The Lattice of Primitive Recursive Clones of Functions Defined on Finite Sets

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Abstract. On the power-set of the set of k-valued logical functions, there is defined a closure operation, called pr-closure. A set of k-valued logical functions is called pr-closed (or pr-clone), if it was a clone, closed under primitive recursion. (Clone is a set of functions containing the projections, and closed under composition of functions.) We prove, that in the case of truth-functions (k=2) there are exactly two pr-clone. In the general case, our main result is to present and prove the basic theorem of pr-completness (similar to the theorem of Rosenberg).

1.Introduction.

Since the seventies and eighties the investigations of clones are a main area of universal algebra and many-valued logics. The inclusion structure of the set of closed classes for two-valued logic was given by Post[P-41] (This is nearly the subclone lattice). All maximal classes in k-valued logics were given for k=3 and k≥3 by Yablonskii and Rosenberg, respectively. [see R-77] According to a result of Yanov and Mučnik [see P-K-79], there are clones infinitely generated (both without and with base) and a continuum of clones in k-valued logics (for k>2). In this monograph, the authors investigated both the maximal and the minimal length of chains of subclone-lattice of k-valued logics, too. The complete lattice of the maximal linear clones in prime-valued logics was given in [B-91] and in [B-D-80].
We defined the pr-clone closure [B-91, B-93] in order to investigate certain subclone lattices. Moreover, it is a usual way in computer science to define functions by primitive recursion and we extended this operation to k-valued logic.

In part 2, after definitions and denotations there are mentioned generalizations and another definition of primitive recursion for finite base set (cyclic recursion).

In part 3 there is given the lattice of pr-clones for truth functions. This lattice is very simple; it is a chain with two elements. Moreover, both of these pr-clones are generated with one element, and each of the truth functions generate one of these pr-clones.

In part 4 there is presented the main result. There are given pr-Sheffer functions with one variable and a Słupecki-kind theorem for pr-clones. We present, that in k-valued logics each function is primitive recursive. As a main result, there are given all of the pr-maximal clones and the basic theorem of pr-completeness.

2. Definitions and notations

Let k>1 fixed, m, n, i be positive integers and l a non-negative integer. Denote $O_K^{(n)}$ the set of n-ary functions over the set K={0,1,…,k-1}. $O_K^{(n)}=\{f:K^n\to K\}$, and $O_K=\cup_{n>0} O_K^{(n)}$. For an arbitrary subset O of the set O_K, let $O^{(n)}=O\cap O_K^{(n)}$. Sometimes there are used the short notations $\tilde{x}=(x_1,x_2,\ldots,x_n)$ and $(\tilde{x},x_{n+1})=(x_1,x_2,\ldots x_n,x_{n+1})$. The mod k addition is denoted by ⊕. The remainder part of this paper, the function and the set O mean an element and a subset of O_K, respectively. Some special functions are named and denoted as follows. The defined concepts are underlayed. The <u>cyclic permutation function s</u> is the unary function such, that $s(x)=x\oplus 1$ for $x\in K$. The <u>n-ary constant functions</u> c_l^n, $l\in K$, and the <u>(n-ary) ith projection function</u> (shortly <u>projection</u>) are defined such that for $\tilde{x}\in K^n$, $c_l^n(\tilde{x})=l$ and $e_i^n(\tilde{x})=x_i$ $(1\le i\le n)$. If n=1, then the upper index is omitted. Therefore, the set of constant functions and projections are C and E, respectively, where $C^{(n)}=\{c_l^n \mid l\in K\}$, $E^{(n)}=\{e_i^n \mid 1\le i\le n\}$.
For a non-empty subset M of K, an n-ary <u>function f is M-preserving</u>, if for $\tilde{x}\in M^n$ holds $f(\tilde{x})\in M$. Denote (O | M) <u>the set of M-preserving functions</u> from O:
$(O \mid M)^{(n)}=\{f \mid f\in O^{(n)}, f(\tilde{x})\in M \text{ for } \tilde{x}\in M^n\}$.
A <u>composition of functions</u> $f\in O_K^{(n)}$ and $g\in O_K^{(m)}$ is the function $g\bullet f\in O_K^{(n+m-1)}$, where
$(g\bullet f)(\tilde{x},x_{n+1},\ldots,x_{n+m-1})=g(f(\tilde{x}),x_{n+1},\ldots,x_{n+m-1})$ for every $x_i\in K$ (i=1,2,…,n+m).
An (n+1)-ary function h is defined by <u>primitive recursion</u> from n-ary function f and (n+2)-ary function g, if h=g$f, where $(g\$f)(\tilde{x},0)=f(\tilde{x})$, and

$$(g\$f)(\tilde{x},s(i))=g(\tilde{x},(g\$f)(\tilde{x},i),i),\ s(i)\in K.$$

Let r_1, j_l unary, r_2, r two-ary, r_3 three-ary and w (k+1)-ary functions given by the following equations. For each $x,y,z,x_1,x_2,\ldots,x_n \in K$ valid

$$r_1(x)=\begin{cases}0, & \text{if } x = 0,\\ x-1, & \text{if } x \neq 0;\end{cases} \qquad j_l(x)=\begin{cases}k-1, & \text{if } x = l,\\ 0, & \text{if } x \neq l;\end{cases}$$

$$r_2(x,y)=\begin{cases}x, & \text{if } y = 0,\\ y-1, & \text{if } y \neq 0;\end{cases} \qquad r_3(x,y,z)=\begin{cases}x, & \text{if } z = 0,\\ y, & \text{if } z \neq 0;\end{cases}$$

$$r(x,y)=\begin{cases}x, & \text{if } y = 0,\\ y, & \text{if } y \neq 0 \text{ and } x = 0,\\ x-1, & \text{if } y \neq 0 \text{ and } x \neq 0;\end{cases} \qquad w(x_1,x_2,\ldots,x_k,l)=x_{l+1},\ l\in K.$$

Let **[ ]** be a closure operation over the power-set of the set O_K (that is, for all subset O' of O_K: $O' \subseteq [O']$; $O'' \subseteq O'$ implies $[O''] \subseteq [O']$ and $[[O']]=[O']$), and let O be a <u>closed class</u>: $[O]=O\subseteq O_K$.
A subset O' of O is **[ ]**-<u>complete in the class O</u>, if **[**O'**]**=O.
The **[ ]**-closed class O'' is <u>maximal in the class O</u>, if $O''\subset O'\subseteq O$ implies **[**O'**]**=O.
The B subset of the class O is a <u>base of the class O</u>, if B is **[ ]**-complete in O and for every proper subset B' of B, **[**B'**]**≠O (that is, B' is not complete in O).
The **[ ]**-closed class is including the set of projections named <u>**[ ]**-clone</u>.
We are interesting in two types of clones. One of them is the <u>compositional closed</u> class. A subset O of the set O_K is said to be a <u>compositional clone</u>, if it contains the projections and it is closed under the functional composition: $f\in O^{(n)}$ and $g\in O^{(m)}$ implies $g\bullet f\in O^{(n+m-1)}$.
A compositional clone O is said to be a <u>pr-clone (primitive recursive clone)</u>, if it is closed under the primitive recursion: $f\in O^{(n)}$ and $g\in O^{(n+2)}$ implies $(g\$f)\in O^{(n+1)}$.
Denote $[O]_{cl}$ and $[O]_{pr}$ the compositional clone closure and the pr-clone closure of the set O.

Remarks

1. Several closure operations yield several kind of clones, therefore different theories .
2. The <u>cyclic primitive recursion</u> yields another, so called <u>cpr-clones</u>.

An (n+1)-ary function h is defined by <u>cyclic primitive recursion</u> from n-ary function f and (n+2)-ary function g, if h=g¢f, where $(g¢f)(\tilde{x},s(i))=g(\tilde{x},(g¢f)(\tilde{x},i),i)$, $i\in K$.

3.The lattice of pr-clones for truth functions (k=2)

Denote $O_{\{0;1\}}$ and O_2 the set of truth functions (logical functions). To see the effectivity of the pr-closure, we present the Post's lattice (of clones) as Fig.1 and the lattice of pr-closed clones (pr-clones) as Fig.2. The latter one is based on the following theorem.

<u>**Theorem 1**</u>. 1. The set $(O_2|\{0\})$ is pr-clone and pr-maximal (in the class O_2).
2. Each {0}-preserving truth function is pr-complete in the class $(O_2|\{0\})$.
3. If a truth function is not {0}-preserving, then it is pr-complete (in O_2).
Proof.

1. It is known [P-41], that $(O_2|\{0\})$ is a maximal compositional clone (in O_2). This clone is closed under primitive recursion, because of $(g\$f)(\tilde{0},0)=f(\tilde{0})=0$, for $f(O_2|\{0\})$. Therefore this is a pr-clone and so it is pr-maximal.
2. Denote ⊕ and . the mod 2 addition and mod 2 multiplication, resp.
   Let g_2, g_3 truth functions be defined by primitive recursion as follows: $g_2(x,0)=e_1(x)$, $g_2(x,1)=e_3^3(x,e_1(x),0)$ and $g_3(x,y,0)=e_1^2(x,y)$, $g_3(x,y,1)=e_2^4(x,y,y,0)$, that is $g_2(x,y)=x\oplus xy$ and $g_3(x,y,z)=x\oplus xz\oplus yz$. According to the definition these functions are elements of every pr-clone. It can be seen, that $g_2(x,x)=c_0(x)$, $g_3(x,y,x)=xy$ and $g_3(x,g_2(y,x),y)=x\oplus y$. It is known [P-41], that the set {x+y,x.y} is a compositional base of the clone $(O_2 \mid \{0\})$. Due to the statement 1 of the theorem, the set E is pr-complete in the set $(O_2 \mid \{0\})$, so the sets $E\cup\{g\}$, $g\in(O_2 \mid \{0\})$ are pr-complete, too.
3. Due to the proof of part 2, each of the pr-clones contains the pr-clone $(O_2 \mid \{0\})$, as a subset.
   As this pr-clone is pr-maximal (statement 1), the statement 3 is true. □

Corollary

There are two pr-clones for truth functions; they are the classes O_2 and (O_2 | {0}). □

The lattices of compositional clones and of pr-clones for truth functions are presented by Fig. 1 and Fig. 2, resp.

4. The maximal pr-clones and the theorem of pr-completeness (case k>2)

It can be seen (part 3), that for truth functions the pr-closure is more effective than the compositional closure: the lattice of the compositional clones is countable [P-41], but the lattice of the pr-clones has only two elements. It is known (and easy to see), that arbitrary $f\in O_K^{(n+1)}$ has a representation of the form:

$f(\tilde{x},z)=\max(\min(j_l(z), f(\tilde{x},l))|l\in K)$.

However, a similar representation is the following (standard representation).

Statement. Every (n+1)-variable function f has a representation of the form:

$f(\tilde{x},z)=w(f(\tilde{x},0),f(\tilde{x},1),\ldots,f(\tilde{x},k-1),z)$. □

Before presenting our results about pr-Sheffer functions and pr-completeness, we prove a lemma.

Lemma 2. 1. Every pr-clone contains the functions c_0, r_1, r_2, r_3, r and w.

2. $c_0\in[r_1]_{cl}$, $r_1\in[r_2]_{cl}$, r, $w\in[r_2,r_3]_{cl}$.

Proof. Let the functions f_1 and f_2 be defined by primitive recursion (over the set E) as follows:

$f_j(x,y,0)=e_1^2(x,y)$, $f_j(x,y,s(i))=e_{2j}^4(x,y,f_j(x,y,i),i)$, i=0,1,2,…,k-2 (or a more compact way: $f_j=e_{2j}^4\$e_1^2$). We obtain: $r_2(x,y)=f_2(x,y,y)$; $r_3=f_1$; $r_1(x)=r_2(x,x)$; $r(x,y)=r_3(x,r_2(y,x),y)$. From the following recursive composition $u_1=r_1$, $u_{m+1}=u_1\bullet u_m$ (m≥1) yields the function c_0: $c_0=u_{k-1}$. The recursive composition $w_2=r_3$, $w_{i+1}(x_1,x_2,\ldots,x_i,y,z)=w_2(x_1,w_i(x_2,\ldots,x_i,y,r_1(z))$, i=2,3,… (not a primitive recursion!) yields the function w; can be seen, that $w(x_1,x_2,\ldots,x_k,z)=w_{k+1}(x_1,x_2,\ldots x_k,z,z)$.

From this construction the statement 2 is obtained. □

Also, the following theorem shows the effectivity of the pr-closure. We mention, that in case of compositional closure the Sheffer functions have at least two variables.

Theorem 3. The functions c_{k-1} and s are pr-Sheffer functions (that is, both of them are pr-complete in O_K).

Proof. Using Lemma 2, every constant function can be generated as follows (similar to the proof of Lemma 2):

$u_1=r_1$, $u_{m+1}=u_1\bullet u_m$, $c_i=u_{k-i-1}\bullet c_{k-1}$;

$v_1=s$, $v_{m+1}=s\bullet v_m$, $c_i=v_i\bullet c_0$, i=0,1,…,k-1 ($u_0=v_0=e_1$).

The statement is obtained by induction on the number n of variables, because of the standard representation

$f(\tilde{x},z)=w(f(\tilde{x},0),f(\tilde{x},1),\ldots,f(\tilde{x},k-1),z)$. □

Corollary. The class C of constant functions is pr-complete. □

Therefore, considered the set $C\cup\{s\}$ as the set of elementary functions in O_K, we obtain that each of the element of the set O_K is primitive recursive function.

According to the definition of the primitive recursion, $(g\$f)(\tilde{0},0)=f(\tilde{0})$, so the 0-preserving class (O_K | {0}) is a pr-clone for k>2, too. Since this class is maximal in O_K as a compositional clone, yields:

Statement. The class of 0-preserving functions is pr-maximal in the class O_K. □

It is known, that every class (O_K | M) for (non-empty) M subsets of K is a clone [R-77]. However, these are not pr-clones, e.g. $r_1\in[O_K |\{1\}]_{pr}$, but $r_1(1)=0$ by Lemma 2. Let $K_i=\{0,1,\ldots,i\}$ for $0\le i\le k-1$ ($K_{k-1}=K$). In the next theorem the subset-preserving clones are characterized in point of wiev pr-completeness and pr-maximality.

Theorem 4. Let M be a non-empty, proper subset of the set K. Then the clone (O_K | M) is

a) maximal pr-clone (in O_K) if $M=K_i$, i=0,1,…,k-2;
b) pr-complete class (in O_K) in other cases.

Proof. For $b\in K\setminus M$ there exists a function $g\in(O_K^{(1)} | M)$ such that g(b)=k-1.

a) According to the definition of the primitive recursion, the classes (O_K | K_i), i=0,1,…,k-2 are pr-clones, however they are not pr-completes, since the function c_{k-1} is not element of them.

For any function $f\in O_K^{(n)}\setminus(O_K | M)$ there exists $\tilde{a}=(a_1,\ldots,a_n)\in M^n$, such that $f(\tilde{a})=b\in K\setminus M$. So $c_{k-1}=g(f(c_{a_1},\ldots,c_{a_n}))\in[(O_K | M)\cup\{f\}]_{pr}$ since $c_{a_1},\ldots,c_{a_n}$, $g\in(O_K | M)$. Therefore, the class (O_K | M) is pr-maximal clone, due to Theorem3.

b) Suppose,that the set M is different from the sets $K_0, K_1,\ldots,K_{k-1}$, so there exists $b+1\in M$, such that $b\in K\setminus M$. Since $c_{b+1}\in(O_K | M)$, so $r_1(c_{b+1})=c_b\in[(O_K | M)]_{pr}$, therefore $g(c_b)=c_{k-1}\in[(O_K | M)]_{pr}$. Due to Theorem 3, the class (O_K | M) is pr-complete. □

The next theorem is similar to the completeness theorem of Rosenberg [R-77].

Theorem 5. (Basic theorem of pr-completeness.) For every integers $k\geq 2$, the set $O\subseteq O_K$ is pr-complete (in O_K) if and only if none of the sets $O\setminus (O_K |K_i)$, $0\leq i\leq k-2$ are empty.

Proof. Clearly, the condition is necessary, since the sets (O_K $|K_i$)are pr-maximal clones for $i<k-1$, by Theorem 4. Sufficient. Suppose, that none of the sets $O\setminus (O_K|K_i)$, $0\leq i\leq k-2$ are empty. So there exists a function $g\in O$, for which $g(c_0,\ldots,c_0)=c_j\neq c_0$, therefore $c_j\in[O]_{pr}$, by Lemma 2. If $j<k-1$, then on a similar way we have get the set $\{g(c_{i_1},\ldots, c_{i_n})|g\in O, \{i_1,\ldots,i_n\}\subseteq K_j\}$ having an element c_l, $l>j$. This way we obtain the function c_{k-1}, at most in steps k-1. Therefore the class O is pr-complete (Lemma 2 and Theorem 3 is used). □

Figure 1

Figure 2

Irodalom

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