# Non-Associative Fuzzy Flip-Flop with Dual Set-Reset Feature

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*Abstract: J-K flip-flops are the most general elementary units in sequential digital circuits. By extending Boolean operations to their respective fuzzy counterparts various fuzzy flip-flops ($F^3$) can be defined. Because of the axiomatic properties of fuzzy operations are considerably weaker then the properties satisfied by Boolean lattices, the minterm and maxterm type definitions of the same $F^3$ are as a rule not equivalent. In former work we found a unique exception where simulation investigations lead to identical results for all parameter combinations examined. The base of this unique $F^3$ is a pair of non-associative fuzzy connectives. In this paper the exact proof is given for the identity of the two definitions, i.e., for the uniqueness of the definition of this special non-associative $F^3$.*

*Keywords: Fuzzy logic, Non-associative operations, J-K flip-flop*

## 1 Introduction

In this paper, we present the preliminary results of analyzing elementary fuzzy 'digital' sequential circuits, i.e. fuzzy flip-flops ($F^3$) based on non-associative fuzzy operations.

In particular, we prove the surprising result, that comparing the equations of the next state of the so called set and reset type [5] version of the modified Fodor fuzzy flip-flop ($F^4$) [2,9], i.e. there is only one $F^4$ and the two formulas are equivalent. This equivalence does not hold for any other $F^3$ defined as far in the literature, this is why the symmetrical combined set-reset type $F^3$ was introduced in [6] etc., in order to maintain the original symmetrical property of the Boolean operations based binary J-K flip-flop.

Before introducing this new $F^3$, an overview of classic and non-associative fuzzy operations (t-norms and co-norms), further of Boolean flip-flops will be given.

# 2 Classic Fuzzy Operations

## 2.1 Fuzzy Intersections (Conjunctions): t-Norms and Fuzzy Unions (Disjunctions): t-Conorms

The intersection (conjunction) of two fuzzy sets *A* and *B* is specified in general by a binary operation on the unit interval; i.e., a function of the form

$$i:[0,1]\times[0,1]\rightarrow[0,1]. \tag{1}$$

For each element *x* of the universal set *X*, this function takes the pair consisting of the element's membership grades in set *A* and in set *B* as its argument, and yields the membership grade of the element in the set constituting the intersection of *A* and *B*. Thus,

$$(A\wedge B)(x)=i[A(x),B(x)] \tag{2}$$

for all $x\in X$.

The fuzzy intersection (t-norm) *i* is a binary operation on the unit interval that satisfies the axiomatic skeleton containing certain boundary conditions, commutativity, monotonicity, and associativity. (See [7, 8].)

Similarly to fuzzy intersection, the general fuzzy union of two fuzzy sets *A* and *B* is specified by the function

$$u:[0,1]\times[0,1]\rightarrow[0,1]. \tag{3}$$

The argument of this function is the pair consisting of the membership grade of some element *x* in fuzzy set *A* and the membership grade of that same element in fuzzy set *B*. The function returns the membership grade of the element in the set $A\vee B$. Thus,

$$(A\vee B)(x)=u[A(x),B(x)] \tag{4}$$

for all $x\in X$.

A fuzzy union (t-conorm, disjunction) $u$ is a binary operation on the unit interval that satisfies its respective boundary conditions, commutativity, monotonicity, and associativity. (Cf. [7, 8].)

While these two operations have a completely dual axiomatic skeleton, they can be defined completely independently from each other.

## 2.2 Fuzzy Complements (Negations)

The complement (negation) of a fuzzy set $A$ is specified by a unary operation on the unit interval:

$$c:[0,1]\rightarrow[0,1]. \tag{5}$$

For each element $x$ this function yields the membership grade of the same element in the complement of the original set, thus

$$(\neg A)(x)=c[A(X)] \tag{6}$$

for all $x \in X$.

Complementation satisfies the boundary conditions and monotonicity, while strong complements (negations) are also involutive.

In the practice very often the t-norm and co-norm are defined in a way where along with a suitable fuzzy complement, such as the very often used Zadeh complement

$$\neg A_Z(x)=n_Z(A(x))=1-A(x), \tag{7}$$

they form a De Morgan-triplet, satisfying

$$\neg(A\wedge B)(x)=((\neg A)\vee(\neg B))(x) \quad \text{and} \tag{8}$$

$$\neg(A\vee B)(x)=((\neg A)\wedge(\neg B))(x).$$

Such triplets often satisfy further interesting axiomatic properties of Boolean operations, but never all of them at the same time.

## 2.3 Non-Associative Operations

The motivation for non-associative fuzzy operations can be found in analyzing the behavior of the connectives in subjective probability or certainty calculation contexts [1].

Let us assume that we know the degrees of certainty (subjective probabilities) $p\ (S_1)$ and $p\ (S_2)$ in two statements $S_1$ and $S_2$, then possible values of $p(S_1 \wedge S_2)$ form an interval

$$p = \left[ max\left( p_1 + p_2 - 1,0 \right), min\left( p_1, p_2 \right) \right]. \tag{9}$$

(These two boundaries represent well known extremes of possible intersections corresponding to the axiomatic skeleton according to Section 2.1.)

As a numerical estimate, it is natural to use the midpoint of this interval:

$$p_1 \wedge p_2 = \frac{1}{2}\left[ max\left( p_1 + p_2 - 1,0 \right) + min\left( p_1, p_2 \right) \right]. \tag{10}$$

Similarly, for the union operation, we can take the midpoint of the corresponding interval:

$$p_1 \vee p_2 = \frac{1}{2}\left[ max\left( p_1, p_2 \right) + min\left( p_1 + p_2, 1 \right) \right]. \tag{11}$$

Selecting the midpoint in this way is not only natural from common sense viewpoint, but it also has a deeper explanation. The subjective probabilities of all four minterm combinations of the statements $S_1$ and $S_2$, these four probabilities should add up to 1. Assuming now that all probability distributions are equally possible, i.e., the 'second order probability' has uniform distribution; the midpoint will have maximum expectation. (For details see e.g. [4].)

Despite the natural interpretation of such midpoint based operations, alas they have a rather disadvantageous (to some extent, counterintuitive) property:

$$\left(p_1 \wedge p_2\right) \wedge p_3 \neq p_1 \wedge \left(p_2 \wedge p_3\right) \quad \text{and} \tag{12}$$

$$\left(p_1 \vee p_2\right) \vee p_3 \neq p_1 \vee \left(p_2 \vee p_3\right),$$

both operations are non-associative!

There are many other possible motivations behind using non-associative norms, such as preference and ranking in multicriteria decision making (see [3]). Some very enlightening examples for real-life related decision situations, where one of the connectives has partial features of its dual pair (an intersection with partial union properties or *vice versa*) have been suggested even much earlier, as e.g. in [13].

Because of the interesting properties of the non-associative norms and their important connections with real life, in this paper we intend to investigate the behavior of an $F^3$ defined by using such non-associative operations. The more so, as this aspect of $F^3$ has been already touched upon [2].

# 3 Binary (Boolean) and Fuzzy Flip-Flops

All types of traditional binary flip-flop circuits, such as the most general J-K flip-flop store a single bit of information. These elementary circuits are also the basic components of every synchronous sequential digital circuit. The next state $Q(t+1)$ of a J-K flip-flop is characterized as a function of both the present state $Q(t)$ and the two present inputs $J(t)$ and $K(t)$. In the next, *J, K* and *Q* will be used instead of $J(t)$, $K(t)$ and $Q(t)$, respectively, as simplified notations. The minterm expression (canonical disjunctive form) of $Q(t+1)$ can be written as

$$Q(t+1) = \overline{J}\,\overline{K}Q + J\overline{K}\,\overline{Q} + J\overline{K}Q + JK\overline{Q}\,, \tag{13}$$

this can be simplified to the minimal disjunctive form

$$Q(t+1) = J\overline{Q} + \overline{K}Q \tag{14}$$

This latter is well-known as the characteristic equation of J-K flip-flops. On the other hand, the equivalent maxterm expression (minimized conjunctive form) can be given by

$$Q(t+1) = (J+Q)(\overline{K}+\overline{Q})\,. \tag{15}$$

These two expressions, (14) and (15), are equivalent in Boolean logic, however there are no such fuzzy operation triplets where these two forms are necessarily equivalent. It is a rather obvious question, which of the two should be considered as the fuzzy extension of the definitive equation of the fuzzy extension of the very fundamental concept of J-K flip-flop. There is no justifiable argumentation that prefers any of these two to the other. Thus, there is no unambiguous way to introduce the concept of fuzzy J-K flip-flop. This is why Hirota and Ozawa [5, 6] proposed two dual definitions of fuzzy flip-flops. They interpreted (14) as 'reset type fuzzy flip-flop':

$$Q_R(t+1) = (J \wedge \neg Q) \vee (\neg K \wedge Q)\,, \tag{16}$$

where the denotations for logic operations stand this time for Zadeh type fuzzy conjunction, disjunction and negation. In a similar way the definition of 'set type fuzzy flip-flop' was obtained by re-interpreting (15) with fuzzy operations:

$$Q_S(t+1) = (J \vee Q) \wedge (\neg K \vee \neg Q)\,. \tag{17}$$

As a mater of course, it is possible to substitute the Zadeh operations by any other reasonable fuzzy operation triplet (e.g. De Morgan triplet), this way obtaining a multitude of various fuzzy flip-flop ($F^3$) pairs, such as the algebraic $F^3$ introduced in [10] and [11]. We examined a large number of operations applicable for constructing $F^3$ in [9]. There are also ways to obtain single $F^3$ definitions by merging the set and reset type versions in a manner so that the merger results in a symmetrical $F^3$ [10]. While such a merger has clear advantages in the sense that the $F^3$ thus defined inherits the ’nice‘symmetric of the original binary flip-flop, it is a disadvantage that such merged $F^3$s cannot be given by a single closed formula and can not be interpreted by any single unconditional fuzzy logic formula.

As it has been pointed out, a multitude of popular fuzzy operation triplets have been investigated as the possible base for defining $F^3$ and, as expected, none of them produced an equivalence of (16) and (17), not even by coincidence. Introducing however a pair of non-associative fuzzy connectives for the same purpose lead to the surprising simulation results pointing at an unexpected equivalence of the corresponding set and reset type $F^3$s. Motivated by the hope to be able to find a single symmetrical definition for an $F^3$ we focus this paper to the mathematical analytic examination of this particular non-associative operations based $F^3$, which will follow in section 4.

## 4 Non-Associative Fuzzy Flip-Flops

In [2] Fodor and Kóczy proposed a pair of non-associative operations for a new class of fuzzy flip-flops. It was stated there that any $F^3$ satisfying:

*P1*: $F(0, 0, Q) = Q$,

*P2:* $F(0, 1, Q) = 0$,

*P3:* $F(1, 0, Q) = 1$,

*P4:* $F(1, 1, Q) = n(Q)$,

*P5:* $F(e, e, Q) = e$,

*P6:* $F(D, n(D), Q) = D$.

Where $e = n(e)$ is the equilibrium belonging to $n$, where $n$ is a strong negation, $D \in (0,1)$. *P*2, *P*3 and *P*6 can be merged into the single property *P*6’:

$F(D, n(D), Q) = D, \quad D \in [0,1]$.

Let the $\phi$ -transform be an automorphism of the unit interval such that

$$Q_R(t+1) = \phi^{-1}\left[\phi(J)(1-\phi(Q)) + \phi(Q)(1-\phi(K))\right]. \tag{18}$$

Similarly, for the $\psi$ -transform

$$Q_S(t+1) = \psi^{-1}\left[\psi(J)(1-\psi(Q)) + \psi(Q)\left(1-\psi(K)\right)\right]. \tag{19}$$

The following equation satisfies all $P_i$-s (i= 1, 6)

$$Q_R\left(t+1\right)=min\left[T\left(J,1-Q\right)+T\left(1-K,Q\right),1\right]=$$

$$=T(J,1-Q)+T(1-K,Q)\,. \tag{20}$$

The formula for the set type F⁴ in [2] is however not the proper dual pair of (20), moreover, it is problematic in the sense of closedness for the unit interval. Thus in [9] we proposed a corrected definition for the set type formula as follows:

$$Q_S\left(t+1\right)=max\left[S\left(J,Q\right)+S\left(1-K,1-Q\right)-1,0\right]=$$

$$=S(J,Q)+S(1-K,1-Q)-1\,. \tag{21}$$

In (20) and (21) T and *S* denote the so called Łukasiewicz norms. From here:

$$Q_R(t+1)=\frac{min\left(J,1-Q\right)+max\left(J-Q,0\right)}{2}+$$

$$\frac{min\left(Q,1-K\right)+max\left(Q-K,0\right)}{2}, \tag{22}$$

$$Q_S\left(t+1\right)=\frac{max(J,Q)+max\left(1-K,1-Q\right)-1}{2}+$$

$$\frac{min\left(J+Q,1\right)+min\left(2-Q-K,1\right)-1}{2}\,. \tag{23}$$

These equations were obtained by combining the standard (Zadeh) and the Łukasiewicz norms by the arithmetic mean in the inner part of the formula. The other parts use Łukasiewicz operations. Let us briefly denote the 'Fodor-Kóczy type fuzzy flip-flop' by $F^4$.

Comparing this corrected form of the set type $F^4$ we come to the surprising result that there is only one $F^4$ in this particular case as the two formulas are equivalent.

This fact was strongly suggested by the simulation results obtained in [9], now we present the exact proof.

## 4.1 Statement

$Q_R^{F^4}=Q_S^{F^4}$ , thus there is only one (symmetrical, corrected) $F^4$.

**Proof**

We prove that

$$Q_R\left(t+1\right)=Q_S\left(t+1\right) \tag{24}$$

for every possible combination of $J, K, Q(t)$.

Substituting (22) and (23) we obtain the following equivalent equation to prove:

$$\begin{aligned} &min(J,1-Q)+max(J-Q,0)+min(Q,1-K)+max(Q-K,0)= \\ &max(J,Q)+max(1-K,1-Q)+min(J+Q,1)+min(2-K-Q,1)-2. \end{aligned} \tag{25}$$

For the 6 values $J, K, Q, \neg J, \neg K, \neg Q$, the total number of all possible combinations is theoretically 3! x $2^3$=48. These 48 cases are not all essentially different. Any variable and its negated are symmetrical to the equilibrium *e*=0.5. Consequently, for describing a case it is sufficient to tell which one of the ponated or negated version of each of the three variables is less or equal then *e*. The 8 main cases to be considered are as follows:

| | |
|---|---|
| $1)J,K,Q$ | $5)J,K,\neg Q$ |
| $2)\neg J,\neg K,\neg Q$ | $6)\neg J,\neg K,Q$ |
| $3)\neg J,K,Q$ | $7)\neg J,K,\neg Q$ |
| $4)J,\neg K,Q$ | $8)J,\neg K,\neg Q$ |

Each of these 8 cases has 3! =6 sub-cases depending on the sequence of these three. In the next table these 48 sub-cases will be discussed so that some sub-cases can be merged in the sense that $Q_R$ and $Q_S$ are identical. The total number of essentially different sub-cases is 13. The first column of the table contains the serial number of the essentially different sub-case; the second column describes the inequality conditions applying for the given essential sub-case, while the third column gives the identical value of $2Q_R$ and $2Q_S$ in the given sub-case. The deduction of these results is omitted here for the sake of saving space.

| Case# | ConditionS | $2Q_R(t+1)=2Q_S(t+1)$ |
|---|---|---|
| 1 | $J,K \le Q,\neg Q$ | $J-K+2Q$ |
| 2 | $J,\neg K \le Q,\neg Q$<br>$K,\neg J \le Q,\neg Q$ | $1+J-K$ |
| 3 | $\neg J,\neg K \le Q,\neg Q$ | $2+J-K-2Q$ |
| 4 | $J,Q \le K,\neg K$ | $J+Q$ |
| 5 | $J,\neg Q \le K,\neg K$ | $1+J-2K+Q$ |
| 6 | $\neg J,Q \le K,\neg K$ | $1+J-Q$ |
| 7 | $\neg J,\neg Q \le K,\neg K$ | $2+J-2K-Q$ |
| 8 | $K,Q \le J,\neg J$ | $2J-K+Q$ |
| 9 | $K,\neg Q \le J,\neg J$ | $1-K+Q$ |

| | | |
|---|---|---|
| 10 | $\neg K, Q \leq J, \neg J$ | $1+2J-K-Q$ |
| 11 | $\neg K, \neg Q \leq J, \neg J$ | $2-K-Q$ |
| 12 | $Q \leq J, K, \neg J, \neg K$ | $2J$ |
| 13 | $\neg Q \leq J, K, \neg J, \neg K$ | $2-2K$ |

Table 1
Essentially different sub-case for the $F^4$

In all cases the third column contains a single expression, so $Q_R^{F^4}$ is always identical with $Q_S^{F^4}$ . Q.e.d.

We have shown that the modified $F^4$ proposed in [9] is indeed a single $F^3$ with nice dual and symmetrical behavior. Figure 1 presents some diagrams illustrating the behavior of $F^4$ for some typical *J*, *K* and *Q*.

**Conclusions**

As far no other $F^3$ could be found with the advantageous property discovered for $F^4$. On the other hand the non-associative behavior of the operation used in the definitions is somewhat inconvenient.

In the future we intend to investigate the behavior of complex fuzzy sequential circuits based on $F^4$ and matching logical connectives. Further we intend to investigate the behavior of other $F^3$s based on various 'famous' non-associative operations well known from the literature (e.g. in [1]). It would be very interesting to find other $F^3$s with similar advantageous properties.

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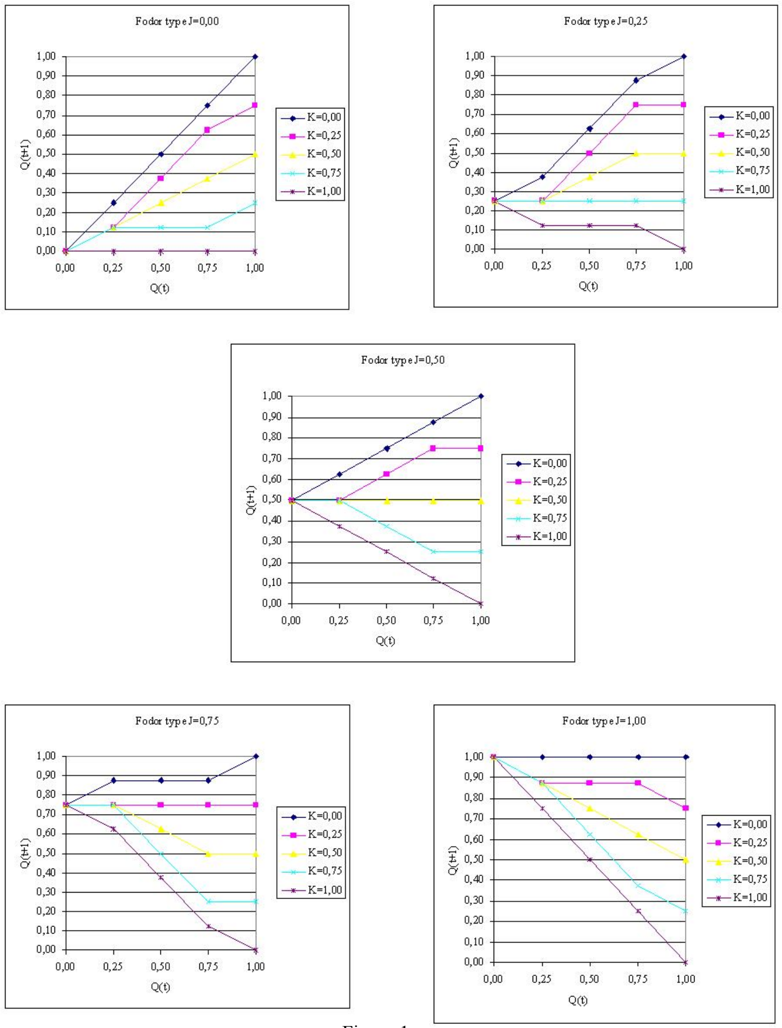


Figure 1

The behavior of $F^4$ for some typical parameter values

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