n} f(\mu_i(x_i))\right), \quad i > 1\ \mu_i \in F, x_i \in \Omega \quad (3)$$

where $f^{(-1)}$ is pseudoinverse of *f* defined by $f^{(-1)}(z) = \begin{cases} f^{-1}(z) & \text{if } z \le f(0) \\ 0 & \text{else} \end{cases}$.

Proof: The proof is made by induction. For n=2 we have Ling's theorem of structure for t-norm continuous and Archimedean. We assume that

$$AND(\mu_1, \ldots, \mu_{n-1})(x_1, \ldots, x_{n-1}) = f^{(-1)}\left(\sum_{i=1}^{n-1} f(\mu_i(x_i))\right), \quad i,n \in IN, \mu_i \in F, x_i \in \Omega,$$

then $AND(\mu_1, \ldots, \mu_n)(x_1, \ldots, x_n) = AND(AND(\mu_1, \ldots, \mu_{n-1}), \mu_n)(x_1, \ldots, x_n) =$

$$= f^{(-1)}\left(f\left(AND(\mu_1, \ldots, \mu_{n-1})(x_1, \ldots, x_{n-1})\right) + f(\mu_n(x_n))\right) =$$

$$= f^{(-1)}\left(f\left(f^{(-1)}\left(\sum_{i=1}^{n-1} f(\mu_i(x_i))\right)\right) + f(\mu_n(x_n))\right) \ i > 1, \mu_i \in F, x_i \in \Omega$$

We have two cases:

a) If $\sum_{i=1}^{n-1} f(\mu_i(x_i)) \le f(0)$ then $f^{(-1)}\left(\sum_{i=1}^{n-1} f(\mu_i(x_i))\right) = f^{-1}\left(\sum_{i=1}^{n-1} f(\mu_i(x_i))\right)$ and $AND(\mu_1,\dots,\mu_n)(x_1,\dots,x_n) = f^{(-1)}\left(\sum_{i=1}^{n} f(\mu_i(x_i))\right)$, $i > 1, \mu_i \in F, x_i \in \Omega$.

b) If $\sum_{i=1}^{n-1} f(\mu_i(x_i)) > f(0)$ then $f^{(-1)}\left(\sum_{i=1}^{n-1} f(\mu_i(x_i))\right) = 0$ and

$$AND(\mu_1,\dots,\mu_n)(x_1,\dots,x_n) = f^{(-1)}\left(\underbrace{f(0) + f(\mu_n(x_n))}_{>f(0)}\right) = 0, \quad i > 1, \mu_i \in F, x_i \in \Omega$$

But $f^{(-1)}\left(\sum_{i=1}^{n} f(\mu_i(x_i))\right) = f^{(-1)}\left(\underbrace{\underbrace{\sum_{i=1}^{n-1} f(\mu_i(x_i))}_{>f(0)} + f(\mu_n(x_n))}_{>f(0)}\right) = 0$. So even in this case $AND(\mu_1,\dots,\mu_n)(x_1,\dots,x_n) = f^{(-1)}\left(\sum_{i=1}^{n} f(\mu_i(x_i))\right)$, $i > 1, \mu_i \in F, x_i \in \Omega$.

Theorem 1 proved that the any fuzzy Archimedean conjunction can be generated by a function f with the properties from theorem.

Proposition 1: If $\exists x_0 \in X$, and $\exists y_0 \in Y$, such that $AND\left(\mu_{\tilde{A}}, \mu_{\tilde{B}}\right)(x_0, y_0) = 1$ then $\mu_{\tilde{A}}(x_0) = \mu_{\tilde{B}}(y_0) = 1$.

Proof: Let $x_0 \in X, y_0 \in Y$ so that $AND\left(\mu_{\tilde{A}}, \mu_{\tilde{B}}\right)(x_0, y_0) = 1$. By $AND\left(\mu_{\tilde{A}}, 1\right)(x, y) = \mu_{\tilde{A}}(x)$ it results that $\mu_{\tilde{A}}(x_0) = 1$ $\left(\mu_{\tilde{B}}(y_0) = 1\right)$ implies $\mu_{\tilde{B}}(y_0) = 1$ $\left(\mu_{\tilde{A}}(x_0) = 1\right)$. Let us assume that $\mu_{\tilde{A}}(x_0) \ne 1, \mu_{\tilde{B}}(y_0) \ne 1$ and $AND\left(\mu_{\tilde{A}}, \mu_{\tilde{B}}\right)(x_0, y_0) = 1$, then $\mu_{\tilde{A}}(x_0) = AND\left(\mu_{\tilde{A}}, 1\right)(x_0, y_0) \ge$

$\ge AND\left(\mu_{\tilde{A}}, \mu_{\tilde{B}}\right)(x_0, y_0) = 1 \Rightarrow \mu_{\tilde{A}}(x_0) = 1$, contradiction.

As a result $\mu_{\tilde{A}}(x_0)=\mu_{\tilde{B}}(y_0)=1$.

Proposition 2 it is a generalization of proposition 1 and it proves that if in a fuzzy rule the premise made of n arity conjunction has value 1 in a certain point, then all the membership functions involved in the premise of that rule will have value 1 in that point:

Proposition 2: If $X_n \subset \Omega, n \in IN$ and $\exists x_0^i \in X_i,\ i=1..n$ so that $AND(\mu_1,\mu_2,...\mu_n)(x_0^1,...,x_0^n)=1\ \forall \mu_n \in F$ then $\mu_1\left(x_0^1\right)=\mu_2\left(x_0^2\right)=$

$=,...,=\mu_n\left(x_0^n\right)=1$.

Proof: The proof is based on the generalization of proposition 1.

Theorem 2 proves that the logical value of a fuzzy conjunction of n arity decreasing with the number of the involved membership functions. This means that the more logical propositions are involved in a fuzzy conjunction, the more its logical value will decrease:

Theorem 2: Let $\mu_k \in F, k=\overline{1,n}$ then $AND(\mu_1,\mu_2,...\mu_n)\leq$

$\leq AND\left(\mu_{i_1},\mu_{i_2},...\mu_{i_l}\right)$ for any $\{i_1,i_2,...,i_l\}\subseteq\{1,2,...,n\}, \mu_i \in F$.

Proof:

$$AND(\mu_1,\mu_2,...\mu_n)=AND(AND(\mu_1,\mu_2,...\mu_{n-1}),\mu_n)\leq$$
$$AND(AND(\mu_1,\mu_2,...\mu_{n-1}),1)=AND(\mu_1,\mu_2,...\mu_{n-1})$$

Due to commutatively we have $AND(\mu_1,\mu_2,...\mu_n)\leq AND\left(\mu_{i_1},\mu_{i_2},...\mu_{i_l}\right)$ (4)

with $\{1,2,...,n\}\supseteq\{i_1,i_2,...,i_l\}$.

Corollary 1:

$$AND(\mu_1,\mu_2,...\mu_n)\leq \min_{\{1,2,...,n\}\supseteq\{i_1,i_2,...,i_l\}}\left\{AND\left(\mu_{i_1},\mu_{i_2},...\mu_{i_l}\right)\right\}\ \forall n,l \in IN, \mu_n \in F \quad (5)$$

The proof is obvious using theorem 2.

Corollary 2:

$$AND(\mu_1,\mu_2,...\mu_n)\leq AND\left(\mu_{i_1},\mu_{i_2}\right)\ \forall \mu_n \in F,\ \{1,2,...,n\}\supseteq\{i_1,i_2\} \quad (6)$$

The proof is obvious using theorem 2.

Corollary 3:

$$AND(\mu_1, \mu_2, ...\mu_n) \le \min\left(\mu_{i_1}, \mu_{i_2}\right) \ \forall \mu_n \in F, \ \{1,2,...,n\} \supseteq \{i_1, i_2\} \quad (7)$$

The proof is obvious, using the proprieties

$AND\left(\mu_{\tilde{A}}, \mu_{\tilde{B}}\right) \le \min\left(\mu_{\tilde{A}}, \mu_{\tilde{B}}\right) \ \forall \mu_{\tilde{A}}, \mu_{\tilde{B}} \in F$ and corolary 2.

From this corollary it immediately results the following:

Corollary 4: $AND(\mu_1, \mu_2, ...\mu_n) \le \mu_i \ \ \forall \mu_n \in F, i = 1..n, \quad n \in IN$ (8)

Corollary 5: If $x_n \in X_n \subseteq \Omega, n \in IN$ and $\exists x_k^0 \in X_k$, with $\mu_k\left(x_k^0\right) = 0 \Rightarrow$

$$\Rightarrow AND(\mu_1, ..., \mu_k, ...\mu_n)\left(x_1, ...x_k^0, ...x_n\right) = 0 \quad (9)$$

Proof: Let $x_n \in X_n \subseteq \Omega, n \in IN$ and $x_k^0 \in X_k$, with $\mu_k\left(x_k^0\right) = 0$.

From corollary 4 we have $AND(\mu_1, \mu_2, ...\mu_n) \le \mu_i \ \ \forall \mu_n \in F, i = 1..n, \quad n \in IN \Rightarrow$

$\Rightarrow AND(\mu_1, ..., \mu_k, ...\mu_n)\left(x_1, ...x_k^0, ...x_n\right) \le \mu_k\left(x_k^0\right) = 0$. So
$AND(\mu_1, ..., \mu_k, ...\mu_n)\left(x_1, ...x_k^0, ...x_n\right) = 0$.

Theorem 3 shows that the larger the number of the positive membership functions involved in the fuzzy disjunction, the closer to 1 the value of the disjunction tends to be.

Theorem 3: If the fuzzy logical conjunction has an Archimedean generator and $\sup\{\mu_n\} < 1$ then $\lim_{n\to\infty}\left(AND(\mu_1, ..., \mu_n)\right) = 0$.

Proof: From [D. Butnariu, E. P. Klement] pag. 24 we have for every constant sequence $(x_n)_{n\in IN} \in [0,1)$ and for every Archimedean t-norm T: $\lim_{n\to\infty} \overset{n}{\underset{i=1}{T}} x_n = 0$. If $\mu_{\sup} = \sup\{\mu_n\}$ then $0 \le \lim_{n\to\infty}\left(AND(\mu_1, ..., \mu_n)\right) \le \lim_{n\to\infty}\left(AND(\mu_{\sup}, ..., \mu_{\sup})\right) = \lim_{n\to\infty} \overset{n}{\underset{i=1}{T}} \mu_{\sup} = 0 \Rightarrow$.

$\Rightarrow \lim_{n\to\infty}\left(AND(\mu_1, ..., \mu_n)\right) = 0$.

Corollary 7: If the fuzzy logical conjunction has an Archimedean generator, and if $(\mu_n)_{n\in IN}$ is a sequence of constant membership functions with $\mu_n = a \in [0,1) \forall n \in IN$, then $\lim_{n\to\infty}(AND(\mu_1,...,\mu_n)) = 0$.

Proof: If $\mu_n = a \in [0,1) \forall n \in IN \Rightarrow \sup\{\mu_n\} < 1$ and from theorem 3 we have $\lim_{n\to\infty} AND(\mu_1, \mu_2, ...\mu_n) = 0$.

Conclusions

The properties of logical fuzzy disjunction proved in this paper represent an important means in developing a fuzzy expert system, fuzzy controllers and in determining its properties.

The fuzzy logical operator AND of n arity can be generalized if $F = \{\mu | \mu : \Omega \to IR^+\}$ and T is a function defined on $IR^+ \times IR^+ \to IR^+$. In the same way there can also be generalized the other logical operators. On the other hand the operator AND can be individualized taking into consideration the class of t-norms used: Schweizer&Sklar [1983], Hamacher [1978], Frank [1979], Yager [1980], Dubois & Prade [1980] or Dombi [1982] [6] (for a better characterization of a t-conorms can be study the paper [8]). Coversely there can be proved the properties of fuzzy logical operator OR, further on establishing properties of the combinations of the two operators [9].

Considering the properties of the logical fuzzy operators of n arity there can be developed methods and techniques of tunning the engines of fuzzy inference.

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