Fuzzy Sets – a Boolean Valued Approach

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Abstract: Though the concept of Boolean valued models is rather old mathematical theory (mid fifties of the last century), it is little or not at all known to the general 'fuzzy' population. Our aim is to reintroduce this theory and to discuss some possibilities of its applications in the field of automated reasoning.

1 Introduction

The fuzzy logic emerges in mid sixties of 20th century in order to capture (mathematically) reasoning about the notions with inherited fuzziness, such as being tall, young, fat, bald etc. Similarly to probabilistic logic, in fuzzy logic we have the real valued truth, i.e. the truth of certain statement can be any real number in the interval [0,1]. However, some classical laws fail in this approach.

Mathematical concept of Boolean valued models allows us to preserve all classical lows and to gain multi (more than two) valued truth. Though this is a rather old theory (mid fifties of the last century), it is almost unknown to the general 'fuzzy' population. We believe that Boolean valued approach can be successfully applied – similar concept with various applications is developed by Dragan Radojević (see [5]).

Concerning this paper, we will reintroduce the concept of |Boolean valued models with particular emphasis on the connection with fuzzy sets and potential applications.

2 Boolean Valued Models

Suppose that **A =** (A, …) is a first order structure of the language L and that **B** = (B, ·, +, ‘, 0, 1) is a complete Boolean algebra. By L(A) we will denote the expansion of the language L with elements of the set A as new symbols for constants. We say that the function || || which maps the set of all sentences of L(A) into B is a **B**-value if the following conditions hold:

1 $\| a = a \| = 1$ for arbitrary elements a and b of A;

2 $\| a = b \| = \| b = a \|$ for any elements a and b of A;

3 $\| a = b \| \cdot \| b = c \| \leq \| a = c \|$ for any elements a, b and c of A;

4 $\| a = b \| \leq \| F(\dots, a, \dots) = F(\dots, b, \dots) \|$ for all elements a, b of A and arbitrary functional symbol F of L;

5 $\| a = b \| \cdot \| R(\dots, a, \dots) \| \leq \| R(\dots, b, \dots) \|$ for any elements a, b of A and arbitrary relational symbol R of L;

6 $\| \neg\varphi \| = \| \varphi \|'$;

7 $\| \varphi \wedge \psi \| = \| \varphi \| \cdot \| \psi \|$;

8 $\| \varphi \vee \psi \| = \| \varphi \| + \| \psi \|$;

9 $\| \forall x\varphi(x) \| = \inf \{ \varphi(a) : a \in A \}$;

10 $\| \exists x\exists\varphi(x) \| = \sup \{ \varphi(a) : a \in A \}$.

We say that (**A**, $\| \ \|$) is a **B**-model of L.

Example 1 Suppose that B is a propositional algebra, i.e. B = {0, 1}. Then the corresponding **B**-value is defined as follows:

$\| \varphi(a_1, \dots, a_n) \| = 1$ if and only if $\varphi(a_1, \dots, a_n)$ is true in **A**.

In this case, **B**-value is the classical satisfaction relation.

Example 2 Suppose that **B** is a Lindenbaum's algebra corresponding to the structure **A**. Then, we define $\| \varphi \|$ as the set of all formulas ψ such that $\varphi\Leftrightarrow\psi$ is true in **A**.

Example 3 Let (**A**, $\| \ \|$) be a **B**-model and let D be a filter of **B**. It is easy to see that a binary relation ~ defined by

$p \sim q$ if and only if there is r in D such that $p\cdot r = q\cdot r$

is a congruence. If $\mathbf{B}_D$ is a corresponding quotient algebra, then the function $\| \ \|_D$ defined by

$\| \varphi \|_D = \{ p\in B : \| \varphi \| \sim p\}$

is a $\mathbf{B}_D$-value. If D is an ultrafilter, then $\mathbf{B}_D$ is isomorphic to the propositional algebra and again we obtain the classical (two valued) model.

Boolean valued models give us layers of mathematical truth, i.e. multi-valued logic. Unlike the case of fuzzy logic, Boolean valued approach preserves all classical laws. Namely, the following theorem holds:

Theorem Suppose that **B** is a complete Boolean algebra, L is a first order language and (**A**, $\| \ \|$) is a **B**-model of L. Then each valid sentence φ of L has a **B**-value equal to 1.

Proof Suppose that φ is a sentence of L such that $\| \varphi \| \neq 1$. Then there is an ultrafilter D of **B** such that $\| \varphi \|$ is not in D. Consequently, $\| \varphi \|_D = 0$, so φ is not true in **A**. Thus, φ is not valid.

To make additional connection to fuzzy logic, suppose that m is a probabilistic real valued (more than two element range) measure on B. Then we can define a fuzzy value of φ as $m(\| \varphi \|)$.

3 Boolean Universe V^B

Application of Boolean valued models to set theory deserves special attention. Though this method is developed for the independence proofs (see [3]), we are particularly interested in making the connection to fuzzy sets and corresponding problems.

The language L of set theory contains single binary relational symbol ∈. Suppose that V is a countable transitive model of set theory, **B** is a complete Boolean algebra in V and that countable limit ordinal δ is the height of V. We define the set V^B of **B**-names (or fuzzy sets) by recursion on δ as follows:

- $V^B{}_0 = 0$;
- $V^B{}_{\alpha+1} = \{ X \in V^B{}_\alpha : X \subseteq V^B{}_\alpha \times P(B) \}$;
- $V^B{}_\alpha = \cup \{ V^B{}_\beta : \beta < \alpha \}$;
- $V^B{}_\delta = \cup \{ V^B{}_\alpha : \alpha < \delta \}$.

We will denote **B**-names by σ and τ, possibly with indices. If a pair (σ, s) is an element of some name τ, then we say that σ is an element of τ with reliability s.

For arbitrary set A ∈ V we define its canonical name Â by

$\hat{A} = \{ (\hat{a}, 1) : a \in A \}$.

Next we will define a **B-**value $\| \ \|$ as follows:

1. $\| \sigma \in \tau \| = \sup \{ \| \sigma = \tau_1 \| \cdot p : (\tau_1, p) \in \tau \}$;
2. $\| \sigma \subseteq \tau \| = \inf \{ p' + \| \sigma_1 \in \tau \| : (\sigma_1, p) \in \sigma \}$;
3. $\| \sigma = \tau \| = \| \sigma \subseteq \tau \| \cdot \| \tau \subseteq \sigma \|$.

It can be shown (see for instance [6]) that defined function is indeed a **B**-value and that each theorem of ZFC has a **B**-value equal to 1. Using the above definition one can also show that the following statements hold:

- If $(\sigma, s) \in \tau$, then $s \leq \| \sigma \in \tau \|$;
- If $a,e \in V$, then $a \in e$ if and only if $\| \hat{a} \in \hat{e} \| = 1$;
- If $a,e \in V$, then $a = e$ if and only if $\| \hat{a} = \hat{e} \| = 1$.

In other words, canonical names represent standard sets.

4 Note on Applicability

Many problems that are solvable in the fuzzy framework may be expressed as a Boolean combination of finite numbers of certain atomic assertions, which may be identified as the initial fuzzy states or values. It is quite natural for **B** to take the free Boolean algebra with the set of all fuzzy values as the set of generators. The difficult part is in the right choice of the **B**-names, i.e. in finding the adequate representation of our particular problem in the above framework. However, we strongly believe that it is effectively possible, especially in the problems related with relational databases, data mining etc.

References

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