# Algorithm for computing the digital convex fuzzy hull

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*Abstract: The problem of finding the digital convex fuzzy hull of a digital fuzzy set, whose support is made of some digital points (centroids) in $\mathbb{Z}^2$, is considered here. A region is $DL-$convex if, for any two pixels belonging to it, there exists a digital straight line between them, all of whose pixels belong to the region. An algorithm how to compute the $DL-$convex hull of a digital region is described. Also we describe algorithm for computing digital convex fuzzy hull of digital fuzzy set, which support is in $\mathbb{Z}^2$, and is DL-convex. The proof that digital convex fuzzy hull is obtained by this algorithm, is also given.*

*Key words and phrases: convexity, digital convex fuzzy hull, DL-convex hull.*

## 1 Introduction

In this article we use the notions and the properties related to them which are given in the paper [3]. Our main problem is how to find convex hull of a given digital set. In order to do so we could compute Euclidean convex hull, but it is not necessary for that set to be digital. So it should be the re-digitized. We can iterative fill concavities by looking at local neighborhoods of border pixels, but we cannot be sure that that set is convex (so it is an approximative method), and these methods do not work if there are several components in a region. Moreover, iterative filling is computationally expensive. Some authors considered this problem by discussing $T-$convex hull and $L-$convex hull using $T-$convexity and $L-$convexity, but the resulting hull may not be digitally connected in certain situations, and a disconnected hull is not attractive. So we define a new kind of digital convexity which is stronger than $T-$convexity and $L-$convexity called $DL-$(digital line) convexity. The $DL-$convex hull of any digital region is always digitally connected.

## 2 On the computation of convex hulls of digital sets

We can say that, no proper $DL-$convex subset of $H_{DL}(R')$ can contain $R'$. Let us consider, how to compute $DL-$convex hull of a given digital region, for which the equivalence of $T-$convexity and $DL-$convexity of an $8-$connected figure gives an attractive solution in most cases.

We can compute the $T-$hull $(H_T(R'))$ of $R'$. By Theorem 3 (see [3, 1]), if $H_T(R')$ is $8-$connected, it must be $H_{DL}(R')$. $H_T(R')$ is computed by constructing the Euclidean convex hull $H(R')$ and then finding all the cellular points contained in $H(R')$. The founded $DL-$hull is unique because Euclidean hull is unique.

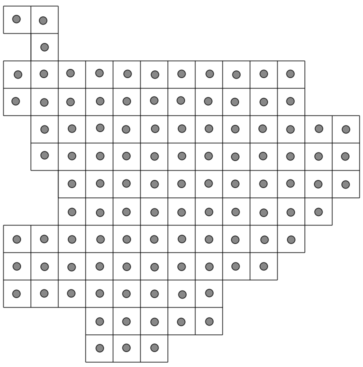

Figure 1 a)

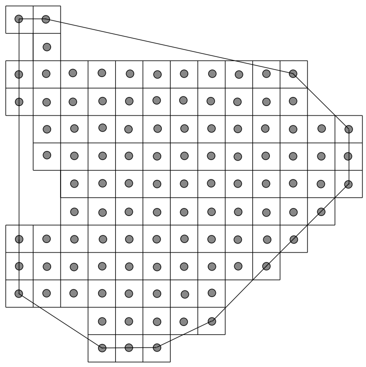

Figure 1 b)

Let us consider the relation between connected components in $H_T(R')$:
1) If $R'$ is connected so is its $T-$hull. If $R'$ is not connected its $T-$hull may still be connected.
2) Let $P$ be a vertex of the Euclidean convex hull polygon of $R'$. If the angle made by two edges meeting at $P$ is divided internally by a horizontal, vertical, or diagonal line (orientation $\pm 45^o$ or $\pm 135^o$) through $P$, no isolated component can exist at $P$. In all other situations, isolated components will exist in $H_T(R')$. This is so because the cellular points in digital space lie on horizontal, vertical or diagonal lines through $P$.
3) If each connected component of $R'$ has a thickness of 2 at every point, $H_T(R')$ must be connected (if a component can contain a square of size at most $t \times t$ cells at a point, then its thickness at that point is $t$).

Our problem becomes more complex if $H_T(R')$ is not connected. To start with, we can connect the components of $H_T(R')$ by digital straight lines.

We can distinguish two situations in which $H_T(R')$ is not connected:

Euclidean hull $H(R')$ is a straight line segment (a degenerate case). In this case we can just join components by a digital straight line segment.

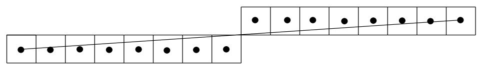

Figure 2

Euclidean hull has no non-zero area (a non-degenerate case), but $H_T(R')$ is not $8-$connected. In this case we "fill" $H(R')$ with pixels to get $H_T(R')$, using the smallest number of pixels we can.

If a vertex of $H(R')$ is isolated in $H_T(R')$ and $H_T(R')$ has $k$ connected components between the edges emanating from this vertex, $(k-1)$ digital straight lines can be drawn to connect these components. Let the result of this operation be the connected region $R''$. Unfortunately $R''$ is not necessarily $T-$convex and hence not $DL-$convex. If $R'$ is $T-$convex, then it is also $DL-$convex, and is $DL-$hull of $R'$. This is so because the deletion of a single point from $R''$ makes it either disconnected or not $T-$convex and thus the minimality condition is satisfied.

If $R''$ is not $T-$convex, we can construct its $T-$hull, say $R'''$. Although $R'''$ is $DL-$convex, it is not necessarily the minimal $DL-$convex set that contains $R'$. We can check the condition of minimality by drawing all the possible digital lines, and computing $T-$hull for each case to see when the $T-$hull is minimal. To limit the computational cost, we can stop at $R'''$ and call it approximate digital hull of $R'$. For a given digital line generation method, and for a fixed method of deciding which points of $R'$ should be joined by digital lines, we get a unique $R'''$.

It should be mentioned that the true $DL-$hull of a disconnected region may not be unique. For example, when $R'$ consists of two isolated points its $DL-$hull is any of the many digital line segments that can be drawn between the points. Uniqueness can be imposed only by using a particular line drawing scheme.

For a digital line generation we use the following line digitization scheme:

Let $f$ be the straight line segment between two points $p$ and $q$. A digital line $D_f$ is defined such that whenever $f$ crosses a grid line, the nearest digital point to the crossing (with a specific tie-breaking rule) is a point of $D_f$, and no other digital point is a point of $D_f$. It can be shown that $D_f$ satisfies the chord property if $f$ is a straight line.

The above discussion suggests an algorithm to compute the $DL-$hull of a region $R'$ using the following steps of

Algorithm 1 (for computing digital convex hull, see [1]):

step 1: Construct the Euclidean convex hull $H(R')$ of $R'$. Add the cellular points of $R'$ to obtain the $T-$hull $H_T(R')$.

step 2: Check $8-$connectivity of $H_T(R')$. If it is connected, define $H_{DL}(R') = H_T(R')$ and go to step 4. Else, check whether $H(R')$ is a line. If it is, generate a digital straight line segment between the extreme cellular points of $H(R')$ and call it $H_{DL}(R')$. Go to step 4. Else, continue.

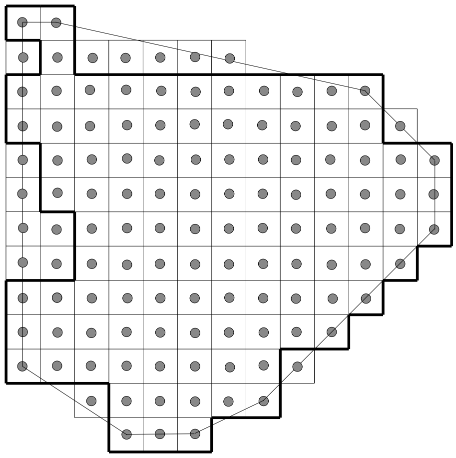

Figure 3

step 3: Find the vertices of $H(R')$ where disconnection occurs. For each such vertex, connect the points that lie between the edges emanating from that vertex by digital straight lines. Let $R''$ be the resulting connected figure. Find the Euclidean convex hull of $R''$. If it does not include any cellular points other than those of $R'$, take $H_{DL}(R') = R''$. Otherwise, find the $T-$hull of $R''$ and call it $H_{DL}(R')$. Continue.

step 4: The $DL-$hull of $R'$ is $H_{DL}(R')$. Return.

Sometimes it is useful to know whether a digital region is convex. In the case of $DL-$convexity, the problem is simpler than finding the $DL-$hull: For a given digital region $R'$ we first test whether it is $8-$connected. If it is not, then $R'$ is not $DL-$convex. Otherwise, we construct the Euclidean convex hull of its cellular points and thus check whether it is $T-$convex. If not, $R'$ is not $DL-$convex. Otherwise, $R'$ is $DL-$convex.

# 3 Digital fuzzy convex hull

Let us now propose a new algorithm for computing digital convex fuzzy hull. Also, we shall prove that the set $\mathcal{A}_{FH}$ obtained by this algorithm is a digital convex fuzzy hull for a given digital fuzzy set $\mathcal{A}$.

Let $\mathcal{A} = \{(x_1, y_1), (x_2, y_2), ...(x_n, y_n)\}$, $n \geq 2$ be support of digital fuzzy set whose membership function is given by $\mu$, which is $DL-$convex. Beside points $A_k(x_k, y_k) \in \mathcal{A}$ we can observe the corresponding points $\tilde{A}_k(x_k, y_k, \mu(A_k))$.

Algorithm 2 (for computing digital convex fuzzy hull):

step 1: Order set $\mathcal{A}$ by non increasing order of membership function $\mu$ for their points and we denote them with $A_1, A_2, ..., A_n$. So, $\mu(A_1) \geq \mu(A_2) \geq ... \geq \mu(A_n)$.

step 2: Observe points $A_1$ and $A_2$. Denote with $p = p(A_1, A_2)$ the line ordered with points $A_1$ and $A_2$.

$$\text{Is it true } \mu(A_1) = \mu(A_2)?$$

If it is, then go to step $3'$.
If it is not, i.e. states $\mu(A_1) > \mu(A_2)$, go to step $3''$.

step $3'$: Observe all points of set $p \cap \mathcal{A}$ whose membership function is equal to $\mu(A_1) = \mu(A_2)$. First we find end points of that set, designate them by $E_1'$ and $E_1''$, and store them in set $\mathcal{E}$ (which we are going to call the set of extreme points), while other points are stored in set $\mathcal{A}_H$ (which we are going to call the set of non extreme points of fuzzy digital convex hull). We remove in $\mathcal{A}_H$ all points $A_p$ of set $\mathcal{A}$ from line segment $E_1'E_1''$ (see Figure 4) for which states $\mu(A_p) < \mu(E_1')$, with improved membership function on $\mu(A_p) = \mu(E_1') = \mu(E_1'')$. Go to step 5.

step $3''$: We move the point $A_1$ from $\mathcal{A}$ and store it in $\mathcal{E}$ with unchanged membership function. Take that $E_1'' \equiv E_1' \equiv A_1$ and $\mu(E_1'') = \mu(E_1') = \mu(A_1)$. Rename indexes of set $\mathcal{A}$ such that $A_1$ be the point with the largest membership function, next point be $A_2$, etc.

step 4: Observe all points of set $p \cap \mathcal{A}$ that have equal membership function as point $A_1$, also consider points $E_1'$ and $E_1''$ ($E_{i-1}'$ and $E_{i-1}''$), versus one of them if they are equal.

From all those points (if they exist) denote with $E_2'$ and $E_2''$ ($E_i'$ and $E_i''$) such points that any other $A_k$ from our set either match with them, or it is between them. We also take that the order $E_2' \preceq E_1' \preceq E_1'' \preceq E_2''$ ($E_i' \preceq E_{i-1}' \preceq E_{i-1}'' \preceq E_i''$), where the relation $\preceq$ is one of relations: $\equiv$ (congruence of points) or $-$ (be between two points) holds.

Store the points $E_2'$ and $E_2''$ ($E_i'$ and $E_i''$) in set $\mathcal{E}$, if they are not already in set $\mathcal{E}$, and if one of them is already extreme point we give it the membership function of the other, i.e. $\mu(E_i') = \mu(E_i'')$. For example $E_3'' \equiv E_2'' \in \mathcal{E}$, we take that $\mu(E_3'') = \mu(E_3')$ (see Figure 4).

Store other points $A_k$ from set $p \cap \mathcal{A}$ in set $\mathcal{A}_H$.

Remove all points $A_q$ from set $\mathcal{A}$ from line segment $E_2'E_2''$ ($E_i'E_i''$) for which states $\mu(A_q) < \mu(E_2')$ ($\mu(A_q) < \mu(E_i')$) in $\mathcal{A}_H$, with improved membership function on $\mu(A_q) = \mu(E_2') = \mu(E_2'')$ ($\mu(A_q) = \mu(E_i') = \mu(E_i'')$).

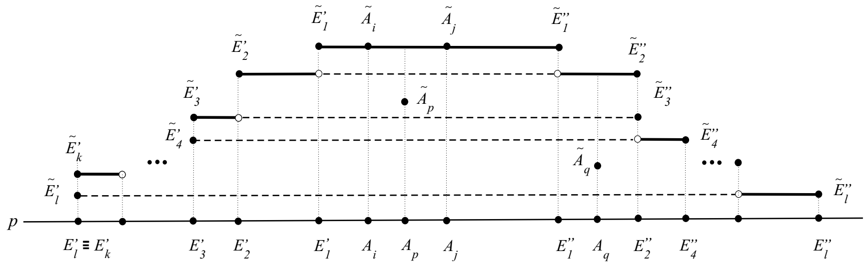


Figure 4

step 5: Rename indexes of set $\mathcal{A}$ such that $A_1$ be the point with the largest membership function, next point be $A_2$, etc.

$$\text{Is it true } A_1 \in p\ ? \tag{1}$$

If it is: go back to step 4, where input is again $A_1$, and output is $E'_i$ and $E''_i$. This iterative cycle repeats until previous condition (1) allows it.
If not:

Two end points ($E'_l$ i $E''_l$ in Figure 4) from $p \cap \mathcal{E}$ denote with $K_1$ i $K_2$. Introduce new set $\mathcal{K}_0 = \{K_1, K_2\}$.

step 6: Join the point $A_1$ with points $K_1$ and $K_2$ from set $\mathcal{K}_0 \subset \mathcal{E}$. Store in $\mathcal{A}_H$, all points of set $\mathcal{A}$ that are in the derived triangle, with improved membership function on $\mu(A_1)$. Store in $\mathcal{E}$ the point $A_1$ with its membership function.

Take that $K_3 \equiv A_1$ and $\mu(K_3) = \mu(A_1)$ and store point $K_3$ in $\mathcal{K}_0$. In that way we get set $\mathcal{K}_1 = \mathcal{K}_0 \cup \{K_3\}$. Mark the derived triangle with $\mathcal{M}(\mathcal{K}_1)$.

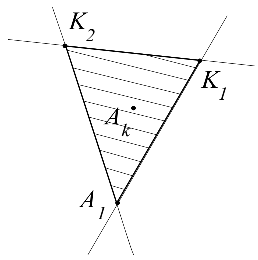


Figure 5

step 7: In the next iteration (and those after if they are needed) we take the first point (from the remaining points, if they exist) from set $\mathcal{A}$. Because of the renaming of indexes of the set $\mathcal{A}$, that point is $A_1$.

If $\mathcal{A} = \emptyset$, then go to step 8.

Store in $\mathcal{E}$ the point $A_1$ with unchanged membership function, and if on the derived line segments or in the interior of the derived polygon in current iteration there are left some of the extreme points with the same membership function $\mu(A_1)$, remove them from set $\mathcal{E}$ in set $\mathcal{A}_H$ with unchanged membership function.

From points of set $\mathcal{K}_2 = \{K_1, K_2, K_3\}$ ($\mathcal{K}_{r-1} = \{K_1, K_2, K_3, ..., K_{i-1}, K_i, K_{i+1}, ..., K_{j-1}, K_j, K_{j+1}, ..., K_m\}$) and point $A_1$ we make another set $\mathcal{K}_4$ ($\mathcal{K}_r$) in which contains those points which make convex hull of set $\mathcal{K}_3 \cup \{A_1\}$

$(\mathcal{K}_{r-1} \cup \{A_1\})$. In Figure 6 a) is $\mathcal{K}_4 = \{K_1, K_2, K_3, A_1\}$, in Figure 6 b) i 6. c) is $\mathcal{K}_4 = \{K_1, K_2, A_1\}$. In derived set $\mathcal{K}_4$ $(\mathcal{K}_r)$ we also rename indexes and denote point $A_1$ with $K_4$ in the case like in Figure 6 a), or with $K_3$ in the cases as in Figures 6 b) and 6 c) (with $K_i$ for some index $i$ in general case).

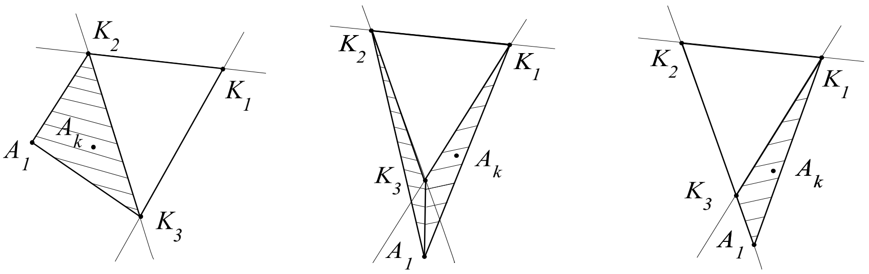


Figure 6 a) Figure 6 b) Figure 6 c)

Denote with $\mathcal{M}(\mathcal{K}_4)$ $(\mathcal{M}(\mathcal{K}_r))$ polygon determined with points from $\mathcal{K}_4$ $(\mathcal{K}_r)$. Store in $\mathcal{A}_H$ all points $A_k$ from $\mathcal{A} \cap (\mathcal{M}(\mathcal{K}_4) \setminus \mathcal{M}(\mathcal{K}_3))$ $(\mathcal{A} \cap (\mathcal{M}(\mathcal{K}_r) \setminus \mathcal{M}(\mathcal{K}_{r-1})))$, with improved membership function from $\mu(A_k)$ on $\mu(A_1)$ if that value is less than $\mu(A_1)$ (because it is possible that $\mu(A_k) < \mu(A_1)$ or $\mu(A_k) = \mu(A_1)$).

If point $A_1$ is colinear with some two neighbor points from $\mathcal{K}_3$ $(\mathcal{K}_{r-1})$, like in Figure 6 c) and the middle point of that three points (in Figure 6 c) is $K_3$) has membership function equal with $\mu(A_1)$ we remove it from $\mathcal{E}$ in $\mathcal{A}_H$ the same as those points from $\mathcal{K}_3$ $(\mathcal{K}_{r-1})$ which belong to interior of polygon $\mathcal{M}(\mathcal{K}_4)$ $(\mathcal{M}(\mathcal{K}_r))$, and have the same membership function as $\mu(A_1)$.

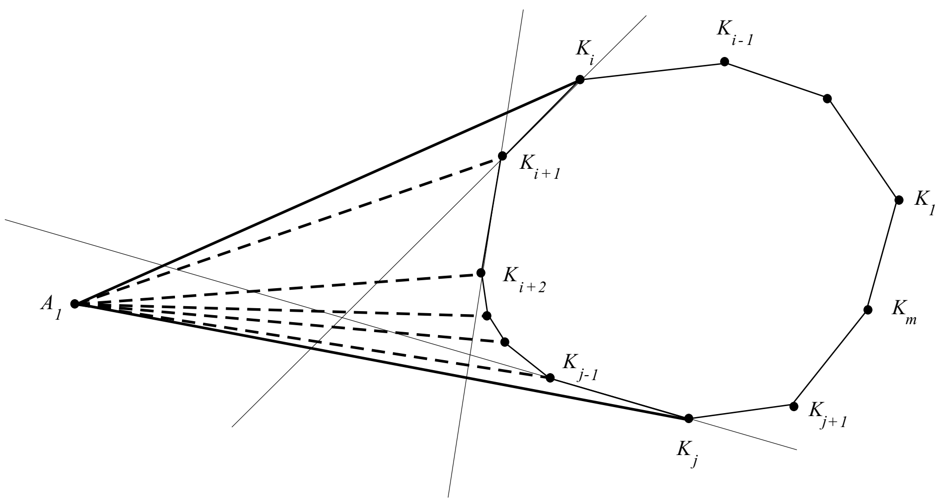


Figure 7

<u>step 8</u>: $\mathcal{A}_{\mathcal{FH}} = \mathcal{E} \bigcup \mathcal{A}_{\mathcal{H}}$. The end.

**Theorem 1** *Let $\mathcal{A}$ be point representation of digital set $H_{DL}(R)$, where $R$ denotes arbitrary digital set, and $H_{DL}(R)$ $DL-$convex hull derived by algorithm 1. Then fuzzy set $\mathcal{A}_{\mathcal{FH}}$ derived by algorithm 2 is digital convex fuzzy hull of set $\mathcal{A}$, i.e. states:*

$$\mathcal{H}_{DLF} = \mathcal{A}_{\mathcal{FH}}.$$

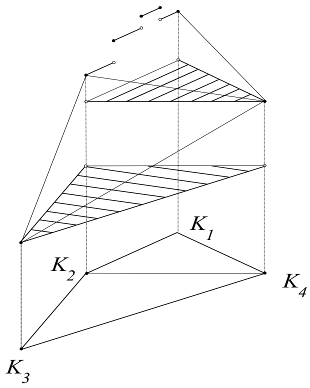


Figure 8

**Proof.** Let $U$ and $V$ be two arbitrary points of set $\mathcal{A}_{\mathcal{FH}}$. Denote with $\mathcal{M}(\mathcal{K}_i)$ a convex polygon constructed over points of set $\mathcal{K}_i$, with the smallest index $i$ that states $U, V \in \mathcal{M}(\mathcal{K}_i)$. Polygon $\mathcal{M}(\mathcal{K}_i)$ is derived from polygon $\mathcal{M}(\mathcal{K}_{i-1})$ by adding point $A_1$ such that convexity is not violated.

Two cases are possible:
1) $U, V \in \mathcal{M}(\mathcal{K}_i) \setminus \mathcal{M}(\mathcal{K}_{i-1})$;
2) $U \in \mathcal{M}(\mathcal{K}_i)$ and $V \in \mathcal{M}(\mathcal{K}_{i-1})$ (or inverse).

1) i) Let $UV \cap \mathcal{M}(\mathcal{K}_{i-1}) = \emptyset$, i.e. all points from line segment $UV$ are in $\mathcal{M}(\mathcal{K}_i)$. If $W \in \mathbb{Z}^2 \cap UV$ digital point (centroid) which is in $UV$, then $\mu(U) = \mu(V) = \mu(W) = \mu(A_1)$, thus inequality

$$\mu(W) \geq \min(\mu(U), \mu(V)) \tag{2}$$

is trivially satisfied.

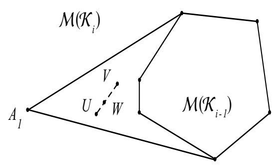


Figure 9 a)

ii) If $UV \cap \mathcal{M}(\mathcal{K}_{i-1}) \neq \emptyset$, then digital point $W$ from line segment $UV$ does not belong to $\mathcal{M}(\mathcal{K}_{i-1})$, i.e. belongs to $\mathcal{M}(\mathcal{K}_i) \setminus \mathcal{M}(\mathcal{K}_{i-1})$. Analogously i) inequality is satisfied.

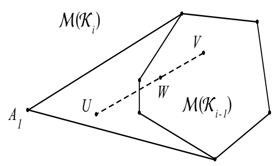


Figure 9 b)

In case that digital point $W$ from line segment $UV$ belongs to $\mathcal{M}(\mathcal{K}_{i-1})$ it has membership function $\mu(W) \geq \mu(A_1)$, so $\mu(W) \geq \mu(A_1) = \mu(U) = \mu(V) = \min(\mu(U), \mu(V))$ states.

2) Let digital point $W$ from line segment $UV$ belongs to $\mathcal{M}(\mathcal{K}_i) \setminus \mathcal{M}(\mathcal{K}_{i-1})$, then $\mu(W) = \mu(A_1) = \mu(U) \leq \mu(V)$ (because by construction for every point $V \in \mathcal{M}(\mathcal{K}_{i-1})$) states $\mu(V) \geq \mu(U)$, where $U \in \mathbb{Z}^2 \setminus \mathcal{M}(\mathcal{K}_{i-1})$), i.e.

$$\mu(W) = \mu(U) = \min(\mu(U), \mu(V)).$$

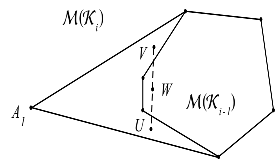


Figure 9 c)

In case that $W \in \mathcal{M}(\mathcal{K}_{i-1})$ we have $\mu(W) \geq \mu(A_1)$ $= \mu(U) = \min(\mu(U), \mu(V))$.

It can happen that both points $U, V$ belong to degenerate polygon $\mathcal{M}(K_0)$, i.e. line segment $K_1K_2$. That means there exist $p, q$ such that $U \in E'_pE''_p$, $V \in E'_qE''_q$. Without loss of generality, let $p \leq q$. Then, by construction $\mu(U) \geq \mu(V)$, since it is $W \in UV$ there exists $r$, $p \leq r \leq q$ such that $W \in E'_rE''_r \subset E'_qE''_q$ which means that $\mu(U) \geq \mu(W) \geq \mu(V)$, i.e. the inequality (2) is valid. $\square$

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