A survey of fuzzy interpolation techniques[1]

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1 Introduction

The original idea of reasoning and control within fuzzy rule bases was proposed by Zadeh [1], and was called the Compositional Rule of Inference (CRI) and had the disadvantage of running directly in the *k*-dimensional input space (where *k* is the number of variables) while being able however to describe multi-dimensional membership function distributions of arbitrary shape. Its modified version, the Mamdani-algorithm [2] applied the projections of the antecedents and thus returned to the form of rules $R_i = A_i \to B$, where $A_i = \prod_{i=1}^{k} A_{im}$, meaning that only such A_i could be used that were the cylindrical closures of some membership functions A_{ik}, each being of the type $A_{im} : X_m \to [0,1]$. This method offers much better computational speed.

The CRI, the Mamdani-algorithm and its variants, the Takagi-Sugeno method [3] use the intersections of the observation with the antecedents in order to determine the output of the system. Therefore, if the rule base contains gaps in between the rule antecedents in at least one dimension, these methods are simply not applicable, because in such a case no firing rule(s), and hence, no actual outcome can be determined. Such a rule base is called sparse. This concept is defined precisely in the next section

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2 Interpolation in sparse fuzzy rule bases

2.1 Sparse rule bases

A common feature of fuzzy rule bases is that the antecedents form a collection of fuzzy sets, where their kernels (either single points or intervals) represent typical values of *x*, and between these kernels the membership functions cover the space, so that for every $x \in X$ there is at least one such antecedent, which is true over *x* at least to a fixed degree $\alpha > 0$. Formally:

Definition 1 If

$$\forall x \in X, i = 1...k, \exists j\ (\mathrm{j} \in \{1...r\}) : A_{ji}(x_i) \geq \alpha > 0$$

then $\{A_{ji}\}(j = 1...r)$ form an α-cover of X_i. If also

$$\forall x \in X, \exists j\ (\mathrm{j} \in \{1...r\}) : A_j(x) \geq \alpha > 0$$

is fulfilled, then $\{A_j\}(\mathrm{j} = 1...\mathrm{r})$ is an α *-cover of X*.

Consequently, if for given α the antecedents form no α-cover, the rule base is α-sparse. If they do not form a cover for any $\alpha > 0$, the rule base is sparse. That is

Definition 2 If

$$\exists x \in X, i = 1...k, \forall j\ (\mathrm{j} \in \{1...r\}) : A_{ji}(x_i) < \alpha \tag{1}$$

for a fixed $\alpha > 0$, then the antecedent system $\{A_j\}(\mathrm{j} = 1...\mathrm{r})$, i.e. the rule base itself is α-sparse. If (1) is true for any $\alpha > 0$ then the system is sparse.

Now, we list some reasons leading to incomplete rule bases. First, reducing the number of the rules in a base and, subsequently, the complexity of the resulting fuzzy system by omitting redundant rules with proper technique can result in incomplete rule base [4]. This research topic, emerged in the early 90s, was the main motivation of the fuzzy interpolation. The use of sparse rule base allows removal of redundant rules with proper techniques even if the resulted rule set contains ``gaps''.

Second, the incomplete knowledge about the modeled system, regardless of the construction of the rule base can result sparse rule bases. Originally, on the basis of Zadeh's concepts, fuzzy systems were constructed from linguistic *IF-THEN* rules provided by a human expert. More recently, learning techniques have increasingly been developed and applied to the construction of fuzzy *IF-THEN* rules from numerical sample data. Both cases of constructing rule bases can result in sparse rule basis. In case of using learning techniques it may happen that the sample data do not sufficiently well represent input parameters which only occur rather infrequently. In the case of rules obtained from human expertise, an incomplete rule base can be the consequence of missing knowledge for certain system configuration.

Third, there also exist motivations starting from dense rule bases which end in incomplete rule bases: by tuning the rules of an originally α-cover type rule base, the rule premises are partially shifted and shrunk and the tuned model can also contain gaps.

Fourth, ``gaps" can be defined between rule bases. Hence, fuzzy interpolation techniques has important role in hierarchically structured systems.

2.2 Fuzzy distances

Instead the intersection operator used in the classical algorithms, a more general notion of the degree of similarity was introduced in [5], based on a general notion of distance.

Definition 3. The family of *α-distances between two convex and normal fuzzy sets* (CNF sets) is

$$\tilde{d}(A_1, A_2) = \{d_{\alpha C}(A_1, A_2), \alpha \in (0,1], C \in \{U, L\}\} =$$
$$= \{\|A_{1\alpha L} - A_{2\alpha L}\|, \|A_{1\alpha U} - A_{2\alpha U}\|, \alpha \in (0,1]\},$$

where subscripts *L* and *U* denoting the minimum and maximum of the respective α-cuts.

For simplicity, the distance belonging to fixed α and *L* or *U* will be denoted by $d_{\alpha C}(A_1, A_2)$. The conditions for the existence of the fuzzy distance set is that both fuzzy sets are CNF, and that they are comparable in the sense of $\prec$.

Using this notion of distance, the fuzzy similarity set can be defined as

$$\tilde{s}(A_1, A_2) = \{s_{\alpha C}(A_1, A_2), \alpha \in (0,1], C \in \{U, L\}\} = \left\{\frac{1}{d_{\alpha C}(A_1, A_2)}\right\},$$

the elements of the similarity degree set being the reciprocals of the elements of the distance set.

2.3 The KH interpolation

The basic idea of the fuzzy rule interpolation is formulated in the *Fundamental Equation of Rule Interpolation* (FERI):

$$D(A^*, A_1) : D(A^*, A_2) = D(B^*, B_1) : D(B^*, B_2)\,.$$

In this equation A^* and B^* denote the observation and the corresponding conclusion, while $R_1 = A_1 \to B_2, R_2 = A_2 \to B_2$ are the rules to be interpolated, such that $A_1 \prec A^* \prec A_2$ and $B_1 \prec B_2$. If *D* denotes the Euclidean distance between two symbols, the solution for B^* results in simple linear interpolation. If $D = \widetilde{d}$ (the fuzzy distance family), linear interpolation between corresponding α-cuts is performed.

A more general form of FERI gives

$$B_\alpha^* = \sum_{i=1}^{r} s_\alpha (A_i, A^*) B_{i\alpha}^*, \alpha \in (0,1],$$

where s_α is some α-cut related similarity degree, e.g., the fuzzy similarity obtained from the reciprocal distances of the α-cuts. This similarity can be considered as an extended "degree of matching", and its value replaces the degrees $A_i(x^*)$ used in the classical fuzzy reasoning algorithms in the fuzzy interpolation techniques.. The first such method was proposed in [6], and known as *KH-interpolation* The conclusion is created by

$$B_{\alpha C}^* = \frac{\sum_{i=1}^{r} \frac{1}{d(A_{\alpha C}^*, A_{i\alpha C})} B_{i\alpha C}}{\sum_{i=1}^{r} \frac{1}{d(A_{\alpha C}^*, A_{i\alpha C})}},$$

where the normalized degree of similarity for fixed α and *C* is the reciprocal distance of the observation from the corresponding antecedent, divided by the sum of all these distances.

The main purpose of fuzzy rule interpolation was the great computational complexity requirement of classical fuzzy reasoning methods \cite{KHsizered}. The rule interpolation is efficient if the shape of the rules is simple, practically piecewise linear, moreover, triangular or trapezoidal, since then the rules, i.e. the fuzzy sets involved in them, can be described with only few characteristic points. (In the latter cases, it is enough to do calculations for $\alpha \in B$, the breakpoint level or characteristic point set of the membership functions, which is four points altogether [6]: $\{0,1;L,U\}$.) It is a natural demand that the method should determine the conclusion based only on a sufficient number of α-cuts, namely, based on the characteristic points (or breakpoint levels) of the involved sets, because otherwise the calculation becomes too ``expensive''. Although it could be expected that the conclusion preserves the linearity of the premises, it is not satisfied in general, i.e., the shape of the conclusion can differ from the shape of the other involved sets. Kóczy and Kovács [7], Kawase and Chen [8], and Shi and Mizumoto [9] examined the condition for preserving linearity for the generated conclusion. The investigations give estimation for the linear deviation error, which turns out to be considerable low for most practical cases. It means that it is sufficient to calculate the conclusion only for characteristic points.

It must be noted, however, that B^* reconstructed directly from the above α-cuts does not always exist, as various abnormalities in the shape of the conclusion might necessitate some transformations, which eventually result in obtaining sub-normal conclusions (cf. [7, 10]). This feature of the KH interpolation stimulated many researcher to improve the original method and to introduce other, conceptually different ones. These approaches are investigated in the next section.

Before turning to the other fuzzy interpolation methods, it should be noted that the linear KH interpolation and its more sophisticated version possesses other advantageous properties such as the mathematical stability, which is equivalent in certain sense with the universal approximation property [11].

3 Alternative fuzzy interpolation methods

The method proposed by Vas, Kalmár and Kóczy [12] decreases the limit of applicability of the method, but does not eliminate it completely. They compute the conclusion based on the distance of the central points and the supports of fuzzy sets.

Conceptually different approaches were proposed by Baranyi, Kóczy and Gedeon in 1996 [13] based on the relation of the fuzzy sets and by Baranyi *et al.* in 1998 [14] based on the semantic and interrelational features of fuzzy sets. They determined the location (central point or most typical point) of the conclusion based on the ratio of the centres of the observation and the antecedents. After all involved sets are rotated by 90º around their centres, and by connecting the corresponding points of antecedents, and consequents, two solids can be formed: one in the input and one in the output dimension. The solids are cut at the centres of the observation and at the determined location of the conclusion, respectively, which results in the set $A^{*'}$ in the input space and in the set $B^{*'}$ in the output space. Then a revision function is used to determine the final conclusion $B^{*'}$

based on the similarity of the observation A^* and the "interpolated" observation $A^{*'}$. These methods have numerous advantages relating to the previous ones as they always give conclusion interpretable as fuzzy set (that is, the abnormal shape of the conclusion is precluded), they can be applied for arbitrary shape fuzzy sets (neither convexity nor normality is prescribed), only the centres of the sets have to be ordered (i.e., some part of the observation can exceed the support of antecedents), further, the versions specialized for piecewise linear fuzzy sets produce piecewise linear fuzzy set as conclusion, hence the shape of the set at hand is preserved. The only problematic point of these methods is that the calculation of the revision function even for the special piecewise linear case needs considerable time, thus one of the most important reasons for inventing fuzzy interpolation technique is violated.

Another fuzzy interpolation technique was proposed in 1996 by Gedeon and Kóczy [15] founded on the preservation of relative fuzziness. This approach can not be applied for certain crisp sets. This method was improved by Kóczy *et al.* in 1997 [16], which is suitable for the above mentioned crisp sets, as well. The authors also showed its immediate connection with the fundamental equation of fuzzy interpolation. These methods are applicable also for CNF sets.

In 1996 Kovács and Kóczy proposed yet another interpolation method based on the approximation of the vague environment of fuzzy rule bases.

Now we will introduced in more detail the modified α-cut based fuzzy interpolation method, also known as MACI method [17].

4 The MACI method

For the sake of simplicity, in this paper only piecewise linear fuzzy sets are considered. (We remark that in [18] a method for arbitrary continuous CNF set is presented.) The method uses the vector representation of fuzzy sets, which assigns to every fuzzy set a vector of its characteristic points. Then fuzzy set *A* is represented by a vector $\underline{a} = [a_{-m}, \ldots, a_0, \ldots, a_n]$ where a_k $(k \in [-m, n])$ are the characteristic points of *A* and a_0 is the reference point of *A* with membership degree one. From this, $\underline{a}_L = [a_{-m}, \ldots, a_0]$, and $\underline{a}_U = [a_0, \ldots, a_n]$ represent the left flank and right flank of *A*, respectively. If *A* is a CNF then, e.g. for the right flank, $a_i \geq a_j$, $i < j \in [0, n]$ should hold with monotone decreasing α levels. When it is not ambiguous we omit subscripts *L* and *U*.

Let us suppose further, that two rules with the observation fulfill the condition concerning the location and the ordering of the involved fuzzy sets. The abnormality is avoided if the characteristic points of the conclusion fulfills the following inequality:

$$b_i^* \geq b_j^* \forall i < j \in [-m, n]. \tag{2}$$

The new method consist of three steps: choosing an appropriate coordinate system for the output space, computing the conclusion (according to the KH interpolation method), and finally this conclusion is transformed back into the original coordinate system. The condition (2) is assured by the choice of an appropriate transformation, which prevents the occurrence of abnormal conclusions. A detailed description of the method can be found in [17]. Using the following notations:

$$\lambda_k = \frac{a_k^* - a_{1k}}{a_{2k} - a_{1k}}$$

and

$${}^{KH}b_k^* = (1 - \lambda_k) b_{1k} + \lambda_k b_{2k}$$

(i.e. the value of the k coordinate is calculated according to the original KH approach), the conclusion of the MACI method is computed by the formulas:

$$b_k^* = {}^{KH}b_k^* + \sum_{i=0}^{k-1} (\lambda_i - \lambda_{i+1})(b_{2i} - b_{1i}), \tag{3}$$

$$b_k^* = {}^{KH}b_k^* + \sum_{i=k+1}^{0} (\lambda_i - \lambda_{i-1})(b_{2i} - b_{1i}), \tag{4}$$

$k \in [0, n]$ for the right flank (3), and $k \in [-m, 0]$, for the left flank (4). For the reference point the two equations give the same output.

Notice that from (3) results that

$$b_k^* - b_{k-1}^* = (1 - \lambda_k)(b_{1k} - b_{1,k-1}) - \lambda_k (b_{2k} - b_{2,k-1}) \tag{5}$$

which applied recursively, for $k-1, \ldots, 2$ leads to

$$b_k^* = {}^{KH}b_0^* + \sum_{i=1}^{k} (1 - \lambda_i)(b_{1i} - b_{1,i-1}) - \lambda_i (b_{2i} - b_{2,i-1})$$

Analogous relations hold for the left flank).

From (5) it is obvious that $b_k^* - b_{k-1}^*$ is positive (it is a linear combination of two positive quantities), and thus the conclusion cannot be abnormal.

It is also very easy to verify that if the observation coincides with one of the antecedents then the conclusion will be exactly the corresponding consequence.

An important step in the algorithm is the assignment of the characteristic values to a certain set of fuzzy sets, whose breakpoint sets are different. Since, in (4) the coefficients λ_k correspond to the input variable, namely, to the *k*th coordinate of the antecedents and the observation, but the calculated value corresponds to the *k*th coordinate of the conclusion, hence, in order to avoid confusion a common breakpoint level set should be determined for both spaces, which is the union of (perhaps different) breakpoint level sets for each variable.

Finally, we remark that multivariable antecedents can be handled analogously as the transformation described in this section affects only the consequent part. Common combined antecedent sets (and observation) can be calculated from the corresponding antecedents (observation) of each variable using Minkowski-type distance, where the weights are identical to 1.

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