# How to costruct left-continuous triangular norms <br> - state of art 2002 * 

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## 1 Introduction

Triangular norms (t-norms for short) play a crucial role in several fields of mathematics and AI. For an exhaustive overview on t-norms we refer to [20]. Recently an increasing interest of left-continuous t-norm based theories can be observed (see e.g. [3, 5, 6, 7, 18]). In this paper we discuss in detail the presently existing construction methods which result in left-continuous triangular norms. The methods are (together with their sources):

- annihilation [12, 2],
- ordinal sum of t-subnorms [11, 9],
- rotation contruction [14, 8],
- rotation-annihilation construction [16],
- embedding method $[17,6]$.

An infinite number of left-continuous triangular norms can be generated with these constructions (and with their combinations), which provides a tremendously wide spectrum of choice for e.g. logical and set theoretical connectives in non-classical logic and in fuzzy theory.

## 2 Preliminaries

A triangular norm (t-norm for short) is a binary operation $T$ (that is, a function $T:[0,1]^{2} \rightarrow[0,1]$ ) such that for all $x, y, z \in[0,1]$ the following four axioms (T1)-(T4) are satisfied:

| $(T 1)$ | Symmetry | $T(x, y)=T(y, x)$ |
| :--- | :--- | :--- |
| $(T 2)$ | Associativity | $T(x, T(y, z))=T(T(x, y), z)$ |
| $(T 3)$ | Monotonicity | $T(x, y) \leq T(x, z)$ whenever $y \leq z$ |
| $(T 4)$ | Boundary condition | $T(x, 1)=x$ |
| $\left(T 4^{\prime}\right)$ | Boundary condition | $T(x, 0)=0$ |
| $\left(T 4^{\prime \prime}\right)$ Range condition | $T(x, y) \leq \min (x, y)$. |  |

As we will se later, in the construction of left-continuous t-norms an essential role is played by t-subnorms:
Definition 1 ([16]) A triangular subnorm (t-subnorm for short) is a function $T:[0,1]^{2} \rightarrow[0,1]$ such that for all $x, y, z \in[0,1]$ axioms (T1), (T2), (T3) and (T4") are satisfied.

Any t-norm is a t-subnorm. We say that a t-subnorm $T$ has zero divisors if there is $x, y \in] 0,1]$ such that $T(x, y)=0$. A t-subnorm is said to be continuous resp. left-continuous if it is continuous resp. left-continuous as a two-place function.

One can define t-subnorms on any $[a, b] \subset \mathbb{R}$ and gets the notion of a t-subnorm on $[a, b]$. Then for any $t$-(sub)norm $T$, the function $T_{[a, b]}:[a, b] \times[a, b] \rightarrow[a, b]$ defined by $T_{[a, b]}(a, b)=a+(b-a) \cdot T\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}\right)$ is a t-(sub)norm on $[a, b]$. If $T_{[a, b]}$ is a $t$-(sub)norm on $[a, b]$ then the function $T:[0,1] \times[0,1] \rightarrow[0,1]$ defined by $T(a, b)=\frac{T_{[a, b]}(a+x(b-a), a+y(b-a))-a}{b-a}$ is a $t$-(sub)norm. Call $T_{[a, b]}$ the linear transformation of $T$ into [ 0,1$]$. Similarly, call $T$ the linear transformation of $T_{[a, b]}$ into $[0,1]$.

[^0]Example 1 Non-trivial examples of t-subnorms are e.g. $T_{\mathbf{P}_{\varepsilon}}(x, y)=\varepsilon \cdot x \cdot y, T_{\mathbf{L}_{\varepsilon}}(x, y)=\max (0, x+y-1-\varepsilon)$, when $\varepsilon$ is a fixed real from $[0,1]$ (see Figure 2) with the exception of $T_{\mathbf{L}_{0}}$ and $T_{\mathbf{P}_{1}}$ which are just $T_{\mathbf{L}}$, the Lukasiewicz t-norm and $T_{\mathbf{P}}$, the product t-norm, respectively.

Example 2 We remark that the construction in [10] (Theorem 2) (see as well [20] (Proposition 11) and [19]) produces t-subnorms if the boundary of the resulted t-norm is not redefined (in the formula which can be found in the cited references). Moreover, if one starts with a left-continuous t-(sub)norm, then the just mentioned construction (again without the separate definition on the boundary) produces a left-continuous t-subnorm.

A negation $([22]) N$ is a non-increasing function on $[0,1]$ with boundary conditions $N(0)=1$ and $N(1)=0$. A negation is called strong if $N$ is an involution, that is, if in addition $N(N(x))=x$ holds for all $x \in[0,1]$. A negation is strong if and only if its graph is invariant w.r.t. the reflection at the median (given by $y=x$ ). A strong negation is automatically a strictly decreasing, continuous function and hence it has exactly one fixed point.

Let $T:[0,1]^{2} \rightarrow[0,1]$ be a function satisfying (T1) and (T3). The implication $I_{T}:[0,1]^{2} \rightarrow[0,1]$ generated by $T$ is given by $I_{T}(x, y)=\sup \{t \in[0,1] \mid T(x, t) \leq y\}$. If $T$ is left-continuous then $I_{T}$ is called the residual implication generated by $T$.

For a left-continuous t-subnorm $T:[0,1]^{2} \rightarrow[0,1]$ define $N_{T}(x)=I_{T}(x, 0)$ for $x \in[0,1]$. If $N_{T}$ is a negation (this holds e.g. if $T$ satisfies (T4); that is $N_{T}$ is always a negation if $T$ is a t-norm) then $N_{T}$ is called the induced negation of $T$.

Let $T:[0,1]^{2} \rightarrow[0,1]$ be a function satisfying (T3), and let $N$ be a strong negation. We say ([16]) that $T$ admits the rotation invariance property with respect to $N$ or rotation invariant w.r.t. $N$ if for all $x, y, z \in[0,1]$ we have $T(x, y) \leq z \Leftrightarrow T(y, N(z)) \leq N(x)$.

## 3 Annihilation

The nilpotent minimum t-norm $T_{\mathbf{M}_{\mathbf{0}}}$ is introduced in [4] in such a way that the values of the minimum t-norm are replaced by 0 under the negation $1-x$. More formally, for $x, y \in[0,1]$ let

$$
T_{\mathbf{M}_{\mathbf{0}}}(x, y)= \begin{cases}0 & \text { if } y \leq 1-x  \tag{1}\\ \min (x, y) & \text { otherwise }\end{cases}
$$

It is observed that the same construction works for any strong negation instead of the standard one $1-x$, and that the costruction doesn't result in a t-norm (in fact, the associativity property is violated) if the minimum t-norm is replaced by the product t-norm. Motivated by this observation the concept of $N$-annihilation ( $N$ being any strong negation) is defined in [12] and a characterization of those continuous t-norms where the annihilated operator is a t-norm is given as follows:

Let $T$ be a t-norm and $N$ be a strong negation. Define the binary operation $T_{(N)}$ (called the $N$-annihilation of $T$ ) as follows:
$T_{(N)}:[0,1] \times[0,1] \rightarrow[0,1] ;$

$$
T_{(N)}(x, y)= \begin{cases}0 & \text { if } x \leq N(y)  \tag{2}\\ T(x, y) & \text { otherwise }\end{cases}
$$

Theorem 1 For any strong negation $N$ and continuous t-norm $T, T_{(N)}$ is a t-norm if and only if $T_{(N)}$ is isomorphic to

$$
T_{\mathbf{J}}(x, y)= \begin{cases}0 & \text { if } x \leq 1-y  \tag{3}\\ \frac{1}{3}+x+y-1 & \text { if } x, y \in\left[\frac{1}{3}, \frac{2}{3}\right] \\ & \text { and } x>1-y \\ \min (x, y) & \text { otherwise }\end{cases}
$$

In this way a new family of left-continuous t-norms with the additional property of strongness of their induced negation is introduced. For a visualization, see Fig. 1.

However it was not published in the paper, there is a more "philosophical" reformulation of the results of [12], which is based on the idea of "level curves": For any $c \in] 0,1]$ call the one-place function $f_{c}(x):=I_{T}(x, c)$, $x \in[c, 1]$ the $c$-level curve of the continuous t-norm $T$. Because of the continuity of $T$ we can infer $T(x, y)=c$ if we have $f_{c}(x)=y$, this explains the name "level curve". Therefore, the $c$-level curve is a part (in fact it is the "upper border") of the $c$-level set $\left\{(x, y) \in[0,1]^{2} \mid T(x, y)=c\right\}$. Further, we say that the negation $N$ "cuts" a $c$-level curve, if there exist $x, y \in] 0,1]$ such that $f_{c}(x)<N(x)$ and $f_{c}(y)>N(y)$. In other words, the graph of the $c$-level curve is NOT entirely in the upper closed subdomain of $[0,1]$ which is determined by the graph of $N$.


Figure 1: The t-norm which is defined in (3)

By using this terminology, we can reformulate the results of [12] as follows: If any of the c-level curves is cut by $N$, then in general $T$ looses its associativity via $N$-annihilation (that is, $T_{(N)}$ is not associative). The only exception is if in the "remaining part" of the c-level curve $T$ coincides with the minimum $t$-norm. More formally, if a $c$-level curve is cut by $N$ then we should have $T(x, y)=\min (x, y)$ whenever $f_{c}(x)=y$ and $x>N(y)$.

In [2] the authors generalize the results of [12] by considering any left-continuous t-norm $T$ and any negation $N$ (i.e., not necessarily strong ones). They characterize those left-continuous t-norms $T$, for which $T_{(N)}$ is a t-norm. They provide a more detailed description for the case when $T$ is a continuous t-norm: Instead of quoting this rather formal result we remark that its equivalent formulation is just the above-presented "philosophical" description (written in italic).

## 4 Ordinal sums of t-subnorms

It is observed in [11] that the well-known ordinal sum theorem of t-norms can be generalized by using t-subnorms as summands. We remark that any further generalization (which still results in t-norms or t-subnorms) is not possible, as it is straightforward to see.

Theorem 2 (Ordinal Sum Theorem for t-subnorms) Suppose that $\left\{\left[a_{i}, b_{i}\right]\right\}_{i \in K}\left(a_{i}<b_{i}\right)$ is a countable family of non-overlapping, closed subintervals of $[0,1]$, denoted by $\mathcal{I}$. With each $\left[a_{i}, b_{i}\right] \in \mathcal{I}$ associate a $t$-subnorm $T_{i}$ where for each $\left[a_{i}, b_{i}\right],\left[a_{j}, b_{j}\right] \in \mathcal{I}$ with $b_{i}=a_{j}$ and with zero divisors in $T_{j}$ we have that $T_{i}$ is a t-norm. Let $T$ be a function defined on $[0,1]^{2}$ by

$$
T(x, y)=\left\{\begin{array}{l}
a_{m}+\left(b_{m}-a_{m}\right) T_{m}\left(\frac{x-a_{m}}{b_{m}-a_{m}}, \frac{y-a_{m}}{b_{m}-a_{m}}\right)  \tag{4}\\
\text { if } \left.(x, y) \in] a_{m}, b_{m}\right]^{2}, \\
\min (x, y) \\
\text { otherwise. }
\end{array}\right.
$$

Then $T$ is a $t$-subnorm and called the ordinal sum of $\left\{\left(\left[a_{i}, b_{i}\right], T_{i}\right)\right\}_{i \in K}$ and each $T_{i}$ is called $a$ summand.
Theorem 3 (Generalized Ordinal Sum Theorem for t-norms) Suppose that $\left\{\left[a_{i}, b_{i}\right]\right\}_{i \in K}\left(a_{i}<b_{i}\right)$ is a countable family of non-overlapping, closed subintervals of $[0,1]$, denoted by $\mathcal{I}$. With each $\left[a_{i}, b_{i}\right] \in \mathcal{I}$ associate a t-subnorm $T_{i}$ where for each $\left[a_{i}, b_{i}\right],\left[a_{j}, b_{j}\right] \in \mathcal{I}$ with $b_{i}=a_{j}$ and with zero divisors in $T_{j}$ we have that $T_{i}$ is a $t$-norm and for $\left[a_{i}, 1\right] \in \mathcal{I}$ we have that $T_{i}$ is a $t$-norm. Let $T$ be a function defined on $[0,1]^{2}$ by (4). Then $T$ is a t-norm.

Example 3 The ordinal sum $\left\{\left([0,0.5], T_{\mathbf{L}}\right),\left([0.5,1], T_{\mathbf{P}_{0.5}}\right)\right\}$ is not a t-norm but a t-subnorm since the "last" summand $T_{\mathbf{P}_{0.5}}$ is not a t-norm. The ordinal sum $\left\{\left([0.2,0.5], T_{\mathbf{P}}\right),\left([0.5,0.8], T_{\mathbf{L}_{0.5}}\right)\right\}$ is a t-norm. Indeed, since $T_{\mathbf{L}_{0.5}}$ has zero divisors the only thing we need to verify that the summand which is just below it (that is, $T_{\mathbf{P}}$ ) is a t-norm. Figure 2 visualizes the above ordinal sums.


Figure 2: $T_{\mathbf{P}_{0.5}}, T_{\mathbf{L}_{0.4}}$, a t-subnorm and a t-norm, which are ordinal sums of t-subnorms, see Examples 1 and 2.

## 5 Rotation

The rotation method is introduced in [14] and a characterization theorem is given in [8]. As in the ordinal sum theorem for t-subnorms, we remark, that it is not possible to provide any further generalization of the method (which still produces t-norms or t-subnorms). The method produces left-continuous (but not continuous) t-norms which have strong induced negations from any left-continuous t-norm $T_{1}$ which either has no zero divisors or all the zero values of its graph are in a sub-square of the unit square (see Fig. 3).

Theorem 4 Let $N$ be a strong negation, $t$ its unique fixed point and $T$ be a left-continuous $t$-norm. Let $T_{1}$ be the linear transformation of $T$ into $\left.\left.[t, 1], I^{+}=\right] t, 1\right]$ and $I^{-}=[0, t]$. Define $T_{\text {Rot }}$ and $I_{T_{\text {Rot }}}$ (of types $[0,1] \times[0,1] \rightarrow[0,1]) b y$

$$
\begin{align*}
& T_{\text {Rot }}(x, y)=\left\{\begin{array}{ll}
T_{1}(x, y) & \text { if } x, y \in I^{+} \\
N\left(I_{T_{1}}(x, N(y))\right) & \text { if }(x, y) \in I^{+} \times I^{-} \\
N\left(I_{T_{1}}(y, N(x))\right) & \text { if }(x, y) \in I^{-} \times I^{+} \\
0 & \text { if } x, y \in I^{-}
\end{array},\right.  \tag{5}\\
& I_{T_{\text {Rot }}}(x, y)= \begin{cases}I_{T_{1}}(x, y) & \text { if } x, y \in I^{+} \\
N\left(T_{1}(x, N(y))\right) & \text { if }(x, y) \in I^{+} \times I^{-} \\
1 & \text { if }(x, y) \in I^{-} \times I^{+} \\
I_{T_{1}}(N(y), N(x)) & \text { if } x, y \in I^{-}\end{cases} \tag{6}
\end{align*}
$$

$T_{\text {Rot }}$ is a left-continuous t-norm if and only if either
C1. T has no zero divisors or
C2. there exists $c \in] 0,1]$ such that for any zero divisor $x$ of $T$ we have $I_{T}(x, 0)=c$.
In this case the induced negation of $T$ is $N$, and the residual implication generated by $T_{\mathbf{R o t}}$ is $I_{T_{\mathbf{R o t}}}$.


Figure 3: The zero values of t -norms which are suitable for the rotation construction

Remark 1 If $T$ is a t-subnorm in Theorem 4 then $T_{\text {Rot }}$ is a left continuous t-subnorm, which is rotation invariant w.r.t. $N$ and the residual implication generated by $T_{\text {Rot }}$ is given by (6).

Example 4 In Figure 4 the rotation of the minimum t-norm and the rotation of the product t-norm can be seen. Observe that the nilpotent minimum t-norm is not else but the rotation of the minimum t-norm. In


Figure 4: A t-norm with zero divisors, its rotation, rotation of minimum t-norm and rotation of product t-norm

Figure 5 the rotations of certain ordinal sums can be seen. On the right-hand side the ordinal sum has two summands: the Łukasiewicz t-norm and the product t-norm. On the left-hand side the summand is the rotation of the product.



Figure 5: Rotations of ordinal sums

## 6 Rotation-annihilation

The rotation-annihilation method has been introduced in [15]. It produces left-continuous (but not continuous) t-norms which have strong induced negations from a pair of certain connectives, as it is given in the following definition. Again, we remark, that it is not possible to provide any further generalization of the method (which still produces t-norms or t-subnorms).

Definition 2 ([12]) Let $N$ be a strong negation and $t$ be its unique fixed point. Let $d \in] t, 1]$. Then $N_{d}$ : $[0,1] \rightarrow[0,1]$ defined by $N_{d}(x)=\frac{N(x \cdot(d-N(d))+N(d))-N(d)}{d-N(d)}$ is a strong negation. Call $N_{d}$ the zoomed d-negation of $N$.

Definition 3 Let $N$ be a strong negation, $t$ its unique fixed point, $d \in] t, 1\left[\right.$ and $N_{d}$ be the zoomed $d$-negation of $N$. Let $T_{1}$ be a left-continuous t-subnorm.
i. If $T_{1}$ has no zero divisors then let $T_{2}$ be a left-continuous t-subnorm which admits the rotation invariance property w.r.t. $N_{d}$. Further, let $I^{-}=\left[0, N(d)\left[, I^{0}=[N(d), d]\right.\right.$ and $\left.\left.I^{+}=\right] d, 1\right]$.
ii. If $T_{1}$ has zero divisors then let $T_{2}$ be a left-continuous t-norm which admits the rotation invariance property w.r.t. $N_{d}$ (it is equivalent to saying that $T_{2}$ is a left-continuous t-norm with strong induced negation equal with $N_{d}$, see [16]). Further, let $\left.I^{-}=[0, N(d)], I^{0}=\right] N(d), d\left[\right.$ and $I^{+}=[d, 1]$.

Let $T_{3}$ be the linear transformation of $T_{1}$ into $[d, 1], T_{4}$ be the linear transformation of $T_{2}$ into $[N(d), d]$ and $T_{5}:[N(d), d]^{2} \rightarrow[N(d), d]$ be the annihilation of $T_{4}$ given by

$$
T_{5}(x, y)= \begin{cases}0 & \text { if } x, y \in[N(d), d] \text { and } x \leq N(y) \\ T_{4}(x, y) & \text { if } x, y \in[N(d), d] \text { and } x>N(y)\end{cases}
$$

Define $T_{\mathbf{R A}}:[0,1] \times[0,1] \rightarrow[0,1]$ by

$$
T_{\mathbf{R A}}(x, y)= \begin{cases}T_{3}(x, y) & \text { if } x, y \in I^{+}  \tag{7}\\ N\left(I_{T_{3}}(x, N(y))\right) & \text { if } x \in I^{+}, y \in I^{-} \\ N\left(I_{T_{3}}(y, N(x))\right) & \text { if } x \in I^{-}, y \in I^{+} \\ 0 & \text { if } x, y \in I^{-} \\ T_{5}(x, y) & \text { if } x, y \in I^{0} \\ y & \text { if } x \in I^{+} \text {and } y \in I^{0} \\ x & \text { if } x \in I^{0} \text { and } y \in I^{+} \\ 0 & \text { if } x \in I^{-} \text {and } y \in I^{0} \\ 0 & \text { if } x \in I^{0} \text { and } y \in I^{-}\end{cases}
$$

Call $T_{\mathbf{R A}}$ the $N$-d-rotation-annihilation of $T_{1}$ and $T_{2}$. If $N(x)=1-x$ (the standard negation) then call $T_{\mathbf{R A}}$ simply the $d$-rotation-annihilation of $T_{1}$ and $T_{2}$.

Theorem 5 (Rotation-annihilation) Let $N$ be a strong negation, $t$ its unique fixed point, $d \in] t, 1\left[\right.$ and $T_{1}$ be a left-continuous t-norm. Take $T_{2}$, depending on the zero divisors of $T_{1}$, as it is taken in Definition 3 and let $T_{\mathbf{R A}}$ be the $N$-d-rotation-annihilation of $T_{1}$ and $T_{2}$.

Finally, define $I_{T_{\mathbf{R A}}}:[0,1] \times[0,1] \rightarrow[0,1]$ by

$$
I_{T_{\mathbf{R A}}}(x, y)= \begin{cases}I_{T_{3}}(x, y) & \text { if } x, y \in I^{+}  \tag{8}\\ N\left(T_{3}(x, N(y))\right) & \text { if } x \in I^{+}, y \in I^{-} \\ 1 & \text { if } x \in I^{-}, y \in I^{+} \\ I_{T_{3}}(N(y), N(x)) & \text { if } x, y \in I^{-} \\ I_{T_{4}}(x, y) & \text { if } x, y \in I^{0} \\ y & \text { if } x \in I^{+}, y \in I^{0} \\ N(x) & \text { if } x \in I^{0}, y \in I^{-} \\ 1 & \text { if } x \in I^{-}, y \in I^{0} \\ 1 & \text { if } x \in I^{0}, y \in I^{+}\end{cases}
$$

Then $T_{\mathbf{R A}}$ is a left-continuous t-norm, its induced negation is $N$, it is rotation invariant w.r.t. $N$, and the residual implication generated by $T_{\mathbf{R A}}$ is given by (8).

Remark 2 If $T_{1}$ is left-continuous t-subnorm in Theorem 5 then $T_{\mathbf{R A}}$ is a left continuous t-subnorm, which is rotation invariant w.r.t. $N$ and the residual implication generated by $T_{\mathbf{R A}}$ is given by (8).

In Figure 6 the rotation-annihilation of $T_{1}$ and $T_{2}$ is presented, where $T_{1}$ is an ordinal sum defined by a Lukasiewicz t-norm and a product t-norm and $T_{2}$ is the rotation of the product. In Figure 6 the rotation-annihilation of $T_{1}$ and $T_{2}$ is presented, where $T_{1}$ is the product t-norm and $T_{2}$ is $T_{\mathbf{L}_{0.5}}$, the rotation invariant t-subnorm given in Section 2.

## 7 Embedding method

### 7.1 Completions of left-continuous monoids

Let $\mathcal{D}=\langle D, \star, \leq, 1\rangle$ be a commutative totally ordered integral monoid. We say that $\star$ is a left-continuous operation on $\mathcal{D}$ if whenever $X \subseteq D$ and $\sup (X)$ exists in $D$, then for every $y \in D$ one has: $y \star \sup (X)=$ $\sup \{y \star x: x \in X\}$. In this case we speak of left-continuous totally ordered commutative integral monoid. Any residuated monoid is left-continuous, the converse is not true in general.

Theorem 6 We can embed any countable left-continuous totally and densely ordered commutative integral monoid $\mathcal{D}$ into a left-continuous t-norm.
(i) Clearly $\mathcal{D}$ is order isomorphic to $\mathbb{Q} \cap[0,1]$. Let $h$ be any order isomorphism from $\mathcal{D}$ onto $\mathbb{Q} \cap[0,1]$, and let - be defined on $\mathbb{Q} \cap[0,1]$ by $x \circ y=h\left(h^{-1}(x) \star h^{-1}(y)\right)$. Then $h$ preserves the monoidal operation, i.e., it is an isomorphism from $\mathcal{D}$ into $\langle Q \cap[0,1], \circ, \leq, 1\rangle$, as desired.
(ii) Define for $\alpha, \beta \in[0,1], T_{4}(\alpha, \beta)=\sup \{h(d \star e): h(d) \leq \alpha$ and $h(e) \leq \beta\}$. Then $T_{4}$ is a left-continuous t-norm on $[0,1]$ which extends $\circ$. Hence $h$ is an embedding of $\mathcal{D}$ into $\left\langle[0,1], T_{4} \leq, 1\right\rangle$, and using the density of $\mathbb{Q}$ in $\mathbb{R}$, we see that $h$ preserves suprema and infima. This ends the construction.


Figure 6: t-norms generated by the rotation-annihilation construction

Definition 4 We say that the t-norm $T_{4}$ defined in the proof of Theorem 6 is the completion of the monoidal operation $\star$.

Theorem 7 We can embed any countable, totally ordered, commutative integral monoid into a left-continuous $t$-norm.

Indeed, let $\mathcal{D}$ be a countable, totally ordered, commutative integral monoid (not necessarily densely ordered and left-continuous). Then we can embed it into a densely and totally ordered left-continuous commutative integral monoid $\mathcal{M}$ with minimum in the following way:

- If $\mathcal{D}$ has no minimum, then add a new element $m$ to be the minimum of the lattice reduct of $\mathcal{D}$ and extend the operation to $m$ in the obvious way (so that it becomes the zero of the multiplication).
- The domain of $\mathcal{M}$ is $\{(m, 1)\} \cup\{(a, q): a \in \mathcal{D}-\{m\}, q \in \mathbb{Q} \cap(0,1]\}$.
- The order is the lexicographic order.
- The monoidal operation $\circ$ is defined by $(a, q) \circ(b, r)=\min \{(a, q),(b, r)\}$ if $a \star b=\min \{a, b\}$, and $(a, q) \circ$ $(b, r)=(a \star b, 1)$ otherwise.

That $\mathcal{M}$ defined in this way is a commutative linearly and densely ordered integral monoid is proved as in [18]. Left-continuity is due to the fact that if $\lim _{n \rightarrow \infty}\left(a_{n}, q_{n}\right)=(a, q)$, then for almost all $n, a_{n}=a$, and $\lim _{n \rightarrow \infty} q_{n}=q$. An application of Theorem 6 ends the construction.

### 7.2 Embedding finite lexicographical products

Theorem $8\left(T_{\langle k\rangle}\right)$ Let $k \in \mathbb{N}$ and $T$ be any t-norm without zero divisors. Let $\oplus_{i}$ be a commutative, disjunctive $\ell$-monoid on $\mathbb{N}$ with zero 0 for $1 \leq i \leq k$.

1. Let $X_{k}=\left\{1-\sum_{i=1}^{k+1} n_{i} \cdot \varepsilon^{i} \mid n_{i} \in \mathbb{N}(1 \leq i \leq k), n_{k+1} \in \mathbb{R}^{+}, \varepsilon>0\right.$ is infinitesimal $\}$. Fix arbitrarily $0=\alpha_{0}<\alpha_{1}<\alpha_{2}<\ldots<\alpha_{k}<\alpha_{k+1}<1$ and let $a_{i}=\varphi_{\alpha_{i-1}}\left(\alpha_{i}\right)(1 \leq i \leq k+1)$. Define a binary operation $\oplus_{T}$ on $\mathbb{R}^{+}$by

$$
r \oplus_{T} s=\log _{a_{k+1}}\left(T\left(a_{k+1}^{r}, a_{k+1}^{s}\right)\right)
$$

and a binary operation $T$ on $X_{k}$ by $T\left(\left(1-\sum_{i=1}^{k+1} n_{i} \cdot \varepsilon^{i}\right),\left(1-\sum_{i=1}^{k+1} m_{i} \cdot \varepsilon^{i}\right)\right)=1-\left(\sum_{i=1}^{k}\left(n_{i} \oplus_{i} m_{i}\right) \cdot \varepsilon^{i}\right)-$ $\left(n_{k+1} \oplus_{T} m_{k+1}\right) \cdot \varepsilon^{k+1}$.
Denote $\phi_{\emptyset}=\left.i d\right|_{j 0,1]}, \phi_{n_{1}, n_{2}, \ldots, n_{k}}=\varphi_{a_{1}, n_{1}}^{-1} \circ \varphi_{a_{2}, n_{2}}^{-1} \circ \ldots \circ \varphi_{a_{k}, n_{k}}^{-1}$ define $\left.\left.\eta_{k}: X_{k} \rightarrow\right] 0,1\right]$ by

$$
\begin{equation*}
\eta_{k}\left(1-\sum_{i=1}^{k+1} n_{i} \cdot \varepsilon^{i}\right)=\phi_{n_{1}, n_{2}, \ldots, n_{k}}\left(a_{k+1} n_{k+1}\right) \tag{9}
\end{equation*}
$$

Then $\eta_{k}$ is an order-preserving bijection from $X_{k}$ to $\left.] 0,1\right]$. Finally, define a binary operation $T_{\langle k\rangle}$ on $\left.] 0,1\right]$ by

$$
\begin{equation*}
T_{\langle k\rangle}(x, y)=\eta_{k}\left(T\left(\eta_{k}^{-1}(x), \eta_{k}^{-1}(y)\right)\right) \tag{10}
\end{equation*}
$$

2. For $x \in] 0,1]$ set $n_{k, 0}(x)=0, x_{k, 0}=x$. Define recursively

$$
\begin{aligned}
n_{k, i}(x) & =\left\lfloor\log _{a_{i}}\left(x_{k, i-1}\right)\right\rfloor, \\
x_{k, i} & =\varphi_{a_{i}}\left(\frac{x_{k, i-1}}{a_{i}^{n_{k, i}(x)}}\right)
\end{aligned}
$$

for $1 \leq i \leq k$ and let $n_{k, k+1}(x)=\log _{a_{k+1}}\left(x_{k, k}\right)$. Define a binary operation $T_{\langle k\rangle}$ on $\left.] 0,1\right]$ by $T_{\langle k\rangle}(x, y)=$ $\phi_{n_{k, 1}(x) \oplus_{1} n_{k, 1}(y), \ldots, n_{k, k}(x) \oplus_{k} n_{k, k}(y)}\left(a_{k+1}^{n_{k, k+1}(x) \oplus_{T} n_{k, k+1}(y)}\right)$.
3. Let $T_{\langle 0\rangle}=T$. For $1 \leq i \leq k$ define binary operations $T_{\langle i\rangle}$ on $\left.] 0,1\right]$ recursively by (let $\bar{i}=k+1-i$, $\bar{x}=\frac{x}{a_{\bar{i}}^{n_{i}(1(x)}}$ and $\bar{y}=\frac{y}{a_{\bar{i}}^{n_{i}, 1(y)}}$ for short) $T_{\langle i\rangle}(x, y)=$

$$
\begin{equation*}
\varphi_{a_{\bar{i}}, n_{i, 1}(x) \oplus_{\bar{i}} n_{i, 1}(y)}^{-1}\left(T_{\langle i-1\rangle}\left(\varphi_{a_{\bar{i}}}(\bar{x}), \varphi_{a_{\bar{i}}}(\bar{y})\right)\right) \tag{11}
\end{equation*}
$$

i. The three definitions for $T_{\langle k\rangle}$ given in 1, 2 and 3. are equivalent.
ii. $T_{\langle k\rangle}$ is a t-norm without zero divisors.
iii. $T_{\langle k\rangle}$ is left-continuous if and only if so does $T$.
iv. $T_{\langle k\rangle}$ is strictly increasing if and only if so does $T$.
v. By using the definition in 3. we have that $\left.T_{\langle k\rangle}\right|_{\left.] \alpha_{1}, 1\right]}$ is order-isomorphic to $T_{\langle k-1\rangle}$.

Remark 3 If $T$ is a left-continuous t-subnorm, or if 0 is not necessarily zero of $\oplus_{i}$ 's and we suppose $0 \oplus_{i} 0=0$ only then everything holds true but the boundary condition of the resulted t-norm is violated. Then we obtain t-subnorms.

Remark 4 As far as we can see Theorem 1 can not be extended so that $T$ is a t-norm with zero divisors without loosing either the associativity or the left-continuity (that is the residuated nature) of the resulted structure.

Corollary $1\left(T_{\langle\oplus\rangle}\right)$ Let $T$ be any t-norm without zero divisors, $\oplus$ be any commutative, disjunctive $\ell$-monoid on $\mathbb{N}$ with zero 0 , $a \in] 0,1\left[\right.$, and $n(x)=\left\lfloor\log _{a}(x)\right\rfloor$. The binary operation $T_{\langle\oplus\rangle}$ on $\left.] 0,1\right]$ given by $T_{\langle\oplus\rangle}(x, y)=$ $a^{n(x) \oplus n(y)} \cdot\left(\varphi_{a}^{-1}\left(T\left(\varphi_{a}\left(\frac{x}{a^{n(x)}}\right), \varphi\left(\frac{y}{a^{n(y)}}\right)\right)\right)\right)=a^{n(x) \oplus n(y)} \cdot\left(a+(1-a) \cdot T\left(\frac{\frac{x}{a^{n(x)}-a}}{1-a}, \frac{\frac{y}{a^{n(y)}-a}}{1-a}\right)\right)=a^{\left\lfloor\log _{a}(x)\right\rfloor \oplus\left\lfloor\log _{a}(y)\right\rfloor}$. $\left(a+(1-a) \cdot T\left(\frac{\frac{x}{a^{\left[\log _{a}(x)\right\rfloor}-a}}{1-a}, \frac{\frac{a^{\left\lfloor\log _{a}(y)\right]}-a}{1-a}}{1-a}\right)\right.$ is a t-norm with out zero divisors. In addition, $T_{\langle\oplus\rangle}$ is left-continuous (resp. strictly increasing on $] 0,1]^{2}$ ) if and only if so does $T$, and $\left.T_{\langle\oplus\rangle}\right|_{] a, 1]}$. is order-isomorphic to $T$.

Remark 5 It is clear from the recursive description of $T_{\langle k\rangle}$ (see eq. (11)) that consecutive applications of Corollary 1 can result in all the t-norms, which can be generated by Theorem 8. Moreover, we see that $T_{\left\langle\oplus_{k}, \ldots, \oplus_{1}\right\rangle}=\left(T_{\left\langle\oplus_{k}, \ldots, \oplus_{i+1}\right\rangle}\right)_{\left\langle\oplus_{i}, \ldots, \oplus_{1}\right\rangle}$

### 7.3 Embedding infinite lexicographical products

In this section we embed commutative, residuated integral $\ell$-monoids of $\times_{i=1}^{\infty} \mathbb{N}$ into $\left.] 0,1\right]$.
Theorem $9\left(T_{\langle\infty\rangle}\right)$ For $i \in \mathbb{N}$ let $\oplus_{i}$ be a commutative, disjunctive $\ell$-monoid on $\mathbb{N}$ with zero 0.

1. Let $X=\left\{1-\sum_{i=1}^{\infty} n_{i} \cdot \varepsilon^{i} \mid n_{i} \in \mathbb{N}, i \in \mathbb{N}, \varepsilon>0 \text { is infinitesimal }\right\}^{1}$. Fix arbitrarily $0=\alpha_{0}<\alpha_{1}<\alpha_{2}<$ $\ldots<\alpha_{i}<\ldots<1$ such that $\lim _{i=1}^{\infty} \alpha_{i}=1$ and let $a_{i}=\varphi_{\alpha_{i-1}}\left(\alpha_{i}\right)(i \in \mathbb{N})$.
Define a binary operation $T$ on $X$ by $T\left(\left(1-\sum_{i=1}^{\infty} n_{i} \cdot \varepsilon^{i}\right),\left(1-\sum_{i=1}^{\infty} m_{i} \cdot \varepsilon^{i}\right)\right)=1-\left(\sum_{i=1}^{\infty}\left(n_{i} \oplus_{i} m_{i}\right) \cdot \varepsilon^{i}\right)$.
[^1]Define $X_{k}, \phi_{n_{1}, n_{2}, \ldots, n_{k}}$ and $\eta_{k}$ as in Theorem 8. Let $\left.\left.\eta_{\infty}: X \rightarrow\right] 0,1\right]$ be given by

$$
\begin{equation*}
\eta_{\infty}\left(1-\sum_{i=1}^{\infty} n_{i} \cdot \varepsilon^{i}\right)=\lim _{k=1}^{\infty} \eta_{k}\left(1-\sum_{i=1}^{k+1} n_{i} \cdot \varepsilon^{i}\right) \tag{12}
\end{equation*}
$$

Then $\eta_{\infty}$ is an order-preserving bijection from $X$ to $\left.] 0,1\right]$. Finally, define a binary operation $T_{\langle\infty\rangle}$ on $\left.] 0,1\right]$ by

$$
\begin{equation*}
T_{\langle\infty\rangle}(x, y)=\eta_{\infty}\left(T\left(\eta_{\infty}^{-1}(x), \eta_{\infty}^{-1}(y)\right)\right) \tag{13}
\end{equation*}
$$

2. For $x \in] 0,1]$ set $n_{0}(x)=0, x_{0}=x$. Define recursively

$$
\begin{aligned}
n_{i}(x) & =\left\lfloor\log _{a_{i}}\left(x_{k, i-1}\right)\right\rfloor, \\
x_{i} & =\varphi_{a_{i}}\left(\frac{x_{i-1}}{a_{i}^{n_{i}(x)}}\right)
\end{aligned}
$$

for $i \in \mathbb{N}, i>0$. Define $\phi_{n_{k}}^{\infty}=\lim _{k=1}^{\infty} \phi_{n_{1}, n_{2}, \ldots, n_{k}}$, and a binary operation $T_{\langle\infty\rangle}$ on $\left.] 0,1\right]$ by

$$
\begin{equation*}
T_{\langle\infty\rangle}(x, y)=\phi_{n_{i}(x) \oplus_{i} n_{i}(y)}^{\infty}(1) \tag{14}
\end{equation*}
$$

3. Let $T$ be an arbitrary left-continuous $t$-norm, and define $T_{\langle i\rangle}$ for $i \in \mathbb{N}$ by (11). Let $T_{\langle\infty\rangle}$ be a binary operation on 10, 1] given by

$$
\begin{equation*}
T_{\langle\infty\rangle}(x, y)=\lim _{i=1}^{\infty} T_{\langle i\rangle}(x, y) \tag{15}
\end{equation*}
$$

i. The three definitions for $T_{\langle\infty\rangle}$ given in 1, 2 and 3. are equivalent.
ii. $T_{\langle\infty\rangle}$ is a strictly increasing, left-continuous $t$-norm without zero divisors.
iii. If $\oplus_{i}=\oplus_{1}$ for $i \in \mathbb{N}, i>0$ then $\left.T_{\langle\infty\rangle}\right|_{]_{\left.\alpha_{1}, 1\right]}}$. is order-isomorphic to $T_{\langle\infty\rangle}$.

Remark 6 It is clear from (15) that consecutive applications of Corollary 1 together with pointwise limit can result in all the t-norms, which can be generated by Theorem 9.

Motivated by Theorems 8 and 9 we shall present further examples together with their 3D plots. We will use the notations introduced until here without making reference to them; but instead of the short notation $T_{\langle k\rangle}$ sometimes we shall use $T_{\left\langle\oplus_{k}, \ldots, \oplus_{1}\right\rangle}$.


Figure 7: 3D plots of $\left(T_{\mathbf{P}}\right)_{\langle+\rangle}$and $\left(T_{\mathbf{P}}\right)_{\langle+,+\rangle}$


Figure 8: 3D plots of $\left(T_{\mathbf{M}}\right)_{\langle+\rangle}$(left) and $\left(T_{\mathbf{o s}}\right)_{\langle+\rangle}$(right)

Example 5 Let $T_{M}$ stands for the minimum operation on $[0,1]$. Define an ordinal sum with one Łukasiewicz summand as follows:

$$
T_{\mathbf{o s}}(x, y)=\left\{\begin{array}{l}
\frac{2}{9}+\frac{5}{9} \cdot \max \left(0, \frac{x-\frac{2}{9}}{\frac{5}{9}}+\frac{y-\frac{2}{9}}{\frac{5}{9}}-1\right) \\
\text { if } x, y \in\left[\frac{2}{9}, \frac{5}{9}\right] \\
\min (x, y) \text { otherwise }
\end{array}\right.
$$

For the 3D plots of $\left(T_{\mathbf{M}}\right)_{\langle+\rangle}$and $\left(T_{\mathbf{o s}}\right)_{\langle+\rangle}$see Fig. 8.
Example 6 Let the operation $\oplus_{\mathrm{x}}$ on $\mathbb{N}$ be given by $x \oplus_{\mathrm{x}} y=(x-1) \cdot(y-1)+1$. The graphs of $\left(T_{\mathbf{P}}\right)_{\left\langle\oplus_{\mathrm{x}}\right\rangle}$ and $\left(T_{\mathbf{P}}\right)_{\left\langle\oplus_{\mathbf{x}}, \oplus_{\mathbf{x}}\right\rangle}$ are presented in Fig. 9 .


Figure 9: $\left(T_{\mathbf{P}}\right)_{\left\langle\oplus_{\mathbf{x}}\right\rangle}$ and $\left(T_{\mathbf{P}}\right)_{\left\langle\oplus_{\mathbf{x}}, \oplus_{\mathbf{x}}\right\rangle}$

Example 7 For the sake of completeness we remark that the left-continuous t-norm which is introduced by Smutna [21] (based on the original idea of Budinčevič and Kurilič [1]) can be constructed by Theorem 9.

## 8 Conclusion

This paper features the presently existing methods that construct left-continuous t-norms. Some of them has the additional advantage that the induced negation of the resulted t-norm is strong, which may be useful in logical


Figure 10: $\left(T_{\mathbf{P}}\right)_{\left\langle+, \oplus_{\mathbf{x}}\right\rangle}$ and $\left(T_{\mathbf{P}}\right)_{\left\langle\oplus_{\mathbf{x}},+\right\rangle}$
applications. By using these methods (consecutive combination of them is as well possible) an infinite number of new left-continuous t-norms can be generated. The resulted operations can be admitted into the attention of researchers of algebra, probabilistic metric spaces, non-classical measures and integrals, non-classical logics, fuzzy theory and its applications.

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[^1]:    ${ }^{1}$ Here sum may be understood formally, think e.g. to vectors with countably infinite integer coordinates ( $n_{1}, n_{2}, \ldots$ ) equipped with the lexicographical order.

