An Axiomatization of the Hybrid Probabilistic-Possibilistic Utility Theory

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Abstract: There is given an axiomatization of the hybrid probabilistic-possibilistic mixture, based on a pair of t-conorm S and t-norm T satisfying (CD) condition, and the corresponding S-measure.

Keywords: utility theory, mixture, triangular norm, triangular conorm, S- measure, conditional distributivity

1 Introduction

Utility theory axiomatic foundations, following von Neumann and Morgenstern [9] are based on the notion of probabilistic mixtures. It has been shown by Dubois et al. [2] that the notion of mixtures can be extended to pseudo-additive measures, specially to possibility measures [3]. Cox's well-known theorem, see [12], which justify the use of probability for treating uncertainty, was discussed in many papers. Recently, there are given some critics on it as well some relaxation on the conditions, which imply that also some non-additive measures can satisfy the required conditions. Relaxing the condition on strict monotonicity to monotonicity on the function which occurs in the conditioning requirement, then the pair of t-conorm S and t-norm T which satisfies (CD) and the corresponding pseudo-additive S-measure satisfy also all other required conditions. The solution to the following question: what else remains possible beyond idempotent (possibilistic) and probabilistic mixtures, is obtained in [4,5], see [12], takes the advantage of a result obtained in [8] on the relaxed distributivity of triangular norm over a triangular conorm (called conditional distributivity).

In this paper there is given an axiomatization of the results related to the generalization of the decision theory to non probabilistic uncertainty based on the characterization of the families of operations involved in generalized mixtures [4,5]. What is obtained is a family of mixtures that combine probabilistic and idempotent (possibilistic) mixtures via a threshold and the corresponding pseudo-additive hybrid idempotent-probabilistic measure satisfies also all other required conditions.

2 Probabilistic and Possibilistic Representation of Utilities

Before we present the hybrid axiomatic framework for utility theory, let us recall and summarize both existing sets of utility axioms: classical, probabilistic, von Neumann and Morgenstern, on one side, and possibilistic framework of [3] for utility theory, on other. Comparing both axiomatization, we came to hybrid one, which generalize possibilistic and probabilistic mixtures. Let X be a set of situations (consequences, outcomes).

2.1 Von Neumann and Morgenstern Axioms of Preference-Probabilistic

Let p be a simple probability measure on X, thus $p=(p(x_1),p(x_2),\ldots,p(x_n))$ where p(xi) are probabilities of outcome $x_i \in X$ occurring, i.e.,

$$p(x_i) \geq 0 \text{ for all } i=1,2,\ldots,n, \text{ and } \Sigma_{i=1}^{n} p(x_i)=1.$$

Define (X) as the set of simple probability measures on X. A particular lottery p is a point in (X). A compound lottery is an operation defined on (X) which combines two probability distributions p and p′ into a new one, denoted $V(p,p';\alpha,\beta)$, with $\alpha,\beta \in [0,1]$ and $\alpha+\beta=1$, and it is defined by

$$V(p,p';\alpha,\beta) = \alpha \cdot p+\beta \cdot p'.$$

Notice that $V(p,p';\alpha,\beta) \in$ (X). Let be a binary relation over (X), i.e.,

$\subset$ (X)$\times$ (X). Hence, we can write $(p,q) \in$, or p q to indicate that lottery q is ‘preferred to or equivalent to’ lottery p.

One of the possible axiom systems for the von Neumann and Morgenstern type utility is:

VNM 1: (X) is equipped with a complete preordering structure.

VNM 2 (Continuity): For p q r $\Rightarrow \exists\alpha$: $q \sim V(p,r;\alpha,1-\alpha)$.

VNM 3 (Independence): $p \sim q \Rightarrow V(p,r;\alpha,1-\alpha) \sim V(q,r;\alpha,1-\alpha)$, $\forall r \in$ (X), $\forall \alpha \in [0,1]$.

VNM 4 (convexity): For $\forall$p q $\Rightarrow$ p $V(p,q;\alpha,1-\alpha)$ q, $\forall \alpha \in (0,1)$.

The theorem below shows that the preference ordering on set of states which satisfies the proposed axioms can always be represented by a utility function.

Representation Theorem (von Neumann, Morgenstern [9]). *A preference ordering relation on (X) satisfies axioms VNM1, VNM2, VNM3 and VNM4 if and only if, there is a real-valued function* $U: (X) \rightarrow$ *such that:*

a) *U represents, i.e.,* $\forall p,q \in (X)$, $p \quad q \Leftrightarrow U(p) \leq U(q)$;

b) *U is affine, i.e.* $\forall p,q \in (X)$, $U(\alpha \cdot p+(1-\alpha)\cdot q) = \alpha \cdot U(p)+(1-\alpha)\cdot U(q)$, *for any* $\alpha \in (0,1)$.

Moreover, U is unique up to a linear transformation.

2.2 Dubois, Godo, Prade and Zapico Axioms of Preferences-Possibilistic

The belief state about such situation in X is the actual one is supposed to be represented by a possibility distribution π. A possibility distribution π defined on X takes its values on a valuation scale V, where V is supposed to be linearly order. V is assumed to be bounded and we take sup(V)=1 and inf(V)=0. Define Pi(X) as set of consistent possibility distributions over X, i.e., $Pi(X)=\{\pi:X \rightarrow V \mid \exists x \in X: \pi(x)=1\}$. The possibilistic mixture is an operation defined on Pi(X) which combines two possibility distributions π and π' into a new one, denoted $P(\pi,\pi';\alpha,\beta)$, with $\alpha,\beta \in V$ and $\max(\alpha,\beta)=1$, and it is defined as:

$$P(\pi,\pi';\alpha,\beta) = \max(\min(\alpha,\pi), \min(\beta,\pi')).$$

Let R be a binary relation over Pi(X), i.e., $R \subset Pi(X)\times Pi(X)$. Hence, we can write $(\pi,\pi') \in R$, or $\pi R \pi'$ to indicate that possibilistic lottery π' is ‘preferred to or equivalent to’ lottery π.

The proposed axiom systems for the Dubois, Godo, Prade, Zapico [3] type optimistic utility is:

DP 1: Pi(X) is equipped with a complete preordering structure R.

DP 2 (Continuity): For $\forall \pi \in Pi(X)$, $\exists \lambda$: $\pi \sim P(\overline{\pi},\underline{\pi};\lambda,1)$, where $\overline{\pi}$ and $\underline{\pi}$ are a maximal and a minimal element of Pi(X) w.r.t. R, respectively.

DP 3 (Independence): $\pi \sim \pi' \Rightarrow P(\pi,\pi'';\lambda,\mu) \sim P(\pi',\pi'';\lambda,\mu)$, $\forall \pi'' \in Pi(X)$, $\forall \lambda,\mu$.

DP 4 (Uncertainty prone): $\pi \leq \pi' \Rightarrow \pi R \pi'$.

The set of axioms DP1, DP2, DP3 and DP4 characterize the preference orderings induced by an optimistic utility.

Representation Theorem (Dubois, Godo, Prade, Zapico [3]). *A preference ordering relation on (X) satisfies axioms DB1, DB2, DB3 and DB4 if and only if, there exist:*

a) *a linearly ordered utility scale U, with inf(U)=0 and sup(U)=1;*

b) *a preference function $u:X\to U$ such that $u^{-1}(1)\neq\emptyset\neq u^{-1}(0)$, and*

c) *an onto order preserving function $h:V\to U$ such that h(0)=0, h(1)=1,*

in such a way that it holds: $\pi R\pi'$ iff $\pi < \breve{\imath}_u \pi'$, where $< \breve{\imath}_u$ is the ordering on Pi(X) induced by the qualitative utility $QU^+(\pi) = \max_{x\in X} \min(h(\pi(x)),u(x))$.

3 Axioms for a Hybrid Probabilistic-Possibilistic Utility Theory

Let S be a t-conorm and let *A* be a σ-algebra of subsets of X. A mapping $m:A\to[0,1]$ is called a pseudo-additive measure (*S-measure*), if $m(\emptyset)=0$, $m(X)=1$ and if for all $A,B\in A$ with $A\cap B=\emptyset$ we have: $m(A\cup B)=S(m(A),m(B))$, see [10]. In order to generalize stated sets of axioms for utility theory, we denote $X=\{x_1,x_2,\ldots,x_n\}$ set of outcomes, $\Delta(X)$ set of S-measures defined on X.

We use now results and notations from papers [4,5]. A *hybrid mixture operation* which combines two S-measures m and m′ into a new one, denoted $M(m,m';\alpha,\beta)$, with

$$(\alpha,\beta) \in \Phi_{S,a} = \{(\alpha,\beta) \mid \alpha,\beta\in(0,1),\ \alpha+\beta=1+a \text{ or } \min(\alpha,\beta)\leq a,\ \max(\alpha,\beta)=1\},$$

where $a\in[0,1]$, is defined by

$$M(m,m';\alpha,\beta) = S(T(\alpha,m),\ T(\beta,m')),$$

where (S,T) is a pair of continuous t-conorm and t-norm, respectively, which satisfy the property of conditional distributivity (CD), i.e., for every x,y,z from [0,1] such that $S(y,z)<1$ we have $T(x,S(y,z))=S(T(x,y),T(x,z))$, see [8].

We propose the following set of axioms for a preference relation $\leq_h$ defined over $\Delta(X)$ to represent optimistic utility:

H1: $\Delta(X)$ is equipped with a complete preordering structure $\leq_h$ (i.e., $\leq_h$ is reflexive, transitive and complete).

H2 (Continuity): If $m <_h m' <_h m''$ then:

- $\exists\alpha\in(a,1)$: $m' \sim_h M(m,m'';1+a-\alpha,\alpha)$, if $m,m',m''>a$;
- $\exists\alpha\in(0,a]$: $m' \sim_h M(m,m'';1,\alpha)$, otherwise.

H3 (Independence): For $\forall m,m',m''\in\Delta(X)$ and for $\forall\alpha,\beta\in\Phi_{S,a}$:

$$m' \leq_h m'' \Leftrightarrow M(m',m;\alpha,\beta) \leq_h M(m'',m;\alpha,\beta).$$

H4 (Uncertainty prone):

- $m \leq_h m' \Rightarrow m \leq_h M(m,m';\alpha,1+a-\alpha) \leq_h m'$, $\alpha\in(a,1)$, if $m,m'>a$;
- $m < m' \Rightarrow m <_h m'$, otherwise.

Now, we define a function of optimistic utility for all $m\in\Delta(X)$ as:

$$U^+(m) = S_{x_i\in X}\big(T\big(m(x_i),u(x_i)\big)\big),$$

where $u{:}X\to U$ is a preference function that assigns to each consequence of X a preference level of U, such that $u^{-1}(1)\neq\emptyset\neq u^{-1}(0)$. It is interesting to notice that U^+ preserves the hybrid mixture in the sense that

$$U^+\big(M\big(m,m';\alpha,\beta\big)\big) = S\big(T\big(\alpha,U^+(m)\big),T\big(\beta,U^+(m')\big)\big) = M\big(U^+(m),U^+(m');\alpha,\beta\big).$$

In the proof of the main representation theorem the crucial is the following lemma. The proofs of the lemma and representation theorem will be published in another paper.

Lemma. *Let $_u$ be the preference ordering on $\Delta(X)$ induced by utility function $U^+(m)=S_{xi\in X}T(m(x_i),u(x_i))$, i.e. $m\ {}_u m'$ if and only if $U^+(m)\leq U^+(m')$. Then the binary relation $_u$ verifies set of axioms {H1, H2, H3, and H4}.*

Representation Theorem (Optimistic Utility)

Let $\Delta(X)$ be a set of S-measures defined on X, and $\leq_h$ a binary preference relation on $\Delta(X)$. Then the relation $\leq_h$ satisfies the set of axioms {H1, H2, H3, H4} if and only if there exist:

- *a linearly ordered utility scale U, with inf(U)=0 and sup(U)=1;*
- *a preference function $u{:}X\to[0,1]$,*

in such a way that $m\leq_h m'$ if and only if $m\ {}_u m'$, where $_u$ is the ordering in $\Delta(X)$ induced by the optimistic utility function defined as:

$$U^+(m) = S_{xi\in X}(T(m(x_i),\ u(x_i)),$$

where (S,T) is a pair of continuous t-conorm and t-norm, respectively, which satisfy the condition (CD).

We will introduce, on the analogous way, the pessimistic criterion in the hybrid utility theory, but first, we have to modify the existing set of axioms. Namely, the axioms H2 and H4 have to be adapted to pessimistic preference criterion.

H2* (Continuity): If $m <_h m' <_h m''$ then:

- $\exists\alpha\in(a,1)$: $m' \sim_h M(m,m'';1+a-\alpha,\alpha)$, if $m,m',m''>a$;
- $\exists\alpha\in(0,a]$: $m' \sim_h M(m,m'';\alpha,1)$, otherwise.

H4* (Uncertainty aversion):

- $m \leq_h m' \Rightarrow m \leq_h M(m,m';\alpha,1+a-\alpha) \leq_h m'$, $\alpha \in (a,1)$, if $m,m'>a$;
- $m < m' \Rightarrow m' <_h m$, otherwise.

Thus, the modified set of axioms, i.e., the set {H1, H2*, H3, H4*} faithfully characterize the preference ordering induced by a pessimistic hybrid utility, which is dual to the optimistic one.

Representation Theorem (Pessimistic Utility)

Let $\Delta(X)$ be a set of S-measures defined on X, and $\leq_h$ a binary preference relation on $\Delta(X)$. Then the relation $\leq_h$ satisfies the set of axioms {H1, H2, H3, H4*} if and only if there exist:*

- *a linearly ordered utility scale U, with info(U)=0 and sup(U)=1;*
- *a preference function ox→[0,1],*

In such a way that $m \leq_h im'$ if and only if $m\ {}_u m'$, where ${}_u$ is the ordering in $\Delta(X)$ induced by the pessimistic utility function defined as:

$U^{-}(m) = 1-S_{xi\in X}(T(m(x_i), 1-u(x_i)),$

Where (S, T) is a pair of continuous t-conform and t-norm, respectively, which satisfy the condition (CD).

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