The Shapley value for capacities and games on set systems

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Abstract: We propose a generalization of capacities which encompass in a large extent the class of Choquet's capacities. Then, we define the class of probabilistic values over these capacities, which are values satisfying classical axioms, the well-known Shapley value being one. Lastly, we propose a value on these capacities by borrowing ideas from electric networks theory.

Keywords: Regular set systems, capacities, Shapley value, probabilistic efficient values, anonimity values, Kirchhoff's laws.

1 Introduction

Capacities [4], also known under the name of fuzzy measures, have become these last two decades an important tool in decision making, able to model in a flexible way the behaviour of the decision maker. Specifically, let us consider a set X of alternatives in a multicriteria decision making problem, where each alternative is described by a set of n real valued scores $(a_1, \ldots, a_n)$. Suppose one wants to compute a global score of this alternative by the Choquet integral w.r.t. a capacity v, namely $\mathcal{C}_v(a_1, \ldots, a_n)$. Then it is well known that the correspondence between the capacity and the Choquet integral is $v(A) = \mathcal{C}_\mu(1_A, 0_{A^c}), \forall A \subseteq N$, where $(1_A, 0_{A^c})$ is an alternative having 1 as score on all criteria in A, and 0 otherwise. Recently, psychology research has shown that most often human decision makers have a different behaviour when faced with alternatives having positive and negative scores, which conducted Grabisch and Labreuche [8] to propose the concept of *bi-capacities* as a generalization of capacities, and later *k-ary capacities* [9].

On the other hand, it may also be interesting to remark that in particular cases, one has not to take into account some alternatives, that is to say the underlying set of feasible coalitions of criteria is only a subset of the power set of N, that amounts to saying that there are forbidden coalitions. We define in the second section regular set systems, which are such structures, and build capacities over.

The notion of Shapley value [12], borrowed from game theory, has become of primary importance for the interpretation of capacities. The Shapley value is a mapping defined on the set of games, which reflects the expected marginal contribution of a player (or criterion) to a random coalition. For example, Edelman deals with the Shapley value in games with forbidden coalitions in the context of voting [6]. In the spirit of Weber's work [13], we define in the third section probabilistic values for games on regular set systems and in the fourth section, we characterize the Kirchhoff's value, named in this way, due to the coefficients of the value satisfying properties of an electrical current. Other analogies between this value and the current flow in an electrical circuit are finally presented in the last section.

In the paper, $N := \{1, 2, \ldots, n\}$ refers to the finite set of criteria. In order to avoid heavy notations, we will often omit braces for subsets, by writing i instead of $\{i\}$ or 123 for $\{1, 2, 3\}$. Furthermore, cardinalities of subsets $S, T, \ldots$ will be denoted by the corresponding lower case letters $s, t, \ldots$

2 Regular capacities

Let us consider $\mathcal{N}$ a subcollection of the power set 2^N of N. We call $(N, \mathcal{N})$ a *set system* on N if $\mathcal{N}$ contains $\emptyset$ and N. A set system is a particular case of a partially ordered set (poset) where the partial order is inclusion. In the sequel, $(N, \mathcal{N})$ always denotes a set system.

Elements of $\mathcal{N}$ are called *coalitions*. For any two coalitions A, B of $\mathcal{N}$, we say that A is *covered* by B, and write $A \prec B$, if $A \subsetneq B$ and $A \subseteq C \subsetneq B$ implies $C = A$.

Definition 1 *$(N, \mathcal{N})$ is a* regular set system *if it verifies the following property: $\forall S, T \in \mathcal{N}$ such that $S \prec T$ in $\mathcal{N}$, then $|T \setminus S| = 1$.*

For any two coalitions S, T in a set system $(N, \mathcal{N})$, a *maximal chain from S to T* is any sequence $(S_0, S_1, \ldots, S_m)$ of elements of $\mathcal{N}$ such that $S_0 = S$, $S_m = T$, and $S_i \prec S_{i+1}$ for every $0 \leq i \leq m-1$. If S and T are not specified, maximal chains are understood to be from $\emptyset$ to N. Note that regular set systems might have also the following definitions:

Proposition 1 *Let $(N, \mathcal{N})$ be a set system. Thus properties* **(i)** *and* **(ii)** *are equivalent.*

(i) $(N, \mathcal{N})$ *is a regular set system.*

(ii) *All maximal chains of* $(N, \mathcal{N})$ *have length* n*, i.e. all maximal chains have exactly* $n+1$ *elements.*

In addition, a regular set system satisfies the following:

(iii) $\forall S \in \mathcal{N} \setminus N, \exists i \in N \setminus S$ *such that* $S \cup i \in \mathcal{N}$ *and* $\forall T \in \mathcal{N} \setminus \emptyset, \exists j \in T$ *such that* $T \setminus j \in \mathcal{N}$*.*

Property **(iii)** is not sufficient to characterize regular set systems, and is actually used by Labreuche in [11]. Note that first property of (iii) is also known under the name *one-point-extension.*

Actually, the set of regular set systems is a very general class embodying some classical structures such as distributive lattices and convex geometries [2]. Let us present them.

A *Jordan-Dedekind poset* is any poset such that all its maximal chains between any two elements have the same length. A *Jordan Dedekind set system* is any Jordan-Dedekind poset which is a set system.

A *convex geometry* is any set system $(N, \mathcal{N})$ verifying:

(C1) One-point-extension (see above)

(C2) Intersection closure: $\forall A, B \in \mathcal{N}, \quad A \cap B \in \mathcal{N}$.

The dual set system of the convex geometry is called *antimatroid*, that is to say **(C1)** is replaced with the second property of Proposition 1-**(iii)** and **(C2)** is replaced with union closure.

A lattice is *distributive* when the infimum and the supremum obey the distributivity law. From any poset, the set of all downsets of $(N, \leq)$ endowed with the inclusion relation is a distributive lattice. Conversely, a fundamental theorem due to Birkhoff [3] says that any distributive lattice $(L, \leq)$ is isomorphic to the set $\mathcal{O}(\mathcal{J})$ of all downsets of the set $\mathcal{J}$ of join-irreducible elements of L. Consequently, any distributive lattice may be given under the isomorphic form $\mathcal{O}(N)$, where $(N, \leq)$ is a poset. We will call this form a *distributive regular set system.* Indeed, it is also known that distributive lattices which are set systems coincide with the class of set systems closed by intersection and union.

We present now the following inclusion diagram where these set systems structures fit into each other (see Fig. 1).

Proposition 2

(1) *The class of Jordan-Dedekind set systems strictly includes regular set systems.*

(2) *The class of regular set systems strictly includes regular set lattices (regular set systems with a lattice structure).*

(3) *The class of regular set lattices strictly includes convex geometries and antimatroids.*

(4) *The intersection of the classes of convex geometries and antimatroids coincides with the class of distributive regular set systems.*

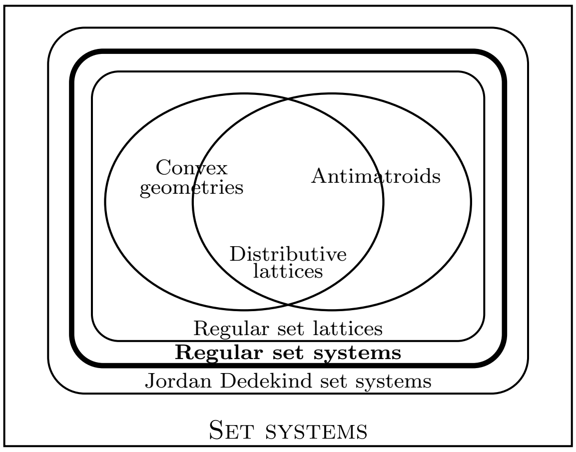

Figure 1: Inclusion diagram of set systems

We define by *regular game* any game defined on a regular set system, that is to say, any mapping v defined over a regular set system $(N, \mathcal{N})$ such that $v(\emptyset) = 0$. If in addition, $A \subseteq B$ implies $v(A) \leq v(B)$ for all $A, B \in \mathcal{N}$, v is called a *regular capacity* on $(N, \mathcal{N})$. We respectively denote by $\mathcal{G}(\mathcal{N})$ and $\mathcal{K}(\mathcal{N})$ the real vector spaces of regular games and regular capacities over the set system $(N, \mathcal{N})$. Whenever $\mathcal{N} := 2^N$, $\mathcal{G}(\mathcal{N})$ is the set of classical cooperative games and $\mathcal{K}(\mathcal{N})$ is the set of classical capacities of Choquet (1953).

We introduce *equidistributed capacities* of $\mathcal{K}(\mathcal{N})$ being regular games v that are both symmetric and additive, that is to say, worths $v(S)$ are proportional to s:

$$\exists \nu \in \mathbb{R} \text{ such that } \forall S \in \mathcal{N},\ v(S) = \nu \cdot s.$$

As a consequence of Proposition 2, the material we propose in what follows, is convenient as well for games on convex geometries [1] and thus for games with precedence of constraints [7], where feasible coalitions of criteria are only the ones that satisfy a given precedence structure on the set of criteria: let $P := (N, \leq)$ be a partially ordered set of criteria, where $\leq$ is a relation of *precedence* in the sense that $i \leq j$ if the presence of j enforces the presence of i in any coalition $S \subseteq N$. Hence, a (valid) *coalition* of P is a subset S of N such that $i \in S$ and $j \leq i$ entails $j \in S$. Hence, the collection $\mathcal{C}(P)$ of all coalitions of P is the collection of all downsets (ideals) of P, which is a distributive lattice.

3 Probabilistic and efficient values

From now on, $(N, \mathcal{N})$ refers to a regular set system. A *value*, or *power index* on $\mathcal{G}(\mathcal{N})$ is a mapping $\Phi : \mathcal{K}(\mathcal{N}) \to \mathbb{R}^n$ that associates to each regular game v a vector $(\Phi^1(v), \dots, \Phi^n(v))$, where the real number $\Phi^i(v)$ represents the importance of player i in the game. The Shapley value Φ_{Sh} for classical cooperative games (or classical capacities) is well known [12].

$$\begin{aligned} &\forall v \in \mathcal{G}(\mathcal{N}), \forall i \in N, \\ &\Phi^i_{Sh}(v) := \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!}(v(S \cup i) - v(S)). \end{aligned} \tag{1}$$

Following the work of Weber [13], Bilbao has defined and axiomatized a class of values for games defined over convex geometries, the *probabilistic values*. It is quite possible to define such values for regular games, and thus for capacities on regular set systems. We propose the following axiomatization, which is also valid in the more general framework of game theory, that is to say, isotonicity of capacities has no inference on the results.

First, we denote by $S + i$ the coalition $S \cup i$ whenever $S \not\ni i$. Thus, writing $S + i \in \mathcal{N}$ infers two relations: $i \notin S$ and $S \cup i \in \mathcal{N}$. Similarly, $S - i$ denotes the coalition $S \setminus i$ and infers $S \ni i$.

Definition 2 *A value Φ on $\mathcal{K}(\mathcal{N})$ is a* probabilistic value *if there exists for each criterion i, a collection of real numbers $\{p^i_S \mid S \in \mathcal{N}, S + i \in \mathcal{N}\}$ satisfying $p^i_S \geq 0$ and $\sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p^i_S = 1$, such that*

$$\Phi^i(v) = \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p^i_S \,(v(S \cup i) - v(S)), \tag{2}$$

for every capacity $v \in \mathcal{K}(\mathcal{N})$.

If no condition is required for real numbers p^i_S, then we call Φ a normal value.

In a capacity, it is assumed that all criteria have to be fully satisfied to form the best alternative. This leads to the problem of distributing the amount $v(N)$ among them. In this case, a value Φ is *efficient* if it verifies:

Efficiency axiom (E): $\forall v \in \mathcal{K}(\mathcal{N}), \quad \sum_{i=1}^n \Phi^i(v) = v(N)$.

We consider also the following axioms.

Linearity axiom (L):
$\forall i \in N, \forall v, w \in \mathcal{K}(\mathcal{N}), \forall \alpha \in \mathbb{R}, \quad \Phi^i(\alpha v + w) = \alpha \Phi^i(v) + \Phi^i(w)$.

A criterion i is said to be *null* in a capacity v, whenever his contributions to all coalitions S are null.

Definition 3 *A criterion $i \in N$ is null for $v \in \mathcal{K}(\mathcal{N})$ if for all $S \in \mathcal{N}$ such that $S + i \in \mathcal{N}$, $v(S \cup i) = v(S)$.*

> **Null axiom (N):** If criterion i is null for capacity v, then $\Phi^i(v) = 0$.

Due to the monotonicity of a capacity, the following axiom naturally holds.

> **Positivity axiom (P):** For any $v \in \mathcal{K}(\mathcal{N})$, all real numbers $\Phi^i(v)$ are non negative.

In the framework of game theory, application of the positivity axiom is also restricted to capacities (isotonic games).

We have the following results, already seen in [1], in the case of convex geometries.

Proposition 3 *Let Φ a value on $\mathcal{K}(\mathcal{N})$. Under axioms **(L)** and **(N)**, Φ is a normal value.*

We present now important results about normal values, already known for convex geometries [1]. Theorem 1 is furthermore presented with weaker axioms.

Theorem 1 *Let Φ be a value on $\mathcal{K}(\mathcal{N})$. Under axioms **(L)**, **(N)**, **(P)** and **(E)**, Φ is a probabilistic and an efficient value.*

Proposition 4 *Let Φ be a normal value on $\mathcal{K}(\mathcal{N})$:*

$$\Phi^i(v) = \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p_S^i \,(v(S \cup i) - v(S)),$$

where p_S^i are real numbers, for all $i \in N$. Then Φ satisfies the efficiency axiom if and only if

$$\sum_{i \in N | i \in \mathcal{N}} p_\emptyset^i = \sum_{i \in N | N \setminus i \in \mathcal{N}} p_{N \setminus i}^i = 1, \tag{3}$$

$$\sum_{i \in N | S-i \in \mathcal{N}} p_{S \setminus i}^i = \sum_{i \in N | S+i \in \mathcal{N}} p_S^i, \tag{4}$$

for all $S \in \mathcal{N} \setminus \{\emptyset, N\}$.

4 The Shapley value for regular capacities

If we focus now on the particular case of classical capacities, we know that Weber has characterized the Shapley value on $\mathcal{K}(2^N)$ as the unique probabilistic value verifying the well known *symmetry axiom*, assuming that the coefficients of the value should not depend on the labelling of the elements of N, that is a very natural property.

Generalizing the symmetry axiom of values defined on larger structures than $\mathcal{K}(2^N)$ is actually a difficult problem since in a set system $(N, \mathcal{N})$, criteria generally cannot be permuted unless the strucure of $(N, \mathcal{N})$ is altered. Moreover, solutions are not a priori unique and are only proposed in cases of capacities (or games) defined over some particular regular set systems. For instance, Grabisch and Labreuche [10] have proposed an *invariance axiom* to define the Shapley value for bi-capacities. Indeed, the domain of bi-capacities with m criteria is isomorphic to $\{-1, 0, 1\}^m$, that we may easily make in correspondance with a regular set system $(N, \mathcal{N})$, where N has $2m$ elements.

We propose now a completely different approach for the axiomatization of the Shapley value on $\mathcal{K}(\mathcal{N})$ from normal values. Consider first the Shapley value on $\mathcal{K}(2^N)$ where $p_S^i := \frac{s!(n-s-1)!}{n!}$ (cf. (1)).

Observing that for any subsets $A, A+i+j \subset N$, the equality $p_A^i + p_{A\cup i}^j = p_A^j + p_{A\cup j}^i$ holds, one may wonder if this property is sufficient to form the Shapley value from a probabilistic and efficient value. Actually, the answer is positive. Furthermore, one can generalize its to any case of capacity defined on a regular set system.

In this perspective, we introduce the following material. Let A, B be any two coalitions of $\mathcal{N}$ such that $A \subseteq B$, and $\mathcal{C} := (S_a, S_{a+1}, \dots, S_b)$ be a maximal chain from A to B. Thus we denote $\sigma^{\mathcal{C}}$ the mapping defined over $\{a+1, \dots, b\}$ by $\sigma^{\mathcal{C}}(i) := S_i \setminus S_{i-1}$. Remark that σ is a permutation of N if $A = \emptyset$ and $B = N$. Besides, for any normal value Φ whose coefficients are given by (2) and any capacity $v \in \mathcal{K}(\mathcal{N})$, we denote the cumulative sum of marginal contributions of criteria of $B \setminus A$ along $\mathcal{C}$ by:

$$m_\Phi^{\mathcal{C}}(v) := \sum_{i=a+1}^{b} p_{S_{i-1}}^{\sigma^{\mathcal{C}}(i)} \left(v(S_i) - v(S_{i-1})\right).$$

Let us consider any equidistributed capacity (Section 2). Thus, *for such a capacity, the cumulative sum of expected marginal contributions of criteria of $B \setminus A$ involved should not depend on the possible considered maximal chains from A to B*, since the path taken from A to B has no effect on the successive increasing worth $v(C)$, $A \subseteq C \subseteq B$.

Indeed, if we assume that $\mathcal{N}$ contains the coalitions $1, 12, 2, 23$ and 123, for instance, then the following equalities should hold for all equidistributed capacity v:

$$\begin{aligned} & p_\emptyset^1\, v(1) + p_1^2\,(v(12) - v(1)) + p_{12}^3\,(v(123) - v(12)) \\ =\; & p_\emptyset^2\, v(2) + p_2^3\,(v(23) - v(2)) + p_{23}^1\,(v(123) - v(23)), \\ & p_2^1\,(v(12) - v(2)) + p_{12}^3\,(v(123) - v(12)) \\ =\; & p_2^3\,(v(23) - v(2)) + p_{23}^1\,(v(123) - v(23)), \end{aligned}$$

that are respectively equivalent to

$$p^1_\emptyset + p^2_1 + p^3_{12} = p^2_\emptyset + p^3_2 + p^1_{23},$$
$$p^1_2 + p^3_{12} = p^3_2 + p^1_{23},$$

since v isequidistributed.

In this spirit, we propose the following axiom.

> **Anonimity axiom (A):**
> For any equidistributed capacity $v \in \mathcal{K}(\mathcal{N})$, for any couple of maximal chains $\mathcal{C}_1, \mathcal{C}_2$ of $(N, \mathcal{N})$, then $m_\Phi^{\mathcal{C}_1}(v) = m_\Phi^{\mathcal{C}_2}(v)$.

We call *anonimity value* any normal value satifying the anonimity axiom. As a consequence, the following fundamental result holds.

Theorem 2 *Let $(N, \mathcal{N})$ be a regular set system. Then there is a unique efficient anonimity value Φ_K on $\mathcal{K}(\mathcal{N})$.*

5 The Shapley value in the framework of network theory

Theorem 1 says that efficient probabilistic values for regular capacities are characterized by the linear axiom, the null axiom, the positivity axiom and the efficiency axiom. Moreover, Proposition 4 enables us to replace the efficiency axiom with properties (3) and (4).

Considering results of previous section, let Φ be a normal value, and let $\mathcal{P} := \{p^i_S \mid i \in N, S \in \mathcal{N} \mid S \cup i \in \mathcal{N}\}$ be the set of associated coefficients. Now, consider the oriented graph $G =: (V, E)$ representing the Hasse diagram of the set system $(N, \mathcal{N})$, where the set of vertices V of G is the set of coalitions of $\mathcal{N}$ and the set of directed edges E of G is given by the relation of precedence induced by inclusion in $\mathcal{N}$, and is identified with the couples $(S, S + i)$ of coalitions of $\mathcal{N}$. Denoting as well by e^i_S the edge $(S, S + i)$, one can attribute to it the weight p^i_S. Note that we can identify maximal chains of $(N, \mathcal{N})$ to maximal paths of the associated graph G to $\mathcal{N}$.

Therefore, if the positivity axiom holds, one can consider the weighted graph $(G, \mathcal{P})$ as a *flow network*, that is to say, a simple, connected, weighted and oriented graph where the weight associated to every directed edge is a non negative number. In a flow network, this worth represents the *capacity* of the edge and is designated as c_{xy} for the edge directed from vertex x to vertex y. In network theory, capacities are maximal amounts of commodity that can be transported along the edge per unit of time.

In addition, we call *flow pattern* in the flow network $(G, \mathcal{P})$ a mapping f defined over E and ranged on $\mathbb{R}+$ such that the following conditions are satisfied:

- For every directed edge (x, y), $f_{xy} \leq c_{xy}$.
- There is a specific vertex s in G, called the *source*, for which

$$\sum_{y \in V|(s,y) \in E} f_{sy} - \sum_{y \in V|(y,s) \in E} f_{ys} = w, \tag{5}$$

  where quantity w is non negative and is called the *flow value*.
- Another specific vertex t in G is called the *sink* and verifies

$$\sum_{y \in V|(t,y) \in E} f_{ty} - \sum_{y \in V|(y,t) \in E} f_{yt} = -w. \tag{6}$$

- All other vertices $x \in V$ verify

$$\sum_{y \in V|(x,y) \in E} f_{xy} - \sum_{y \in V|(y,x) \in E} f_{yx} = 0. \tag{7}$$

As a consequence, Proposition 4 says that for any probabilistic and efficient value Φ, in the corresponding flow network $(G, \mathcal{P})$, the assignment to every directed edge $e_S^i = (S, S+i)$ of its capacity p_S^i is a flow pattern. Particularly, by assigning 1 to the flow value, thus (6) and (5) express (3), and (7) expresses (4). Besides, the source and the sink are obviously $s = \emptyset$ and $t = N$.

Therefore, if we focus our observations on electrical networks which are particular flow networks, thus in this case, the edges of the graph G are called *branches*, whereas the vertices are named *nodes*. Moreover, the commodity flowing in the branches is called *electrical current*.

In this context, the flow pattern property on G is known under the name of *first Kirchhoff's law*, expressing that the algebraic sum of the currents entenring a node, is zero. Therefore, we propose

Node axiom (Nd): For any regular capacity v, (3) and (4) hold.

Actually, this axiom is just the first Kirchhoff's law fitted for values, expressing in addition a unitary flow value.

Now, in the graph G, we call *mesh*, or *closed circuit*, any sequence $(e_1, \ldots, e_m)$ of E such that the e_j's are different and two consecutive edges are incident, as e_1 and e_m. In terms of electrical networks, the *second Kirchhoff's law* expresses that the algebraic sum of the potential drops along any mesh is null, where the *potential drop* of a branch is a worth proportional to the current flowing whithin and depending on a particular quantity: the *resistance* of the branch. Actually, the potential drop V_{AB} of a branch AB of resistance R_{AB}, in which a current I flows is given by the *Ohm's law*: $V_{AB} = R_{AB} \cdot I$.

Thus we establish a straight link between the second Kirchhoff's law and the anonimity axiom just as we have established one between the first Kirchhoff's law and the efficiency axiom. In our framework, all edges of the corresponding electrical network of $(N, \mathcal{N})$ should be considered on equal terms, thus they should not depend on any resistance. Consequently, we propose the following transposition of the second Kirchhoff's law for values.

Let Φ be a normal value on $\mathcal{K}(\mathcal{N})$ and p_S^i be the associated coefficients of (2). For any mesh $\mathcal{M} := (e_1, \dots, e_m)$ on $(N, \mathcal{N})$, we denote $p(e_j)$, $j = 1, \dots, m$, the coefficient of Φ associated to e_j, that is to say, $p(e_j) := p_S^i$ where $\exists S \in \mathcal{N}, \exists i \in N$ such that e_j is directed from S to $S + i$. Furthermore, let $V(e_j)$ be the real number defined by

$$V(e_j) = \begin{cases} \ p(e_j) & \text{if in the path } \mathcal{M},\ e_j \text{ is} \\ & \quad\text{directed in accordance} \\ & \quad\text{with } \subseteq, \\ -\ p(e_j) & \text{otherwise.} \end{cases}$$

Mesh axiom (M): For any mesh $(e_1, \dots, e_m)$ of $\mathcal{N}$, then $\sum_{j=1}^{m} V(e_j) = 0$.

Thus, as Proposition (4) shows that under being a normal value, the efficiency axiom is equivalent to the node axiom (corresponding to the first Kirchhoff's law), we prove the following result, corresponding to the second Kirchhoff's law.

Proposition 5 *Let Φ be a normal value on $\mathcal{K}(\mathcal{N})$:*

$$\Phi^i(v) = \sum_{S \in \mathcal{N} | S + i \in \mathcal{N}} p_S^i \,(v(S \cup i) - v(S)),$$

where p_S^i are real numbers, for all $i \in N$. Then the anonimity axiom is equivalent to the mesh axiom.

Therefore, to sum up, we could say with a slight abuse of language that the unique efficient anonimity value on $\mathcal{K}(\mathcal{N})$, is also the unique normal value satifying the Kirchhoff's laws in the sense that it satisfies the node axiom and the mesh axiom. That is why we may call Φ_K the *Kirchhoff's value* or also the *Shapley-Kirchhoff value.*

Finally, we should mention a negative result, that is there are some regular set systems, for which the Kirchhoff's value does not satisfy the positivity axiom. The result is interesting since trivial examples of such regular set systems do not exist: the positivity axiom appears to be satisfied in so far as the regular set system is not "too peculiar". Certainly, being peculiar does not fall within mathematics, thus getting conditions on $(N, \mathcal{N})$ so that positivity axiom holds remains an open problem.

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