The Lattice of Primitive Recursive Clones of Functions Defined on Finite Sets

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Abstract. On the power-set of the set of k-valued logical functions, there is defined a closure operation, called pr-closure. A set of k-valued logical functions is called pr-closed (or pr-clone), if it is a clone, closed under primitive recursion. (Clone is a set of functions containing the projections, and closed under composition of functions.) We prove, that in the case of truth-functions (k=2) there are exactly two pr-clone. In the general case, our main result is to present and prove the basic theorem of pr-completeness (similar to the theorem of Rosenberg).

1. Introduction
Since the seventies and eighties the investigations of clones are a main area of universal algebra and many-valued logics. The inclusion structure of the set of closed classes for two-valued logic was given by Post\cite{P-41} (This is nearly the subclone lattice). All maximal classes in k-valued logics were given for k=3 and k≥3 by Yablonskii and Rosenberg, respectively. \cite{R-77} According to a result of Yanov and Mučnik \cite{P-K-79}, there are clones infinitely generated (both without and with base) and a continuum of clones in k-valued logics (for k>2). In this monograph, the authors investigated both the maximal and the minimal length of chains of subclone-lattice of k-valued functions, too. The complete lattice of the maximal linear clones in prime-valued logics was given in \cite{B-D-80}.

We defined the pr-clone closure \cite{B-D-80} in order to investigate certain subclone lattices. Moreover, it is a usual way in computer science to define functions by primitive recursion and we extended this operation to k-valued logic.

In part 2, after definitions and denotations there are mentioned generalizations and another definition of primitive recursion for finite base set (cyclic recursion).

In part 3 there is given the lattice of pr-clones for truth functions. This lattice is very simple; it is a chain with two elements. Moreover, both of these pr-clones are generated with one element, and each of the truth functions generate one of these pr-clones.

In part 4 there is presented the main result. There are given pr-Sheffer functions with one variable and a Slupecki-kind theorem for pr-clones. We present, that in k-valued logics each function is primitive recursive. As a main result, there are given all of the pr-maximal clones and the basic theorem of pr-completeness.

2. Definitions and notations
Let k>1 fixed, m, n, i be positive integers and l a non-negative integer. Denote \(O_k^n\) the set of n-ary functions over the set \(K=\{0,1,\ldots,k-1\}\). \(O_k^n=\{f:K^n\to K\}\), and \(O_k=\cup_{n=0}^\infty O_k^n\). For an arbitrary subset \(O\) of the set \(O_k\), let \(O^{(O)}=O\cap O_k^n\). Sometimes there are used the short notations \(\bar{x}=(x_1,x_2,\ldots,x_n)\) and \((\bar{x},x_{m+1})=(x_1,x_2,\ldots,x_n,x_{m+1})\). The \(\mod k\) addition is denoted by \(\oplus\). The remainder part of this paper, the function and the set \(O\) mean an element and a subset of \(O_k\), respectively. Some special functions are named and denoted as follows. The defined concepts are underlayed. The cyclic permutation function \(\bar{x}\) is the unary function such, that \(s(x)=x\oplus 1\) for \(x\in K\).

The n-ary constant functions \(c^n_i\), 1\(\in\)K, and the \((n\text{-ary})\ \bar{\pi}\) projection function (shortly projection) are defined such that for \(\bar{x}\in K^n\), \(c^n_i\ (\bar{x})=1\) and \(e^n_i\ (\bar{x})=x_i\ (1\leq i\leq n)\). If \(n=1\), then the upper index is omitted. Therefore, the set of constant functions and projections are \(C\) and \(E\), respectively, where \(C^{(C)}=\{c^n_i\mid 1\in K\}\) and \(E^{(E)}=\{e^n_i\mid 1\leq i\leq n\}\).

For a non-empty subset \(M\) of \(K\), an n-ary function \(f\) is M-preserving, if for \(\bar{x}\in M^n\) holds \(f(\bar{x})\in M\). Denote \((O\mid M)\) the set of M-preserving functions from \(O\):

\[\{(O\mid M)\}^{(O\mid M)}=\{f\mid f\in O^n, f(\bar{x})\in M\ \text{for} \ x\in M^n\}.\]

A composition of functions \(f\in O_k^n\) and \(g\in O_k^m\) is the function \(g\cdot f\in O_k^{n+m}\), where

\[(g\cdot f)(x_{i+1},\ldots,x_{i+m})=g(f(\bar{x}),x_{i+1},\ldots,x_{i+m})\text{ for every }x_i\in K\ (i=1,2,\ldots,n+m).\]

An \((n+1)\text{-ary}\) function \(h\) is defined by primitive recursion from \(f\)-ary function \(f\) and \((n+2)\text{-ary}\) function \(g\), if \(h=g\cdot f\), where \((g\cdot f)(\bar{x},s(i))=g(\bar{x},g\cdot f(\bar{x},i),i), s(i)\in K\).
Let \( r_1, \) \( r_2, \) \( r_3 \) unary, \( r_2, \) \( r_2 \) two-ary, \( r_3 \) three-ary and \( w \) \((k+1)\)-ary functions given by the following equations. For each \( x,y,z,x_1,x_2,...,x_n \in K \) valid

\[
\begin{align*}
  r_1(x) &= \begin{cases} 
    0, & \text{if } x = 0, \\
    x - 1, & \text{if } x \neq 0;
  \end{cases} \\
  j(x) &= \begin{cases} 
    k - 1, & \text{if } x = 1, \\
    0, & \text{if } x \neq 1;
  \end{cases} \\
  r_2(x,y) &= \begin{cases} 
    x, & \text{if } y = 0, \\
    y - 1, & \text{if } y \neq 0;
  \end{cases}
\end{align*}
\]

Let \( | \cdot | \) be a closure operation over the power-set of the set \( O_k \) (that is, for all subset \( O' \) of \( O_k \), \( O' \subseteq O' \) implies \([O'] \subseteq [O] \) and \([O']) \subseteq [O'] \), and let \( O \) be a closed class: \([O] = O \subseteq O_k \).

A subset \( O' \) of \( O \) is \( | \cdot | \)-complete in the class \( O \), if \([O'] = O \).

The \( | \cdot | \)-closed class \( O' \) is maximal in the class \( O \), if \([O'] = O \).

The \( | \cdot | \)-closed class is including the set of projections named \( | - \)-clone.

We are interested in two types of clones. One of them is the compositional closed class. A subset \( O \) of the set \( O_k \) is said to be a compositional clone, if it contains the projections and it is closed under the functional composition: \( f \in O^{(n)} \) and \( g \in O^{(m)} \) implies \( g \circ f \in O^{(n+m-1)} \).

A compositional clone \( O \) is said to be a pr-clone (primitive recursive clone), if it is closed under the primitive recursion: \( f \in O^{(n)} \) and \( g \in O^{(n+2)} \) implies \( (g \circ f)(x) = g(x, (g \circ f)(x), i), i \in K \).

Denote \( [O]_l \) and \( [O]_p \) the compositional clone closure and the pr-clone closure of the set \( O \).

**Remarks**

1. Several closure operations yield several kind of clones, therefore different theories.

2. The cyclic primitive recursion yields another, so called cpr-clones.

An \((n+1)\)-ary function \( h \) is defined by cyclic primitive recursion from \( n \)-ary function \( f \) and \((n+2)\)-ary function \( g \), \( h = \text{g} \circ \text{f} \), where \( (\text{g} \circ \text{f})(x, s(i)) = (\text{g}(x, (\text{g} \circ \text{f})(x, i), i), i \in K \).

**3. The lattice of pr-clones for truth functions (k=2)**

Denote \( O_{0,1} \) and \( O_2 \) the set of truth functions (logical functions). To see the effectivity of the pr-closure, we present the Post’s lattice (of clones) as Fig.1 and the lattice of pr-closed clones (pr-clones) as Fig.2. The latter one is based on the following theorem.

**Theorem 1.**

1. The set \( (O_2 \setminus \{0\} \) is pr-cloned and pr-maximal (in the class \( O_2 \).

2. Each \( \{0\} \)-preserving truth function is pr-complete in the class \( (O_2 \setminus \{0\}) \).

3. If a truth function is not \{0\}-preserving, then it is pr-complete (in \( O_2 \).

**Proof.**

1. It is known [P-41], that \( (O_2 \setminus \{0\} \) is a maximal compositional clone (in \( O_2 \). This clone is closed under primitive recursion, because of \( (g \circ f)(0,0) = f(0) = 0 \), for \( f(O_2 \setminus \{0\} \). Therefore this is a pr-cloned and so it is pr-maximal.

2. Denote \( \circ \) and \( \cdot \). the mod 2 addition and mod 2 multiplication, resp.

Let \( g_2, g_3 \) truth functions be defined by primitive recursion as follows: \( g_2(x,0) = e(x), g_2(x,1) = e_1(x, e(x,0)) \)

and \( g_4(x,0) = e_2(x, y), g_4(x,1) = e_4(x, y, 0), \) that is \( g_2(x, y) = x \circ y \) and \( g_4(x, y, z) = x \circ y \circ z \). According to the definition these functions are elements of every pr-cloned. It can be seen, that \( g_2(x,0) = c_0(x) \), \( g_4(x, y, 0) = x \circ y \) and \( g_4(x, y, z) = x \circ y \circ z \). It is known [P-41], that the set \( \{x, y, x, y\} \) is a compositional base of the clone \( (O_2 \setminus \{0\} \). Due to the statement 1 of the theorem, the set \( E \) is pr-complete in the set \( (O_2 \setminus \{0\} \), so the sets \( E \cup | \cdot | \subseteq (O_2 \setminus \{0\} \) are pr-complete, too.

3. Due to the proof of part 2, each of the pr-clones contains the pr-cloned \( (O_2 \setminus \{0\} \), as a subset.

As this pr-cloned is pr-maximal (statement 1), the statement 3 is true.
Corollary
There are two pr-clones for truth functions; they are the classes O₂ and \( O₂ \) .

The lattices of compositional clones and of pr-clones for truth functions are presented by Fig. 1 and Fig. 2, resp.

4. The maximal pr-clones and the theorem of pr-completeness (case k>2)
It can be seen (part 3), that for truth functions the pr-closure is more effective than the compositional closure: the lattice of the compositional clones is countable [P-41], but the lattice of the pr-clones has only two elements. It is known (and easy to see), that arbitrary \( f \in O^{k+1}_K \) has a representation of the form:

\[
\hat{f}(\bar{x},z) = \max(\min(\bar{j}, \hat{f}(\bar{x},1)).
\]

However, a similar representation is the following (standard representation).

Statement. Every \((n+1)\)-variable function \( f \) has a representation of the form:

\[
\hat{f}(\bar{x},z) = w(f(\bar{x},0),f(\bar{x},1),...,f(\bar{x},k-1),z).
\]

Before presenting our results about pr-Sheffer functions and pr-completeness, we prove a lemma.

Lemma 2. 1. Every pr-clone contains the functions \( c₀, r₁, r₂, r₃, r \) and \( w \).

2. \( c₀ = \{r₁\}, c₁ = \{r₂\}, r \in \{r₁,r₂\}, w \in \{r₁,r₂\} \).

Proof. Let the functions \( f₁ \) and \( f₂ \) be defined by primitive recursion (over the set E) as follows:

\[
f₁(x,y,0) = e₁²(x,y), f₁(x,y,i) = e₂² \cdot x \cdot y, 0 ≤ i ≤ k-2.
\]

We obtain:

\[
r₁(x,y) = f₁(x,y), r₁(x) = f₁(x,x); r₁(y) = f₁(x,y,y).
\]

From the following recursive composition \( u₁ = r₁ \), \( u₂ = u₁ \cdot u₁ \) \((n≥1)\) yields the function \( c₀ = u₁ \cdot u₁ \). The recursive composition \( w₁ = r₁ \), \( w₂ = w₁ \cdot w₁ \) \((i=2,3,...)\) yields the function \( w \); can be seen, that \( w(x₁,x₂,...,xₙ,z) = w₁(x₁,x₂,...,xₙ,z) \).

From this construction the statement 2 is obtained.

Also, the following theorem shows the effectivity of the pr-closure. We mention, that in case of compositional closure the Sheffer functions have at least two variables.

Theorem 3. The functions \( c_{k,1} \) and \( s \) are pr-Sheffer functions (that is, both of them are pr-complete in \( O_K \)).

Proof. Using Lemma 2, every constant function can be generated as follows (similar to the proof of Lemma 2):

\[
\begin{align*}
u₁ = r₁, \quad vₙ+₁ = u₁ \cdot vₙ, \quad c₁ = u₁ \cdot c₁; \\
\end{align*}
\]

\[
\begin{align*}
v₁ = s, \quad vₙ+₁ = s \cdot vₙ, \quad c₁ = v₁ \cdot c₁.
\end{align*}
\]

The statement is obtained by induction on the number \( n \) of variables, because of the standard representation

\[
f(\bar{x},z) = w(f(\bar{x},0),f(\bar{x},1),...,f(\bar{x},k-1),z).
\]

Corollary. The class \( C \) of constant functions is pr-complete.

Therefore, considered the set \( C \cup \{s\} \) as the set of elementary functions in \( O_k \), we obtain that each of the element of the set \( O_k \) is primitive recursive function.

According to the definition of the primitive recursion, \((g\$f)(\bar{0},\bar{0}) = f(\bar{0})\), so the 0-preserving class \((O_k \cup \{0\})\) is a pr-clone for \( k>2 \), too. Since this class is maximal in \( O_k \) as a compositional clone, yields:

Statement. The class \( O_k \) is pr-maximal in the class \( O_K \).

It is known, that every class \( (O_k \cup \{0\}) \) for (non-empty) \( M \) subsets of \( K \) is a clone [R-77]. However, these are not pr-clones, e.g. \( r₁ \in O_k \cup \{0\} \), but \( r₁(1) = 0 \) by Lemma 2. Let \( K = \{0,1,...,i\} \) \((0≤i≤k-1 \ (K_{k-1}=K)) \). In the next theorem the subset-preserving clones are characterized in point of wiev pr-completeness and pr-maximality.

Theorem 4. Let \( M \) be a non-empty, proper subset of the set \( K \). Then the class \((O_K \cup \{0\})\) is

a) maximal pr-clone (in \( O_k \)) if \( K = M \), \( i=0,1,...,k-2 \);

b) pr-complete class (in \( O_k \)) in other cases.

Proof. For \( b \in K \setminus M \) there exists a function \( g ∈ (O^{(1)}_k \cup \{0\}) \) such that \( g(b) = k-1 \).

a) According to the definition of the primitive recursion, the classes \((O_K \cup \{0\})\), \( i=0,1,...,k-2 \) are pr-clones, however they are not pr-complete, since the function \( c_{k-1} \) is not element of them.

For any function \( f ∈ O^{(0)}_K \cup \{0\} \) there exists \( \bar{a} = (a₁,...,aₙ) \in M^n \), such that \( f(\bar{a}) = b \in K \setminus M \). So \( c_{k-1} \cdot g(f(c_{a₁},...,c_{aₙ})) = (O_K \cup \{0\}) \cup \{f\} \) since \( c_{a₁},...,c_{aₙ} \cdot g ∈ (O_K \cup \{0\} \cup \{f\}) \). Therefore, the class \((O_K \cup \{0\})\)
is pr-maximal clone, due to Theorem 3.

b) Suppose that the set $M$ is different from the sets $K_0, K_1, \ldots, K_{k-1}$, so there exists $b+1 \in M$, such that $b \in K\setminus M$.

Since $c_{b+1} \in (O_K | M)$, so $r_i(c_{b+1}) = c_b \in [(O_K | M)]_i$, therefore $g(c_b) = c_{k-1} \in [(O_K | M)]_{k-1}$. Due to Theorem 3, the class $(O_K | M)$ is pr-complete.

The next theorem is similar to the completeness theorem of Rosenberg [R-77].

**Theorem 5.** (Basic theorem of pr-completeness.) For every integers $k \geq 2$, the set $O \subseteq O_K$ is pr-complete (in $O_K$) if and only if none of the sets $O \setminus (O_K | K_i)$, $0 \leq i \leq k-2$ are empty.

Proof. Clearly, the condition is necessary, since the sets $(O_K | K_i)$ are pr-maximal clones for $i < k-1$, by Theorem 4.

Sufficient. Suppose, that none of the sets $O \setminus (O_K | K_i)$, $0 \leq i \leq k-2$ are empty. So there exists a function $g \in O$, for which $g(c_0, \ldots, c_0) = c_j \neq c_0$, therefore $c_j \in [O]_{pr}$, by Lemma 2. If $j < k-1$, then on a similar way we have get the set \{g(c_{i_1}, \ldots, c_{i_n}) | g \in O, \{i_1, \ldots, i_n\} \subseteq K_j\} having an element $c_l$, $l > j$. This way we obtain the function $c_{k-1}$, at most in steps $k-1$. Therefore the class $O$ is pr-complete (Lemma 2 and Theorem 3 is used).
Irodalom


