Disjunctive and Conjunctive Normal Forms in Fuzzy Logic

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Abstract

When performing set-theoretical operations, such as intersection and union, on fuzzy sets, one can opt not to consider exact formula, but to leave the results less specific (in particular, interval-valued) by using both disjunctive and conjunctive representations (normal forms) of the underlying logical operations. We investigate which De Morgan triplets are suitable for this transformation (i.e. really yield intervals) and reveal the importance and unique role of the Łukasiewicz-triplet in the theory of fuzzy normal forms. To conclude we extend binary fuzzy normal forms to higher dimensions.

Keywords: Fuzzy normal form, Interval-valued fuzzy set.

1 Introduction

In many cases, crisp models are too 'poor' to represent the 'human way of thinking'. Fuzzy sets provide a widely accepted solution to that end. Typical to fuzzy set theory is the large set of options (logical operations, shapes of membership functions, parameters) that are available to the user. A unique and definite definition of the intersection of two fuzzy sets, for instance, cannot be expected. Because of this and other reasons, Türkşen proposes to leave the result of such logical operations unspecified, to some extent, by drawing upon the theory of fuzzy normal forms. In this way, he created interval-valued fuzzy sets [6]. But is this way of generating interval-valued fuzzy sets meaningful?

In Section 2, the generalization from Boolean to fuzzy normal forms is briefly recalled. Some results of Bilgiç [2] are disclaimed and some new properties concerning the relationship between the fuzzy normal forms are stated. In Section 3, we investigate which De Morgan triplet is the most appropriate to construct interval-valued fuzzy sets and give a new characterisation of the L-triplet. So far we dealt with binary fuzzy normal forms only. Section 4 extends this notion to higher dimensions. We give some examples and characterisations of De Morgan triplets for which the difference between the fuzzy normal forms is independent of the Boolean function. A summary and some suggestions of further research are given in Section 5.

Before we start our study, we first fix some notations concerning De Morgan triplets. For a t-norm T, a t-conorm S and two negators N_1 and N_2 the two laws of De Morgan are given by

$$N_1(S(x,y)) = T(N_1(x), N_1(y))\,, \tag{1}$$

$$N_2(T(x,y)) = S(N_2(x), N_2(y))\,. \tag{2}$$

We say that $\langle T, S, N\rangle$ is a De Morgan triplet if N is a strict negation and (1) is satisfied with $N_1 = N$. Let ϕ be an automorphism of the unit interval and N be the standard negator, then the triplets[1] $\langle (T_{\mathrm{M}})_\phi\,, (S_{\mathrm{M}})_\phi\,, N_\phi\rangle$, $\langle (T_{\mathrm{P}})_\phi\,, (S_{\mathrm{P}})_\phi\,, N_\phi\rangle$, $\langle (T_{\mathrm{L}})_\phi\,, (S_{\mathrm{L}})_\phi\,, N_\phi\rangle$, $\langle (T_{\mathrm{D}})_\phi\,, (S_{\mathrm{D}})_\phi\,, N_\phi\rangle$ and $\langle (T^{n\mathrm{M}})_\phi, (S^{n\mathrm{M}})_\phi, N_\phi\rangle$[2] will be called respectively (M, ϕ)-, (P, ϕ)-, (L, ϕ)-, (D, ϕ)- and (nM, ϕ)-triplets. In case ϕ is the identity mapping, we talk about the M-, P-, L-, D- and nM-triplet.

Table 1: Boolean normal forms

No	$D_B = C_B$
1	$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y') = 1$
2	$0 = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y')$
3	$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) = x \vee y$
4	$x' \wedge y' = (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y')$
5	$(x' \wedge y) \vee (x \wedge y') \vee (x' \wedge y') = x' \wedge y'$
6	$x \wedge y = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y)$
7	$(x \wedge y) \vee (x' \wedge y) \vee (x' \wedge y') = x' \vee y$
8	$x \wedge y' = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y')$
9	$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y') = x \vee y'$
10	$x' \wedge y = (x \vee y) \wedge (x' \vee y) \wedge (x' \vee y')$
11	$(x \wedge y) \vee (x' \wedge y') = (x \vee y') \wedge (x' \vee y)$
12	$(x \wedge y') \vee (x' \wedge y) = (x \vee y) \wedge (x' \vee y')$
13	$(x \wedge y) \vee (x \wedge y') = (x \vee y) \wedge (x \vee y')$
14	$(x' \wedge y) \vee (x' \wedge y') = (x' \vee y) \wedge (x' \vee y')$
15	$(x \wedge y) \vee (x' \wedge y) = (x \vee y) \wedge (x' \vee y)$
16	$(x \wedge y') \vee (x' \wedge y') = (x \vee y') \wedge (x' \vee y')$

[1] Note: we use notations from [4]

[2] By T^{nM} we mean the nilpotent minimum introduced by Fodor [3].

2 Fuzzy normal forms of binary Boolean functions

In a Boolean algebra every function can be represented by its disjunctive (D_B) and conjunctive (C_B) normal form. Although they are identical, both forms are computed in a different way.

Consider the Boolean algebra $B(\{0,1\}, \vee, \wedge, ')$. The disjunctive and conjunctive normal forms of an n-ary Boolean function f are given by

$$D_B(f)(x_1, ..., x_n) = \bigvee_{f(e_1,...,e_n)=1} x_1^{e_1} \wedge ... \wedge x_n^{e_n} \tag{3}$$

and

$$C_B(f)(x_1, ..., x_n) = \bigwedge_{f(e_1,...,e_n)=0} x_1^{e_1'} \vee ... \vee x_n^{e_n'}, \tag{4}$$

where $x^e = x$ if $e = 1$ and $x^e = x'$ if $e = 0$.
When working with two variables only, the Boolean normal forms for the sixteen Boolean functions are given in Table 1.

We can fuzzify these definitions by replacing $(\wedge, \vee, ')$ by a triplet (T, S, N). The corresponding disjunctive and conjunctive normal forms are denoted by D_F and C_F.

Table 2: Disjunctive and Conjunctive fuzzy normal forms

No	D_F	C_F
1	$S[T(x,y), T(x,y^N), T(x^N,y), T(x^N,y^N)]$	1
2	0	$T[S(x,y), S(x,y^N), S(x^N,y), S(x^N,y^N)]$
3	$S[T(x,y), T(x,y^N), T(x^N,y)]$	$S(x,y)$
4	$T(x^N,y^N)$	$T[S(x,y^N), S(x^N,y), S(x^N,y^N)]$
5	$S[T(x,y^N), T(x^N,y), T(x^N,y^N)]$	$S(x^N,y^N)$
6	$T(x,y)$	$T[S(x,y), S(x,y^N), S(x^N,y)]$
7	$S[T(x,y), T(x^N,y), T(x^N,y^N)]$	$S(x^N,y)$
8	$T(x,y^N)$	$T[S(x,y), S(x,y^N), S(x^N,y^N)]$
9	$S[T(x,y), T(x,y^N), T(x^N,y^N)]$	$S(x,y^N)$
10	$T(x^N,y)$	$T[S(x,y), S(x^N,y), S(x^N,y^N)]$
11	$S[T(x,y), T(x^N,y^N)]$	$T[S(x,y^N), S(x^N,y)]$
12	$S[T(x,y^N), T(x^N,y)]$	$T[S(x,y), S(x^N,y^N)]$
13	$S[T(x,y), T(x,y^N)]$	$T[S(x,y), S(x,y^N)]$
14	$S[T(x^N,y), T(x^N,y^N)]$	$T[S(x^N,y), S(x^N,y^N)]$
15	$S[T(x,y), T(x^N,y)]$	$T[S(x,y), S(x^N,y)]$
16	$S[T(x,y^N), T(x^N,y^N)]$	$T[S(x,y^N), S(x^N,y^N)]$

The main point of study so far has been the relationship between D_F and C_F. The following claims can be found in [2]:

1. If (T, S, N) is a triplet such that T is a t-norm, S is a t-conorm and N is a negator, then $D_F(.)$ cannot be equal to $C_F(.)$.

2. Consider a De Morgan triplet $\langle T, S, N \rangle$, with involutive negator N. If one of the following inequalities

$$S[T(x,y), T(x,y^N)] \leq x\,, \tag{5}$$

$$S[T(x,y), T(x^N,y)] \leq x\,, \tag{6}$$

$$S[T(x,x^N), T(x^N,y)] \leq x\,, \tag{7}$$

holds for all $(x, y) \in [0, 1]^2$, then

$$S[T(x,y), T(x,y^N), T(x^N,y)] \leq S(x,y)\,, \tag{8}$$

$$S[T(x,y), T(x^N,y^N)] \leq T[S(x,y^N), S(x^N,y)]\,, \tag{9}$$

$$S[T(x,y), T(x,y^N)] \leq T[S(x,y), S(x,y^N)]\,, \tag{10}$$

are satisfied for all $(x, y) \in [0, 1]^2$.

3. If in a De Morgan triplet $\langle T, S, N \rangle$, T and S are continuous and N is an involutive negator, then $D_F \leq C_F$ for the sixteen Boolean functions.

Note that, when using an involutive negator N and a De Morgan triplet $\langle T, S, N \rangle$, the three inequalities (8)–(10) are equivalent to $D_F \leq C_F$.

Unfortunately in the last two claims above the author jumped into conclusions.

1. It is obvious that in the crisp case inequalities (6) and (7) are not true. In the fuzzy case, it indeeds hold that

$$(5) \quad \Longrightarrow \quad (8), (9), (10).$$

For every (M, ϕ)-, (P, ϕ)-, (L, ϕ)- and (D, ϕ)-triplet inequality (5) holds. Consequently $D_F \leq C_F$, when working with these particular De Morgan triplets. One easily verifies that for a (nM, ϕ)-triplet (8)–(10) hold, although (5) is false.

2. Consider the ordinal sum

$$T \approx \left(\left\langle 0, \frac{1}{3}, T_{\mathrm{P}} \right\rangle, \left\langle \frac{1}{3}, 1, T_{\mathrm{L}} \right\rangle \right).$$

Let N be the standard negator, then $\langle T, S, N \rangle$ fulfils the conditions of the third claim. However, as shown in Figure 1, there exists $(x, y) \in [0, 1]^2$ and a Boolean function f (the logical equivalence) such that

$$C_F(f)(x, y) < D_F(f)(x, y).$$

C. and E. Walker have even shown [7, 8] that $D_F \leq C_F$ does not hold for every nilpotent or every strict De Morgan triplet.

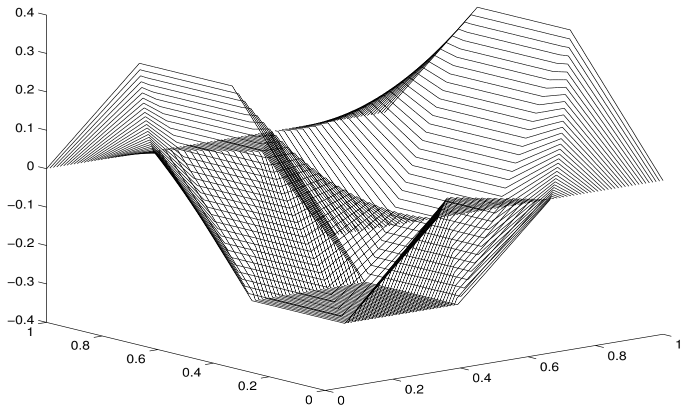

Figure 1: $C_F - D_F$ for row 11 of Table 2

Let us now focus on inequalities (8)–(10). The question arises whether some of these inequalities can be turned into equalities.

Proposition 1 *For any De Morgan triplet $\langle T, S, N\rangle$, the equality*

$$S[T(x,y), T(x,y^N), T(x^N,y)] = S(x,y)$$

cannot hold for all $(x,y) \in [0,1]^2$.

Proposition 2 *Let N be a negator that has a fixpoint, T an arbitrary t-norm and S an arbitrary t-conorm. Then the equality*

$$S[T(x,y), T(x,y^N)] = T[S(x,y), S(x,y^N)]$$

cannot hold for all $(x,y) \in [0,1]^2$.

Proposition 3 *Let N be a negator with fixpoint a, T a t-norm and S a t-conorm such that for all $x \in [0,a]$:*

$$\begin{cases} T(x,a) = \min(x,a)\,, \\ S(x,a) = \max(x,a)\,. \end{cases}$$

Then the equality

$$S[T(x,y), T(x^N,y^N)] = T[S(x,y^N), S(x^N,y)]$$

holds for all $(x,y) \in [0,1]^2$.

Note that these three propositions imply that, for a continuous De Morgan triplet, equality for all $(x,y) \in [0,1]^2$ cannot occur in (8) and (10). Moreover, if the unique fixpoint a of the negator is an idempotent element of T, then equality in (9) holds for all $(x,y) \in [0,1]^2$.

3 Interval-valued fuzzy sets

If a fuzzy set is constructed from other fuzzy sets by means of logical connectives for which $D_F \leq C_F$, Türkşen [6] proposes to create an interval-valued fuzzy set ($IVFS$)

$$IVFS(.) = [D_F(.), C_F(.)]$$

in order to model uncertainty concerning the exact membership degrees. We now would like to know whether this replacement is meaningful.

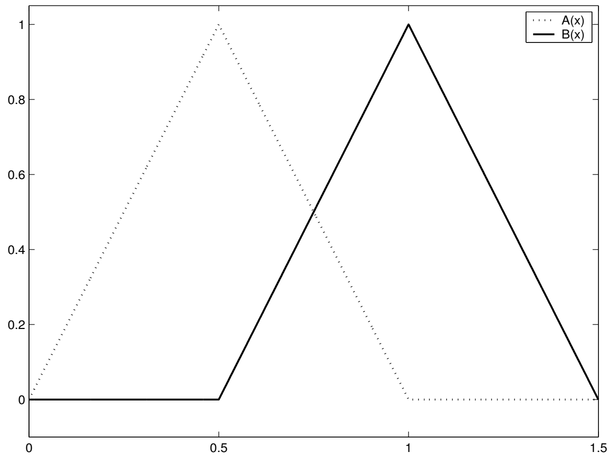

Figure 2: Fuzzy sets A and B on $[0, \frac{3}{2}]$

Consider the fuzzy sets A and B in Figure 2. The $IVFS$ of the intersection of both fuzzy sets is given by

$$IVFS(A \cap B)(x) =$$
$$[D_F(\wedge)(A(x), B(x)), C_F(\wedge)(A(x), B(x))],$$

for all $x \in [0, \frac{3}{2}]$. Figure 3 plots the disjunctive and conjunctive fuzzy normal forms of $A \cap B$ for respectively the M-, P- and L-triplet.
The upper and lower bounds of the $IVFS$ for the M- and P-triplet show a different convexity behaviour. It is clear that these t-norms are not suited for further use. In case of the L-triplet the disjunctive fuzzy normal form becomes zero due to the law of contradiction. One can wonder here wheter the constructed interval is really meaningful.

Let us now take a deeper look at the shape and width of the general interval $[D_F, C_F]$ constructed by (8), (9) and (10), when using the M-, P- and L-triplet.

From Figure 4 it follows that in case of the L-triplet the difference between the disjunctive and conjunctive normal form is always the same.

Theorem 1 *For the* L-*triplet*

$$C_F(f)(x, y) - D_F(f)(x, y) = 2 \min(1 - x, 1 - y, x, y),$$

for all $(x, y) \in [0, 1]^2$ *and any binary Boolean function* f.

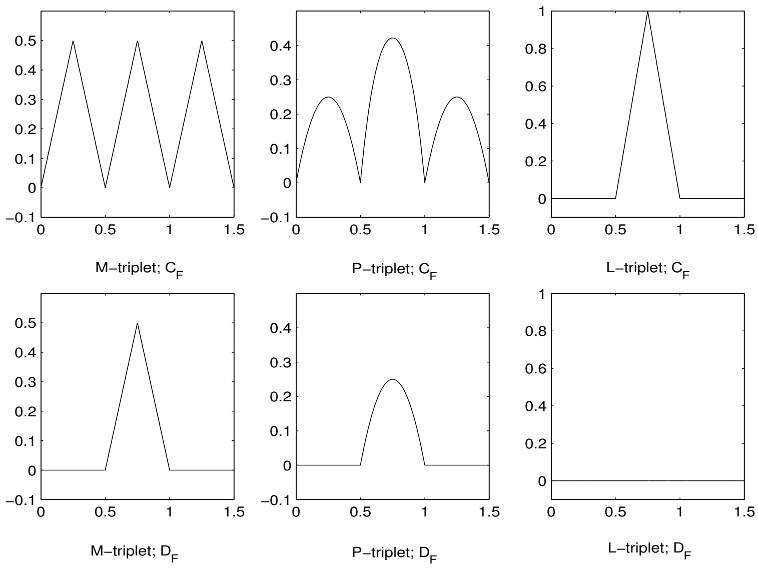

Figure 3: Disjunctive and conjunctive normal forms of $A \cap B$

Conversely, assuming that the difference between C_F and D_F is independent of the Boolean function f we get the following result.

Theorem 2 *Let $\langle T, S, N\rangle$ be a De Morgan triplet, N an involutive negator with fixpoint a and suppose that the diagonal δ_T is continuous on $]a, 1]$. If $C_F(f) - D_F(f)$ is independent of the binary Boolean function f, then N has to be the standard negator.*

If the t-norm in the foregoing theorem is continuous, we can even determine the De Morgan triplet in a unique way.

Theorem 3 *Let $\langle T, S, N\rangle$ be a continuous De Morgan triplet with involutive negator N. If $C_F(f) - D_F(f)$ is independent of the binary Boolean function f, then $\langle T, S, N\rangle$ must be the* L*-triplet.*

4 Fuzzy normal forms of n-ary Boolean functions

So far, all authors restrict themselves to normal forms of binary Boolean functions. For n-ary disjunctive and conjunctive normal forms we obtain 2^{2^n} different expressions (which depend on the n-ary Boolean function used). Because this large amount of normal forms is not easy to work with, we express D_F and C_F for a De Morgan triplet $\langle T, S, N\rangle$, with N an involutive negator, in the following natural way:

$$D_F(f)(\mathbf{x}) = S\{f(\mathbf{e})\ T(\mathbf{x}^{\mathbf{e}}) \mid \ \mathbf{e} \in \{0,1\}^n\}$$

$$C_F(f)(\mathbf{x}) = T\{[(1 - f(\mathbf{e}))\ T(\mathbf{x}^{\mathbf{e}})]^N \mid \ \mathbf{e} \in \{0,1\}^n\}$$

where $\mathbf{x} \in [0,1]^n$ and $\mathbf{x}^{\mathbf{e}} = (x_1^{e_1}, ..., x_n^{e_n})$.

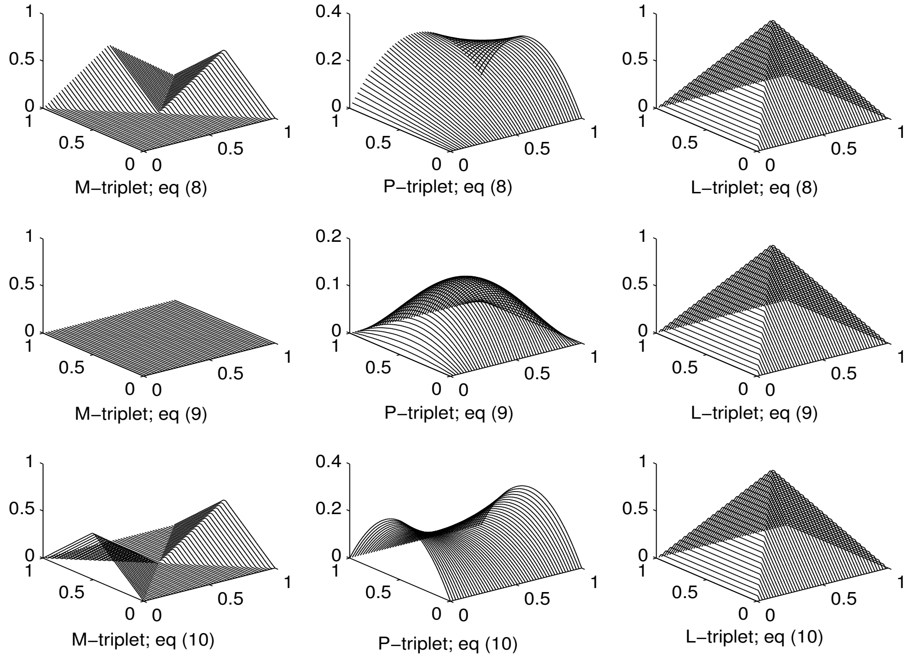

Figure 4: Difference between C_F and D_F

First of all we want to know whether $D_F \leq C_F$ still holds for all (M, ϕ)-, (P, ϕ)-, (L, ϕ)-, (D, ϕ)- and (nM, ϕ)-triplets.

Theorem 4 *For any* (M, ϕ)-, (P, ϕ)-, (L, ϕ)-, (D, ϕ)- *and* (nM, ϕ)*-triplet it holds that*

$$D_F(f) \leq C_F(f)$$

for any n-ary Boolean function f.

Similarly as in the 2-dimensional case, the L-triplet plays a prominent role in the general case.

Theorem 5 *For the* L*-triplet:*

$$C_F(f)(\mathbf{x}) - D_F(f)(\mathbf{x}) = 1 - \sum_{\mathbf{e} \in \{0,1\}^n} T_{\mathrm{L}}(\mathbf{x}^{\mathbf{e}}),$$

for all $\mathbf{x} \in [0,1]^n$ and any n-ary Boolean function f.

Note that for $n = 2$

$$2 \min(1-x, 1-y, x, y) = 1 - \sum_{\mathbf{e} \in \{0,1\}^2} T_{\mathrm{L}}(x^{e_1}, y^{e_2}).$$

Theorem 5 is therefore a generalization of Theorem 1.

When we work with n-ary Boolean functions, Theorems 2 and 3 extend in the following natural way.

Theorem 6 *Let $\langle T, S, N\rangle$ be a De Morgan triplet and N an involutive negator with fix-point a. If $C_F(f) - D_F(f)$ is independent of the n-ary Boolean function f, then:*

1. *If $T(x, ..., x)$ is continuous on $]a, 1]$, then N has to be the standard negator.*
2. *If T is continuous, then $\langle T, S, N\rangle$ is the* L*-triplet.*

We need to consider the separate cases in the theorem for it is not true that, if the difference between the disjunctive and conjunctive fuzzy normal form is independent of the Boolean function f, the De Morgan triplet used must be the L-triplet. The following theorems illustrate this statement.

Theorem 7 *For the* nM*-triplet,* $\mathbf{x} \in [0, 1]^n$, $I = \{1, ..., n\}$ *and an arbitrary n-ary Boolean function f it holds that:*

1. *If $(\forall i \in I)(x_i \neq \frac{1}{2})$, then there exists a unique n-tuple $\mathbf{q} \in \{0, 1\}^n$ such that $x_i^{q_i} < \frac{1}{2}$, $\forall i \in I$. It then holds that*

$$C_F(f)(\mathbf{x}) - D_F(f)(\mathbf{x}) = \begin{cases} \max_i(x_i^{q_i})\,, & \text{if } \max_i(x_i^{q_i}) = \max_{j\neq i}(x_j^{q_j})\,, \\ 0\,, & \text{else}\,. \end{cases}$$

2. *If $(\exists! i \in I)(x_i = \frac{1}{2})$, then*

$$C_F(f)(\mathbf{x}) - D_F(f)(\mathbf{x}) = 0\,.$$

3. *If $(\exists (i, j) \in I^2)(i \neq j \ \wedge \ x_i = x_j = \frac{1}{2})$, then*

$$C_F(f)(\mathbf{x}) - D_F(f)(\mathbf{x}) = 1\,.$$

It follows from Theorem 6 that the nM-triplet is the only (nM, ϕ)-triplet for which $C_F(f) - D_F(f)$ is independent of the n-ary Boolean function f.
Remark that, if the first condition of Theorem 6 is false, it is still possible for a De Morgan triplet with involutive negator, that $C_F(f) - D_F(f)$ does not depend on the Boolean function f used. The following theorem illustrates this claim.

Theorem 8 *For any* (D, ϕ)*-triplet,* $\mathbf{x} \in [0, 1]^n$, $I = \{1, ..., n\}$ *and any n-ary Boolean function f it holds that:*

$$C_F(f)(\mathbf{x}) - D_F(f)(\mathbf{x}) = \begin{cases} 1\,, & \text{if } (\exists (i, j) \in I^2)(i \neq j \ \wedge \ x_i, x_j \notin \{0, 1\})\,, \\ 0\,, & \text{else}\,. \end{cases}$$

Figure 5 shows the results of the last two theorems in the 2-dimensional case.

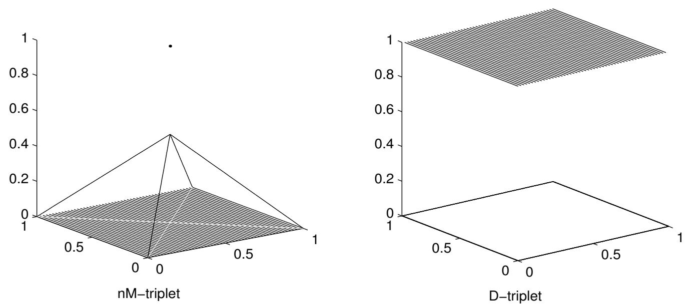

Figure 5: Difference between C_F and D_F for binary Boolean functions.

5 Conclusion and further research

In this paper we have investigated the relationship between the n-ary disjunctive and conjunctive fuzzy normal forms of Boolean functions. It seems that $D_F \leq C_F$ only holds in some cases. In particular, the inequality is fulfilled for every (M, ϕ)-, (P, ϕ)-, (L, ϕ)-, (D, ϕ)- and (nM, ϕ)-triplet.

When working with a continuous De Morgan triplet and an involutive negator, the difference between the conjunctive and disjunctive normal form will be independent of the Boolean function f if and only if we are dealing with the L-triplet.

It is interesting to know for which De Morgan triplets inequality (9) becomes an equality and for which non-continuous De Morgan triplets $C_F(f) - D_F(f)$ is only a function of the variable $\mathbf{x} \in [0, 1]^n$.

Once these problems are solved we can take a deeper look into the interval-valued preference structures, introduced by Bilgiç [2].

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