Boolean Frame is Adequate for Treatment of Gradation or Fuzziness Equally as for Two-Valued or Classical Case

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Abstract: Boolean frame are all axioms and theorems of Boolean algebra. Everything which satisfies all Boolean axioms and theorems is inside a Boolean frame and/or is an element of corresponding Boolean algebra. Common for all known approaches for treatment gradation is the fact that either they are not complete (from the logical point of view) or they are not in the Boolean frame. Here is given Interpolative Boolean algebra (IBA) as a consistent MV realization of finite (atomic) Boolean algebra. Since, axioms and lows of Boolean algebra are actually meter of value independent structure of IBA elements, all axioms and all laws of Boolean algebra are preserved in any type of value realization (two-valued, three-valued, …, [0, 1]). To every element of IBA corresponds generalized Boolean polynomial with ability to process all values of primary variables from real unit interval [0, 1].

Keywords: Boolean algebra, Interpolative realization of Boolean algebra – IBA, Structure of Boolean algebra element, Principle of structural functionality, Generalized Boolean polynomial, Generalized product, Excluded midle

1 Introduction

In many real applications the classical two-valued (‘black and white’) realization of Boolean algebra [1] is not adequate. L. Zadeh, after his famous and distinguished contribution in the modern control theory, has ingeniously recognized the necessity of gradation in relations generally (theory of sets – fuzzy sets [2], logic – fuzzy logic [3], relations – fuzzy relations [4]).

Conventional fuzzy approaches rely on the same principle as many-valued (MV) logics [5]. MV-logics are similar to classical logic because they accept *the principle of truth-functionality* [5]. A logic is truth functional if the truth value of a compound sentence depends only on the truth values of the constituent atomic sentences, not on their meaning or structure. The consequences of this direction are in the best way described by Lukasiewicz, the innovator of MV-logic: *‘Logic* (truth functional) *changes from its very foundations if we assume that in addition to truth and falsehood there is also some third logical value or several such values, …'* [6]. Either simple MV-logics based on truth functional principle are not in the Boolean frame and/or they are not realization of Boolean algebra. According to [7] fuzzy logic is based on truth functionality, since: *'This is very common and technically useful assumption'*. A contrary example: 'Our world is a flat plate' was also a very common and technically useful assumption in the Middle Ages!?

One 'argument' for destroying the Boolean frame in treating gradation (MV-case) can be the definition of Boolean axioms of contradiction and excluded middle according to Aristotle: *The same thing cannot at the same time both belong and not belong to the same object and in the same respect* (Contradiction) … *Of any object, one thing must be either asserted or denied* (Excluded middle).

If the goal is mathematics for gradation then it seems 'reasonable' to leave these axioms as inadequate and accept the principle of truth functionality with all consequences or to go to the very source of Boolean algebra idea.

It is interesting that in his seminal paper [1] G. Boole has said: ***'… the symbols of the*** *(logic)* ***calculus do not depend for their interpretation upon the idea of quantity…'*** and only '***in their particular application…, conduct us to the quantitative conditions of inference***'. So, according to G. Boole, the principle of truth functionality is not a fundamental principle (and as a consequence this principle can't be the basis of any generalization).

A very important question is: ***Can fuzziness and/or gradation be realized in a Boolean frame as a realization of Boolean algebra?*** We have obtained a ***positive answer*** to this question as an unexpected result, during solving the problem of fuzzy measure (or capacity) meaning in decision making by theory of capacity [8].

The new approach to treating gradation in logic, theory of sets, relations etc., is based on interpolative realization of finite Boolean algebra (IBA) [9].

IBA has two levels: (a*) Symbolic or qualitative* – is a matter of finite Boolean algebra and (b) *Semantic or valued* – is in a general case, a matter of interpolation.

Structure of any IBA element and/or *principle of structural functionality* [10] are a bridge between two levels and they are the basis of generalization, since they are value independent. The structure of analyzed element determines which *atom* (from a finite set of IBA atoms) is included in it and/or which is not. The principle of structural functionality: *The structure of any IBA combined element can be directly calculated on the basis of structures of its constituents*, using homomrphic mapping IBA on two-element Boolean algebra. Similarly, according to the principle of value (truth in the case of logic) functionality value of combined element of BA can be directly calculated on the basis of values of its components, in a two-valued case. So, the set of structures as a value independent homomrphic image of IBA elements is isomorphism to the set of values of IBA elements in a two-valued case.

In the new approach we keep still all Boolean axioms and identities which follow from these axioms. Now Contradiction is given by the following definition: *the same atomic property cannot at the same time both belong and not belong to the analyzed property…*and Excluded middle***:*** *For any property one atomic property must be either asserted or denied.* From new generalized definitions of contradictions and excluded middle follow Aristotle's definitions as a special classical two-valued case. Value realization of these two axioms in a general case is: (as a consequence of the fact that the sum of the intensity values of atomic property is identical to 1 for any analyzed object) the intensity of realization of analyzed property for any object is the complement of its non-realization (or the realization of its negation as a property).

IBA has a crucially different approach to gradation compared to fuzzy approaches. Technically, as any element of finite Boolean algebra can be represent in a canonical disjunctive form it can also be represented in the form of a corresponding generalized Boolean polynomial. A generalized Boolean polynomial can process values from a real unit interval [0, 1]. So, all laws of Boolean algebra are preserved in the case of gradation.

The main characteristics of the new approach will be illustrated on Interpolative sets as consistent realization of idea of fuzzy sets;

2 Interpolative Boolean Algebra

Interpolative Boolean algebra (IBA) is devoted to the treatment of generalized valued realization [9]. Generalization means that elements of IBA can have more than two values (as in classical case) including general case the hole real unit interval [0, 1]. Interpolative Boolean algebra has a finite number of elements. Finite number constraint should be seen in accordance with the fact that gradation gives the incomparably lager 'power' to any element of an algebra. IBA attains much more in the sense of valued realization effects (adequate descriptions in a general case) with a smaller number of elements (so, with lower complexity). IBA is an atomic algebra (as a consequence of the finite number of elements). Atoms as the simplest elements of algebra have a fundamental role in the new approach.

All laws of Boolean algebra are value indifferent and they are the objects of ***the symbolic level*** of IBA. One of the basic notions from symbolic level is ***the structure*** of IBA element, which determines the atoms of which the analyzed element is built up and/or is not built up. On IBA ***valued level*** the elements of IBA are value realized so that all laws are preserved from symbolic level. The structure is value independent and it is the key of preserving the Boolean laws from a symbolic level on a valued level.

2.1 Symbolic Level

A symbolic or qualitative level is value independent and as a consequence, it is the same for all realizations on valued level: classical (two-valued), three-valued,… and a generalized MV-case. A symbolic or qualitative level is independent of the rank, or type, of the analyzed relation on value level: null-ary (in logic), unary (in the theory of sets), binary (in the graph theory, preference structures etc.) until general n-ary relations. On a symbolic or qualitative level: the main notion is a *finite set of elements with corresponding Boolean operators – atomic Boolean algebra*. This set is generated by a set of *primary elements – context* or *set generator* of analyzed Boolean algebra. No primary element can be realized as a Boolean function of the remaining primary elements. An *order relation* on this level is based only on the operator of *inclusion*. The *atomic element* of Boolean algebra is the simplest in the sense that it doesn't include in itself any other element except itself and a trivial zero constant. Meet (conjunction, intersection) of any two atomic elements is equal to a zero constant. Any element from analyzed Boolean algebra can be represented by a disjunctive canonical form: join (disjunction, union) of relevant atoms. The *structure* of analyzed element determines which atom is relevant (or included in it) and/or which is not relevant (or not included in it). The *structure function* of analyzed element is actually the characteristic function of the subset of its relevant atoms from the set of all atoms. Calculus of structure is based on *the principle of structural functionality*. This value independent principle is formally similar to the principle of 'truth functionality', but fundamentally different since truth functionality is value dependent (the matter of valued level) and is actually valid only for two-valued case. The principle of structural functionality as value independent is the fundamental characteristic. The principle of *truth functionality* on a value level is only isomorphism of the principle of structural functionality and valid only for a classical (two valued) case.

The axioms and identities of IBA are the same as in Boolean algebra, given in paragraph 2. The axioms and identities of IBA are qualitatively (independently of IBA realization nature) and value irrelevant (independently of the number of possible values in valued realizations).

IBA is on a symbolic level identical to Boolean algebra with a finite number of elements.

$$\langle BA, \cap, \cup, C \rangle$$

Algebraic structure with: *BA* is a set with a finite number of elements, two binary operators $\cap$ and $\cup$, and one unary operator C, for which all axioms and identities from 2. are valid.

The element of IBA set in a symbolic way represents everything what on a valued level can characterize, image, determine, assert … the analyzed object in a qualitative sense. IBA element on a symbolic level is treated independently of its potential realization both in a qualitative (property, characteristic, relation,…) and quantitative sense (intensity of value realization). So two IBA-s are the same, on this level, if they have same number of elements. IBA elements are mutually different in their complexity and/or structure on which is based the partial order of IBA elements, based on the relation of inclusion.

The Basic operations on IBA elements are the basic Boolean operations.

Primary IBA elements have the property that any of them can't be expressed only on the basis of the remaining primary IBA elements applying Boolean operation and they form a set $\mathbf{\Omega} \subset \mathbf{BA}$ of primary IBA elements. Set $\mathbf{\Omega}$ generates a set of IBA elements, which is denoted by $\mathbf{BA}(\mathbf{\Omega})$ as a consequence. If $n = |\mathbf{\Omega}|$ then $2^{2^n} = |\mathbf{BA}(\mathbf{\Omega})|$.

Atomic IBA element is a total meet of primary elements $a_i \in \mathbf{\Omega}$ and/or their complements so that every element occurs only once as a primary element or as its complement:

$$\alpha(S)(a_1,...,a_n) = \bigcap_{a_i \in \mathbf{\Omega}} \pi_S(a_i),$$

$$\pi_S(a_i) = \begin{cases} a_i, & a_i \in S \\ Ca_i, & a_i \notin S \end{cases}, \quad S \in \mathbf{P}(\mathbf{\Omega}).$$

To any element $S \in \mathbf{P}(\mathbf{\Omega})$ of a power set of a primary set (subset of primary set), there corresponds one atomic element $\alpha(S)(a_1,...,a_n)$. Atomic elements are the simplest elements of IBA because they don't include in themselves anything except their selves and a trivial IBA element '0'. As a consequence intersection of two different atoms is identically equal to 0:

$$\alpha(S_i)(a_1,...,a_n) \cap \alpha(S_j)(a_1,...,a_n) = 0, \; i \neq j.$$

Universe of atomic elements $U_\alpha(\mathbf{\Omega})$ is a set of all atomic elements (generated by primary elements).

$$U_\alpha(\mathbf{\Omega}) = \{\alpha(S)(a_1,...,a_n) \,|\, S \in \mathbf{P}(\mathbf{\Omega}), \; a_i \in \mathbf{\Omega}\}.$$

Based on the definition, a cardinal number of the universe of atomic elements is $2^{|\mathbf{\Omega}|}$ in the case when $|\mathbf{\Omega}|$ is the number of primary elements.

Structure of any IBA element $\varphi \in \mathbf{BA}(\Omega)$ is a set $\sigma'(\varphi)$ of atomic elements contained in analyzed IBA element φ. To every element of $\sigma'(\varphi)$ corresponds an element of power set $\sigma(\varphi) \in \mathbf{P}(\mathbf{P}(\Omega))$.

Characteristic function of structure $\chi_{\sigma(\varphi)}$ or shortly **structural function** of any IBA element $\varphi \in \mathbf{BA}(\Omega)$ is given by the following expression:

$$\chi_{\sigma(\varphi)}(S) = \begin{cases} 1, & S \in \sigma(\varphi) \\ 0, & S \notin \sigma(\varphi) \end{cases}; \ S \in \mathbf{P}(\Omega),$$

where: Ω is a set of IBA primary elements and $\mathbf{P}(\Omega)$ is a power set of Ω.

Structural function $\chi_{\sigma(\varphi)}$ of IBA element φ determines the inclusion (value 1) and/or non inclusion (value 0) of analyzed atomic element in it. It is clear that the following equality holds:

$$\chi_{\sigma(\varphi)}(S) = \left\| \varphi\left(\chi_S(a_1), \ldots, \chi_S(a_n)\right) \right\|$$

where: $\left\| \varphi\left(\chi_S(a_1), \ldots, \chi_S(a_n)\right) \right\| \in \{0,1\}$ is value of $\varphi \in \mathbf{BA}(\Omega)$ for

$$\chi_S(a_i) = \begin{cases} 1, & a_i \in S \\ 0, & a_i \notin S \end{cases}; \ S \in \mathbf{P}(\Omega), \ a_i \in \Omega.$$

***Comment*:** It is interesting that J. Boole in seminal paper ‘Calculus of logic’ used an expression analogous to expression $\left\| \varphi\left(\chi_S(a_1), \ldots, \chi_S(a_n)\right) \right\|$ and called it ‘mod’.

The value of structural function is determined on a symbolic level and of course it doesn’t depend on the values of IBA elements.

The characteristic elements of IBA are ‘0’ and ‘1’.

IBA element ‘0’ is trivial in the sense that it doesn’t include in itself any atomic element.

$$\chi_{\sigma(0)}(S) = 0, \quad S \in \mathbf{P}(\Omega).$$

IBA element ‘1’ is the most complex in the sense that its structure is equal to the universe of atomic elements.

$$\chi_{\sigma(1)}(S) = 1, \quad S \in \mathbf{P}(\Omega).$$

So, **'1'** includes in themselves (itself) all atomic elements from Ω and as a consequence it includes any IBA element too.

The structural function is ***homomrphic mapping*** of IBA elements:

$$\chi_{\sigma(\varphi\cap\psi)}(S) = \chi_{\sigma(\varphi)}(S) \wedge \chi_{\sigma(\psi)}(S),$$
$$\chi_{\sigma(\varphi\cup\psi)}(S) = \chi_{\sigma(\varphi)}(S) \vee \chi_{\sigma(\psi)}(S),$$

$$\chi_{\sigma(C\varphi)}(S) = \neg\chi_{\sigma(\varphi)}(S),$$
$$\forall S \in \mathbf{P}(\Omega).$$

Since the structure is a result of homomrphic mapping of IBA elements, ***all axioms and identities of Boolean algebra are valid for the structure*** of IBA elements.

Any IBA element $\varphi \in \mathrm{BA}(\Omega)$ can be expressed in ***a disjunctive canonical form*** as a union of relevant IBA atomic elements, determined by its structure:

$$\varphi(a_1,...,a_n) = \bigcup_{S\in\mathbf{P}(\Omega)} \chi_{\sigma(\varphi)}(S)\alpha(S)(a_1,...,a_n)$$

where: Ω is a set of primary IBA elements and $\mathbf{P}(\Omega)$ a power set of Ω .

Structure of atomic element $\alpha(S)(a_1,...,a_n)$ has only one element; it contains only itself, and is given by the following expression:

$$\chi_{\sigma(\alpha(S))}(SS) = \begin{cases} 1, & S = SS \\ 0, & S \neq SS \end{cases}; \; S,\, SS \in \mathbf{P}(\Omega).$$

Structure of primary element $a_i \in \Omega$ contains all atomic elements in which the analyzed primary element figures affirmatively (not as a complement) and is given by the following expression:

$$\chi_{\sigma(a_i)}(S) = \begin{cases} 1, & a_i \in S \\ 0, & a_i \notin S \end{cases}; \; S \in \mathbf{P}(\Omega).$$

Principle of structural functionality: Structure of any IBA elements can be directly calculated on the basis of structures of its components. This principle is a fundamental principle contrary to the famous truth functional principle.

The structural table of IBA elements represents the dependence in the sense of inclusion of any IBA element from IBA atomic elements.

The truth table is a matter of values, so it is even not possible on a symbolic level. The structural table is value independent and as a consequence the same for all value realizations. So, the structural table is a fundamental notion contrary to the truth table which is correct from the point of view of BA, only for a two-valued case. ***The structure, as value indifferent, preserves all laws from a symbolic level on a valued level.***

On a valued level the IBA is treated from the aspect of its value realization.

2.2 Value Level

On a valued level the IBA is value realized or, for short, realized. To elements of Boolean algebra from a symbolic level on value level can correspond: null-ary relation (truth), unary relations (properties), binary relations and n-ary relations, etc. On a value level a result from a symbolic level is concretized in the sense of value. In a classical case there are only two-valued but in a more general case there are three and more values up to the case of all values from a real unit interval [0, 1]. An element from a symbolic level on a value level preserves all its characteristics (described by Boolean axioms and laws) throughout corresponding values. For example, to the order, which is determined by inclusion on a symbolic level, corresponds the order on the valued level, determined by relation 'less or equal'. Any element from a symbolic level has its value realization on elements of analyzed universe. The value of analyzed element from a symbolic level on any element of universe is obtained by superposition of value realizations of its relevant atomic elements for this element of universe. The value of atomic element for the analyzed element of universe is a function – *Generalized Boolean polynomial*, of the values of primary elements realization for this element and a chosen operator of *generalized product*. The value realization of atomic elements from a symbolic level for any element of universe is non-negative and their sum is equal to 1. All tautologies and contradictions from a symbolic level are tautologies and contradictions, respectively, on the valued level.

2.2.1 Basic Notions of Valued Level

A value realization of any IBA element $\varphi \in \mathrm{BA}(\Omega)$ is the *value* or *intensity* $\varphi^v(x)$ of owning a property which this element represents from the analyzed object x.

The universe of IBA realization is a finite or infinite set X of members on which IBA elements are realized on members (as object) of X themselves (unary relation – property)

$$\varphi^v : X \to [0, 1],\ \varphi \in \mathrm{BA}(\Omega)\ ,$$

or on their ordered n-tuples (nary relation):

$$\varphi^v : X^n \to [0, 1],\ \varphi \in \mathrm{BA}(\Omega),\ n \geq 2\ .$$

In the case of null-ary relation (propositional calculus for example) an object is the IBA element itself.

Generalized value realization means that any IBA element φ on a valued level obtains the value – intensity of realization φ^v. Intensity or gradation here means that the set of possible values can have more than two members including the most general case when the set of possible values is a real unit interval [0, 1].

Gradation in the value realization of IBA elements offers a much more efficient description (determination, characterization, specification, etc.) of analyzed objects. With a small number of properties (IBA elements) equipped with gradation on a valued level one can do much more than with a large number of two-valued properties.

The possibility that with fewer properties (IBA elements) one can do more (in the sense of expressivity) is the basic motive for introduction and treatment of MV in Boolean algebra and/or for development of IBA. So, gradation is the basis for balance between simplicity and efficiency in real problems.

A disjunctive canonical form for any IBA element $\varphi \in \mathbf{BA}(\boldsymbol{\Omega})$, (from a symbolic level)

$$\varphi(a_1,...,a_n) = \bigcup_{S \in \mathbf{P}(\boldsymbol{\Omega})} \chi_{\sigma(\varphi)}(S)\alpha(S)(a_1,...,a_n)$$

has its value interpretation: *Generalized Boolean polynomial.* Boolean polynomials are defined in [1] by J. Boole.

Generalized Boolean Polynomial

The value or intensity φ^v of any IBA element $\varphi \in \mathbf{BA}(\boldsymbol{\Omega})$ on the analyzed object is equal to the sum of values of relevant atomic IBA elements contained in this element for the analyzed object:

$$\varphi^{\otimes}(a_1,...,a_n) = \sum_{S \in \mathbf{P}(\boldsymbol{\Omega})} \chi_{\sigma(\varphi)}(S)\alpha^{\otimes}(S)(\|a_1\|,...,\|a_n\|)$$

where: $\alpha^{\otimes}(S)(\|a_1\|,...,\|a_n\|)$ is a value realization of atomic function $\alpha(S)(a_1,...,a_n)$.

The structure function is the same on both levels (symbolic and valued).

Value realization of atomic function $\alpha^{\otimes}(S)(\|a_1\|,...,\|a_n\|)$ determines the intensity of atomic function $\alpha(S)(a_1,...,a_n)$ on the basis of the values of primary IBA elements $\|a_i\| \in [0, 1]$, $a_i \in \boldsymbol{\Omega}$, and in the most general case:

$$\alpha^{\otimes}(S): [0, 1]^{|\boldsymbol{\Omega}|} \to [0, 1], \quad S \in \mathbf{P}(\boldsymbol{\Omega}).$$

The expression for value realization of atomic function is:

$$\alpha^{\otimes}(S)\left(\|a_1\|,\ldots,\|a_n\|\right)=\sum_{C\in\mathbf{P}(\boldsymbol{\Omega}\setminus S)}(-1)^{|C|}\bigotimes_{a_i\in S\cup C}\|a_i\|$$

where: $\otimes$ is a ***generalized product operator***.

Generalized product $\otimes$ is any function $\otimes:[0,1]\times[0,1]\to[0,1]$ that satisfie all four conditions of ***T-norms*** [12]: Commutativity, Associativity, Monotonicity, 1 as identity and plus one additional condition:

Non-negativity condition*:*

$$\sum_{C\in\mathbf{P}(\boldsymbol{\Omega}\setminus S)}(-1)^{|C|}\bigotimes_{a_i\in S\cup C}\|a_i\|\geq 0\ ,$$

where: $\boldsymbol{\Omega}=\{a_1,\ldots,a_n\}$, $S\in\mathbf{P}(\boldsymbol{\Omega})$.

The role of additional axiom ‘non-negativity’ is to ensure non-negative values of valued atomic functions.

Comment: *In the case when the set of primary IBA elements is* $\boldsymbol{\Omega}=\{a,b\}$ *the constraint on non-negativity is satisfied when the operator of generalized product satisfied the following non equality:*

$$max\left(\|a\|+\|b\|-1,\ 0\right)\leq\|a\|\otimes\|b\|\leq min\left(\|a\|,\ \|b\|\right).$$

It is easy to show that in the case of three primary IBA elements the Lukasiewicz T-norm is not a candidate for the operator of generalized product $\otimes$.

Normalized value of valued atomic functions: is a very important characteristic (condition) of IBA value atomic functions. This condition is defined by the following expression:

$$\sum_{S\in\mathbf{P}(\boldsymbol{\Omega})}\alpha^{\otimes}(S)\left(\|a_1\|,\ldots,\|a_n\|\right)=1\ .$$

This condition in the two-valued case is trivial since only one atomic element is realized (equal to 1), and all others are not realized (equal to 0) for any analyzed object. So, in a general case (contrary to a two-valued case) all atoms can be simultaneously realized for the analyzed object or all of them can simultaneously have the values of valued atomic functions greater than zero, but so that their sum is equal to 1.

So, value of any Boolean variable (relation) as a function of primary variables is given by the following expression:

$$\varphi^{\otimes}\left(\|a_1\|,\ldots,\|a_n\|\right)=\sum_{S\in\mathbf{P}(\boldsymbol{\Omega})}\chi_{\sigma(\varphi)}(S)\sum_{C\in\mathbf{P}(\boldsymbol{\Omega}\setminus S)}(-1)^{|C|}\bigotimes_{a_i\in S\cup C}\|a_i\|.$$

Properties of generalized valued realizations:

From the fact that the structure of IBA element ‘1’ includes all IBA atomic elements and on the basis of normalized value of valued atomic functions condition, it follows that: a value realization of IBA element ‘1’ is equal to 1 for any analyzed object.

Because the structure of IBA element ‘0’ does not include any IBA atom it follows that: a value realization of IBA element ‘0’ is equal to 0 for any analyzed object.

Since the laws of ***Commutativity, Associativity, Absorption and Distributivity*** are valid for the structure of IBA elements and on the basis of valued representation of disjunctive canonical form and from the property that two Boolean expressions are equal if they have the same structure, it follows that all these laws (Commutativity, Associativity, Absorption and Distributivity) ***are valid in the case of a generalized valued realization too***.

The laws of ***Excluded middle*** and ***Contradiction*** (***Complements***) are valid for the structure of IBA elements (symbolic level). The structure of the union of any IBA element and its complement is equal to the structure of IBA element ‘1’. The structure of intersection of any IBA element and its complement is equal to the structure of IBA element ‘0’. As a consequence these laws ***are valid for a generalized value realization (valued level) too***.

Comment: Simultaneous appearance of two IBA elements (properties) in the same object in a general case doesn’t mean automatically their intersection. So, one object can have some property (IBA element) with some intensity, simultaneously this object has a property which is a complement of the analyzed property with intensity as a complement value, but the value of their intersection is always equal to 0. So simultaneousness and intersections in a general case are not synonyms (as in a classical case).

Since the laws: ***Idempotency, De Morgan’s laws, Boundedness, Involution*** hold for structures of IBA elements the same laws are valid on the value level for generalized valued realization of IBA elements too. So, since the structures of IBA elements are obtained by homomrphic mapping of IBA elements, it follows that the structures preserve all laws (Boolean laws) on which IBA are based. Since the laws of value realization are direct consequences of laws which are valid for the structures of IBA elements, it follows that all laws from symbolic level (Boolean laws) are preserved on a valued level in all possible valued realizations including the most general for which the set of possible values is a real unit interval [0,1].

3 Example of IBA Application

Theory of Interpolative sets (I-sets) is the realization of fuzzy sets idea in Boolean frame. The main characteristics of application of IBA algebra in the theory of I-sets is illustrated on the example of two I-sets *A* and *B*. The generalized Boolean polynomials of corresponding sets are:

1 $$(A \cap B)^{\otimes}(x) = A(x) \otimes B(x),$$

2 $$\left(A \cap B^c\right)^{\otimes}(x) = A(x) - A(x) \otimes B(x),$$

3 $$\left(A^c \cap B\right)^{\otimes}(x) = B(x) - A(x) \otimes B(x),$$

4 $$\left(A^c \cap B^c\right)^{\otimes}(x) = 1 - A(x) - B(x) + A(x) \otimes B(x),$$

5 $$A(x) = A(x),$$

6 $$B(x) = B(x),$$

7 $$\left((A \cap B) \cup \left(A^c \cap B^c\right)\right)^{\otimes}(x) = 1 - A(x) - B(x) + 2A(x) \otimes B(x),$$

8 $$\left(\left(A \cap B^c\right) \cup \left(A^c \cap B\right)\right)^{\otimes}(x) = A(x) + B(x) - 2A(x) \otimes B(x),$$

9 $$\left(B^c\right)^{\otimes}(x) = 1 - B(x)$$

10 $$\left(A^c\right)^{\otimes}(x) = 1 - A(x)$$

11 $$(A \cup B)^{\otimes}(x) = A(x) + B(x) - A(x) \otimes B(x),$$

12 $$\left(A^c \cup B\right)^{\otimes}(x) = 1 - A(x) + A(x) \otimes B(x),$$

13 $$\left(A \cup B^c\right)^{\otimes}(x) = 1 - B(x) + A(x) \otimes B(x),$$

14 $$\left(A^c \cup B^c\right)^{\otimes}(x) = 1 - A(x) \otimes B(x),$$

15 $$\left(A \cap A^c\right)^{\otimes}(x) = 0, \qquad \left(B \cap B^c\right)^{\otimes}(x) = 0,$$

16 $$\left(A \cup A^c\right)^{\otimes}(x) = 1, \qquad \left(B \cup B^c\right)^{\otimes}(x) = 1.$$

Realization of the all possible set function for the given I-sets *A* and *B*, in the case when the generalized product is given as *min* function is represented in Fig. 3.

It is clear that all properties of the classical set algebra are preserved in the case of I-set algebra and/or by using I-set approach (realization of IBA algebra) one can treat gradation in the Boolean frame, contrary to all fuzzy sets approaches.

Conclusions

Interpolative Boolean algebra (IBA) is a consistent MV realization of finite (atomic) Boolean algebra. In new approach, based on IBA, to every element of any finite Boolean algebra, corresponds generalized Boolean polynomial with ability to process all values of primary variables from real unit interval [0, 1]. Since, axioms and lows of Boolean algebra are actually meter of value independent structure of IBA elements, all axioms and all laws of Boolean algebra are preserved in any type of value realization (two-valued, three-valued, …, [0, 1]). Possibility of approaches based on IBA are illustrated (a) on generalized preference structure – as straightaway generalization of classical result and (b) on interpolative sets as consistent realization of idea of fuzzy sets – all laws of set algebra are preserved in general case, contrary to conventionally fuzzy approaches.

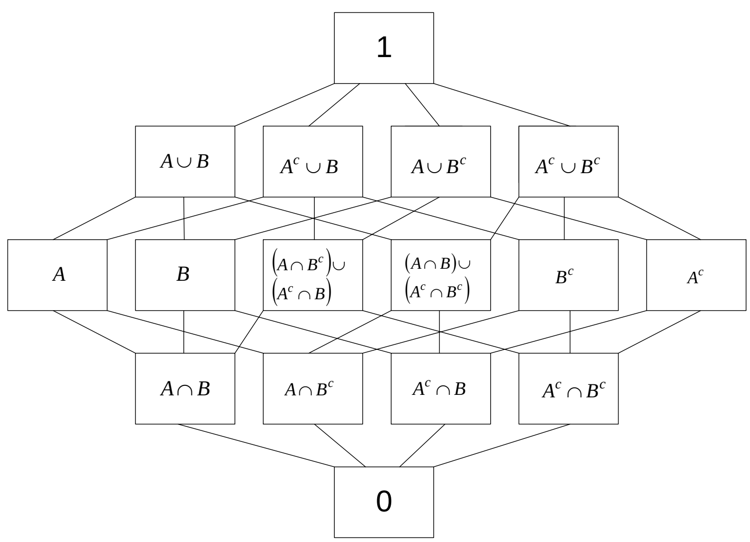

Figure 1

Boolean lattice generated by primary sets $\Omega = \{A, B\}$

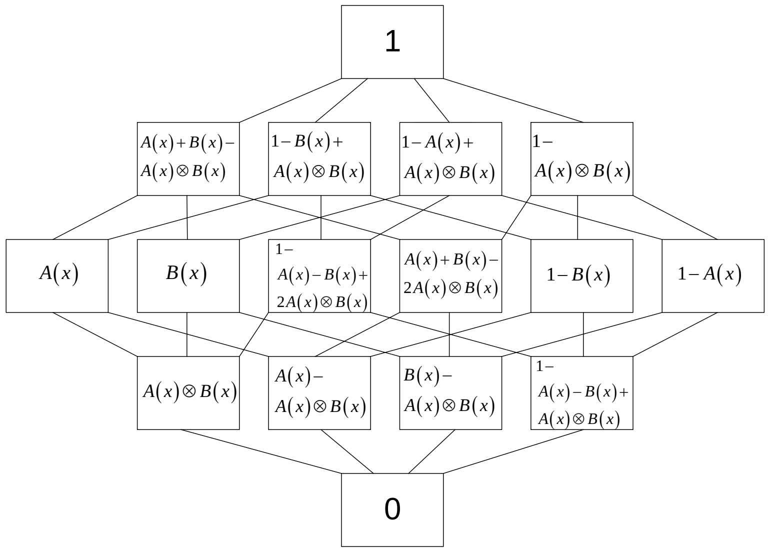

Figure 2

Value realization of lattice generated by $\Omega = \{A,\ B\}$ for $x \in X$

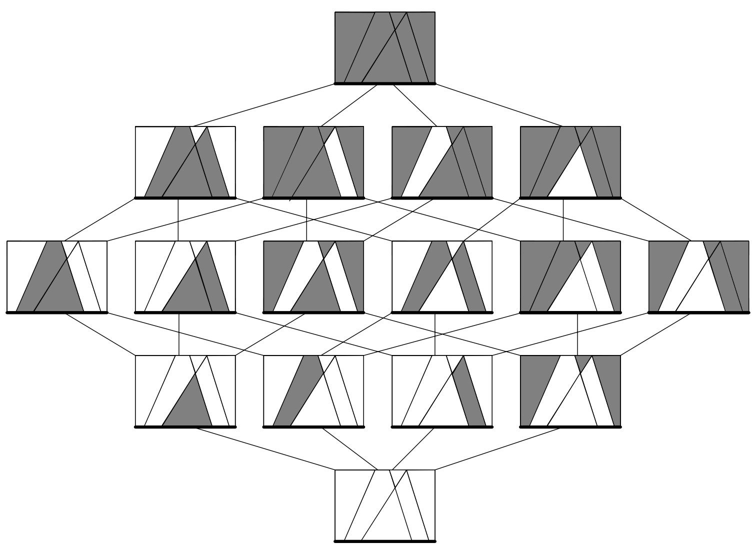

Figure 3

Partially ordered I-sets generated by two I-sets $\Omega = \{A, B\}$

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