# Fuzzy Truth Functions

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Abstract. *In this paper we introduce and study fuzzy truth functions as a generalization of logical functions. We can use this generalization for every logical function and the generalized special functions as fuzzy disjunction, fuzzy conjunction and fuzzy negation have the usual properties (commutativity, associativity, nondecreasivity and nonincreasivity, involution, resp.). The fuzzy truth functions are characterized as polynomial functions being linear for each of their variables*.

Keywords: fuzzy truth function, linear polynomial for each variables.

## 1. Introduction

It is known that the fuzzy system theory is developed to define and solve problems concerning uncertain information [Za]. A fuzzy subset **A** of a nonempty set **X** is characterized by its membership function A:**X**→[0,1] and A(x) is interpreted as the degree of membership of element x in fuzzy set **A** for each x∈**X**. Originally the fuzzy set theory was based on the fuzzy disjunction, fuzzy conjunction and fuzzy negation as membership functions max(A,B), min(A,B) and 1-A, resp.

Several generalizations of these connectives preserving the main properties e.g. commutativity, associativity, monotonity and boundary conditions were given. The appropriate definition of connectives (conjunction, disjunction, negation, etc.) is a basic problem in fuzzy logic and its applications. An axiomatic system for basic functions was given in in [BG]. Earlier a similar system was investigated in [SS] (and by Menger in 1942). These triangular norms (t-norms and t-conorms or s-norms) were considered as solutions of a system of functional equations and inequalities. In the remainder part of this paper denote I the closed unit interval I:=[0,1] and $I^m$ the m-times Cartesian product : $I^m:=I\times I\times\ldots\times I$ (m is a positive integer).

A function t:I×I→I is a *t-norm* if it is commutative, associative, nondecreasing and t(x,1)=x for all x∈I. A function s: I×I→I is a *t-conorm* or *s-norm* if it is commutative, associative nondecreasing and s(x,0)=x for all x∈I.

By Bellman and Giertz, the *fuzzy conjunction $t_0$* and *the fuzzy disjunction $s_0$* are derived from the t-norm and the s-norm such a way, that the *boundary conditions* are

$t_0(1,1)=1$, $t_0(0,0)=t_0(0,1)(=t_0(1,0))=0$ and

$s_0(0,0)=0$, $s_0(0,1)(=s_0(1,0))=s_0(1,1)=1$

instead of t(x,1)=x, and s(x,0)=x, resp. If the idempotency and continuity are required as axioms for these functions too, then $t_0$ and $s_0$ are the unique min and max, resp. [KF,Ba].

Recently several generalizaton of t-norm and s-norm are introduced. The uninorms and nullnorms are investigated in [FYR], the I-fuzzy system is defined and investigated in [Ko], and the distance-based functions are introduced and investigated in [Ru].

The third base function n:I→I is a generalization of the *fuzzy negation* if n is a nonincrease function and satisfying the boundary conditions n(0)=1, n(1)=0. This n is a *strict negation*, if it is decreasing and continuous. The function n is a *strong negation* if it is strict and involutive: n(n(x))=x for all x∈I.

By generalizations the boundary conditions are necessary being they the truth table of the functions conjunction, disjunction and negation, resp. The important properties of truth functions as theorems in the crisp theory can be used to define fuzzy operations. But sometimes another properties of truth functions are more interesting.

Axiomatic skeletons for three fuzzy functions were given as definition of fuzzy function of type negation, disjunction and conjunction. This is not a suitable way to define fuzzy generalizations of logical functions since these function types have only the boundary conditions as common property. Often it is reasonable requirement the function is continuous and representable with a formula similar to the formula representing the logical function in question. In this paper it is investigated the class of functions having representations similar to the *canonical disjunctive normal form of truth functions.*

## 2. Some properties of truth functions

For positive integer m, the m-places truth functions are the functions of type $f:\mathbf{B}^m \to \mathbf{B}$, where $\mathbf{B}$ is the two-element set $\{0,1\}$ and $\mathbf{B}^m := \mathbf{B}\times\ldots\times\mathbf{B}$ is the m-times Cartesian product. It is well known [Ád], that every truth function is representable by using the truth functions negation *n*, conjunction *k*, disjunction *d*, projection $e_2^2$ and composition of functions in finite times. The mentioned truth functions are defined by real polynoms as follows:

$n(x):=1-x$, $k(x,y):=xy$, $d(x,y):=x+y-xy$, $x,y\in\mathbf{B}$,

$e_i^m(x_1,x_2,\ldots,x_m):=x_i$ $(1\le i\le m)$, $x_1,x_2,\ldots,x_m\in\mathbf{B}$.

As the functions disjunctions and conjunctions have the property commutativity and associativity, it can be defined by composition their m-places variant, e.g. $d_m(x_1,\ldots,x_m):=d(d_{m-1}(x_1,\ldots,x_{m-1}),x_m)$, $m\ge 2$, $d_2=d$. For the seak of clearity, it is introduced the following two notations: $x^i := e_{2-i}^2(x,n(x))$, $i=0,1$.(These are the functions n and $e_1^1$, with uniform notation. The last function is n(n).) The mentioned representation is presented as follows.

*Statement.*Let m be a positive integer. Every m-places truth function f can be given (in a unique way) in the following form (canonical disjunctive normal form, **CDNF**):

$f(x_1,\ldots,x_m)=d(k(x_1^{a_1},\ldots,x_m^{a_m},f(a_1,\ldots,a_m))|(a_1,\ldots,a_m)\in\mathbf{B}^m)$.

The *proof* is easy. Because $x^a=1$ if $x=a$ and $x^a=0$ else, so $x_1^{a_1}\ldots x_m^{a_m}=1$ if $(x_1,\ldots,x_m)=(a_1,\ldots,a_m)$ and $x_1^{a_1}\ldots x_m^{a_m}=0$ else. Therefore the value of only one term of the disjunction differs from 0 by arbitrary place $(b_1,\ldots,b_m)\in\mathbf{B}^m$ and this value is $f(b_1,\ldots,b_m)$. □

*Remark.* It is easy to see that this proof is valid, if it is written $\Sigma$ instead of disjunction d (and product instead of conjunction):

$f(x_1,\ldots,x_m)=\Sigma(\prod_{j=1}^m x_j^{a_j}\, f(a_1,\ldots,a_m)|(a_1,\ldots,a_m)\in\mathbf{B}^m)$.

*Example 1:* Let f be the equivalence function: f(0,0)=f(1,1)=1 and f(0,1)=f(1,0)=0.
According to the Remark we have:

$f(x,y)=x^0y^0f(0,0)+x^0y^1f(0,1)+x^1y^0f(1,0)+x^1y^1f(1,1)=x^0y^0+x^1y^1=(1-x)(1-y)+xy=1-x-y+2xy$.

## 3. Characterization of fuzzy truth functions

An extension of truth functions is defined, for which, similarly to the truth functions, the functions have the polynomial representation given in Remark. The extension means that the range and the domain of the functions are the unit interval I:=[0,1] and the Cartesian product $I^m$, resp.

The extended projection $e_1$ and negation n are fuzzy functions, because
$e_1(x)=xe_1(1)+x^0e_1(0)=x$, $x\in I$
$n(x)=xn(1)+x^0n(0)=x^0$, $x\in I$.

The two functions $x^a$ according to the two value of the parameter $a\in\{0,1\}$, is defined on the set I in the same way as was by truth functions, therefore $x^1=x$ and $x^0=1-x$, $x\in I$.

Due to investigations on infinite valued logics of Łukasiewicz and others, it is known that it does not exist finite set of functions of type $I^m\to I$ to generate all the functions of this type. But several interesting questions raise.

In furthers it is written "$I_m$-function" (sometimes simply I-function) instead of "function of type $I^m\to I$".

Let m be a positive integer. An $I_m$-function f is called *fuzzy truth function* (shortly: *FT-function*), if for every $(x_1,...,x_m)\in I^m$ satisfies:

$$f(x_1,...,x_m)=\Sigma\left(\prod_{j=1}^{m} x_j^{a_j} f(a_1,...,a_m)|(a_1,...,a_m)\in \mathbf{B}^m\right). \tag{1}$$

The value of variable $x_j$ can be interpreted as fuzzy-degree of the event that "the value of the $j^{th}$ variable is 1".

Denote $\alpha_m$ the *constant $I_m$-function* with value $\alpha$:

for every $\alpha\in I$ and $(x_1,...,x_m)\in I^m$: $\alpha_m(x_1,...,x_m)=\alpha$.

Denote $\mathbf{LI}_m$ and $\mathbf{FT}_m$ the set of $I_m$-functions beeing linear polynomials of each their variable, and the set of FT-functions with m variables, resp. The m-places functions f and g are called *I-equivalent*, (notation: $f==^Ig$) if their I-restrictions are the same function: $f(x_1,...,x_m)=g(x_1,...,x_m)$, if $(x_1,...,x_m)\in I^m$. For example, the boundary conditions for $t_0$ and $s_0$ means, that $t_0$ and $s_0$ are **B**-equivalent to t and s , resp.

In the next theorem there are presented all FT-functions with one- and two-variables, and all constant FT-functions.

*Theorem 1.* (i) Every constant $I_m$-function (m=1,2,...) is FT-function.

(ii) The following three sets are pairwise I-equivalent:

(a) $\mathbf{L}_1:=\{\alpha+ß x|\alpha,\alpha+ß\in I\}$,

(b) $\mathbf{LI}_1$,

(c) $\mathbf{FT}_1$.

(iii) The following three sets are pairwise I-equivalent:

(a) $\mathbf{L}_2:=\{\alpha+ß_1x_1+ß_2x_2+\gamma x_1x_2|\{\alpha,\alpha+ß_1,\alpha+ß_2,\alpha+ß_1+ß_2+\gamma\}\subseteq I\}$,

(b) $\mathbf{LI}_2$,

(c) $\mathbf{FT}_2$.

The proof of theorem needs two lemmas.

*Lemma 1*: The functions satisfying condition (1) are linear polynomials of each their variable.

*Proof:* Each term of the right side of equality (1) is a product such that their factor depending on $x_j$ is either $x_j$ or $1-x_j$, $j=1,2,...,m$. □

*Lemma 2*: $\Sigma(\prod_{j=1}^{m}\left(x_j^{a_j}\right)|(a_1,...,a_m)\in\mathbf{B}^m)={}^{I}1_m(x_1,...,x_m)$.

*Proof*: We use induction on m. In the case m=1 the statement is clear: $x^0+x^1=(1-x)+x={}^{I}1_1(x)$.

Suppose that the statement is true for a positive integer m. Proving that it is true for the number m+1 can be presented as follows:

$$\Sigma(\prod_{j=1}^{m+1}\left(x_j^{a_j}\right)|(a_1,...,a_{m+1})\in\mathbf{B}^{m+1})=\Sigma((x_{m+1}^{0}+x_{m+1}^{1})\prod_{j=1}^{m}\left(x_j^{a_j}\right)|(a_1,...,a_m)\in\mathbf{B}^m)$$

$$={}^{I}1_1(x_{m+1})\Sigma\prod_{j=1}^{m}\left(x_j^{a_j}\right)|(a_1,...,a_m)\in\mathbf{B}^m)={}^{I}1_1(x_{m+1})1_m(x_1,...,x_m)=1_{m+1}(x_1,...,x_{m+1}).$$ □

*Proof of Theorem 1*. (i) Substitute $\alpha$ for $f(a_1,...,a_m)$ at the right side of the equality (1) and use Lemma 2: $\Sigma(\prod_{j=1}^{m}\left(x_j^{a_j}\right)\alpha|(a_1,...,a_m)\in\mathbf{B}^m)={}^{I}\alpha 1_m(x_1,...,x_m)=\alpha_m(x_1,...,x_m)$.

(ii) $\mathbf{L}_1\subseteq\mathbf{LI}_1$, because $\alpha$, $\alpha+ß\in I$ implies: the function $\alpha+ßx$ is monotone nondecreasing and decreasing, if $ß\geq 0$ and $ß<0$, respectively, therefore their maximal and minimal values are $\alpha+ß$, $\alpha$ and $\alpha$, $\alpha+ß$, resp.

Proof of $\mathbf{LI}_1\subseteq\mathbf{FT}_1$. It is sufficient to prove that every linear function $f(x)=\alpha+ßx$ satisfy the equality (1):

$\Sigma(x^a f(a)|a\in\mathbf{B})=x^0\alpha+x^1(\alpha+ß)=(1-x)\alpha+x(\alpha+ß)=\alpha+ßx=f(x)$.

Proof of $\mathbf{FT}_1\subseteq\mathbf{L}_1$. It is sufficient to prove that every 1-place function g satisfying (1) is a linear one. But $\Sigma(x^a g(a)|a\in\mathbf{B})=x^0g(0)+x^1g(1)=(1-x)g(0)+xg(1)=\alpha+ßx$, where $\alpha:=g(0)$ and $ß:=g(1)-g(0)$.

(iii) Proof of $\mathbf{L}_2\subseteq\mathbf{LI}_2$. Clearly, the elements of $\mathbf{L}_2$ are linear polynomials of both variables. For a fixed $x_0\in I$, the one variables polynomial $(\alpha+\beta_1x_0)+(\beta_2+\gamma x_0)y$ is I function by (ii) if and only if $\{\alpha+\beta_1x_0, (\alpha+\beta_1x_0)+(\beta_2+\gamma x_0)\}\subseteq I$. Using again the statement (ii) for polynomials $\alpha+\beta_1x$ and $(\alpha+\beta_1x)+(\beta_2+\gamma x)=(\alpha+\beta_2)+(\beta_1+\gamma)x$, the conditions for coefficients is resulted as it is given in (a).

Proof of $\mathbf{LI}_2\subseteq\mathbf{FT}_2$. The elements of both sets are I-functions. Arbitrary element of $\mathbf{LI}_2$ is an FT-function, since $\mathbf{LI}_2\ni\alpha+\beta_1x+\beta_2y+\gamma xy=(1-x)(1-y)f(0,0)+(1-x)yf(0,1)+x(1-y)f(1,0)+xyf(1,1)=$ $=f(x,y)\in\mathbf{FT}_2$, where $f(0,0)=\alpha$, $f(0,1)=\alpha+\beta_2$, $f(1,0)=\alpha+\beta_1$, $f(1,1)=\alpha+\beta_1+\beta_2+\gamma$.

Proof of $\mathbf{FT}_2\subseteq\mathbf{L}_2$. Let $f(x,y)\in\mathbf{FT}_2$. By the previous equality, $f(x,y)=\alpha+\beta_1x+\beta_2y+\gamma xy$, where $\alpha=f(0,0)$, $\beta_1=f(1,0)-f(0,0)$, $\beta_2=f(0,1)-f(0,0)$, $\gamma=f(1,1)+f(0,0)-f(0,1)-f(1,0)$. F is I-function, so the conditions of (a) is fulfilled.

This cyclic inclusions among the three sets yields that the sets are pairwise equal. □

The general case is more complicated, of course. For fixed positive integer m denote $\sum_m\left(\bullet\,|\,i_1<i_2<...<i_k\right)$ the summation of $(\bullet)$ for all k-element subset of the index-set $\{1,2,\ldots,m\}$. For example, in case m=4 and k=2,
$\sum_4\left(\alpha_{i_1i_2}x_{i_1}x_{i_2}|\,i_1<i_2\right)=\alpha_{12}x_1x_2+\alpha_{13}x_1x_3+\alpha_{14}x_1x_4+\alpha_{23}x_2x_3+\alpha_{24}x_2x_4+\alpha_{34}x_3x_4$, and
$\sum_4\left(\alpha_{1234}x_1x_2x_3x_4\right)=\alpha_{1234}x_1x_2x_3x_4$. Let h be an m-places, linear polynomial function of each variable with real coefficients, that is

$$h(x_1,\ldots,x_m)=\alpha_0+\sum_m\left(\alpha_i x_i\,|\,i\right)+\sum_m\left(\alpha_{i_1i_2}x_{i_1}x_{i_2}|\,i_1<i_2\right)+\ldots$$
$$+\sum_m\left(\alpha_{i_1\ldots i_k}x_{i_1}\ldots x_{i_k}|\,i_1<\ldots<i_k\right)+\ldots+\alpha_{12\ldots m}x_1x_2\ldots x_m.$$

Denote $\mathbf{H}_m$ the set of these m-places polynomials.

*Lemma 3*. For arbitrary $h \in \mathbf{H}_m$, $h(x_1,\ldots,x_m)=(1-x_1)h(0,x_2,\ldots,x_m)+x_1 h(1, x_2,\ldots x_m)$.

*Proof*. Clearly, for each elementh of $\mathbf{H}_m$ is valid: $h(x_1,\ldots,x_m)= r(x_2,\ldots,x_m) +x_1\ s(x_2,\ldots x_m)$, where $r,s \in \mathbf{H}_{m-1}$. Substituting $x_1=0$ and $x_1=1$, the equality yields
$r(x_2,\ldots,x_m)=h(0,x_2,\ldots,x_m)$ and
$s(x_2,\ldots x_m)=h(1,x_2,\ldots,x_m)-h(0,x_2,\ldots,x_m)$, resp. Therefore the statement is true. □

*Theorem 2*. The following three sets are pairwise I-equivalent:

(a) $\mathbf{L}_m:=\{h | h \in \mathbf{H}_m, \{h(a_1,\ldots,a_m) | (a_1,\ldots,a_m) \in B^m\} \subseteq I\}$,

(b) $\mathbf{LI}_m$,

(c) $\mathbf{FT}_m$.

The *proof* is similar to the proof of Theorem 1, but it needs induction on m. The Lemma 1 and Lemma 2 are presented in general case. □

# 4. References


Ád A. Ádám, Truth Functions. Hungarian Academic Press (1968).

Ba J. Bagyinszki, Fuzzy Logic. TEMPUS S JEP No 07759-94, 88p, (1996).

BB J. Bagyinszki, A. Bagyinszkiné-Orosz, On R-fuzzy Petri nets and their languages. Conf. on System Theoretical Aspects in Computer Science, DM-82, 30-41, (1982).

BG R. Bellman, M. Giertz, On the analytic formalism of the theory of fuzzy sets. Information Sciences 5, 149-156, (1973).

DL J. Dombi, Gy. Lencsés, On the Boolean structure of fuzzy logical systems: a computer example. Acta Cybernetica 10/4, 317-322, (Szeged 1992).

FYR J. Fodor, R. Yager, A. Rybalov, Structure of Uninorms. Internat. J. Uncertainty, Fuzzyness and Knowledge Based Systems 5, 411-427, (1997).

Fu R. Fullér, Introduction to Neuro-Fuzzy Systems. Physica-Verlag, Heidelberg, 289p, (2000).

KF G. J. Klir, T. A. Folger, Fuzzy Sets. Uncertainty and Information, Prentice Hall, Englewood Cliffs (1988).

Ko L.T. Kóczy, I-fuzzy structure: the world of strictly monotonous norms. in:Wang and Loe, Between Mind and Computer: Fuzzy Science and Engineering, World Sci. 105-137, (1993).

Ru I. J. Rudas, Distance-based Evolutionary Operations. Proc. 1st Internat. Symp. of Hungarian Researchers, Computational Intelligence, Budapest Polytechnic, 125-139, (2000).

SS B. Schweizer, A. Sklar, Probabilistic Metric Spaces. North-Holland (1983).

Za L.A. Zadeh, Fuzzy sets. Information and Control 8, 338-353, (1965).