The opposite of quasi-triangular fuzzy number

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Abstract: This paper presents the geometrical interpretation of the quasi-triangular fuzzy number, being also shown that the opposite of quasi-triangular fuzzy number is a quasi-triangular fuzzy number with centre symmetrical to the origin and spread rotated with 90^0 .

1 Introduction

The concept of quasi-triangular fuzzy numbers generated by a continuous decreasing function g was introduced first by M. Kovács [4]. The shortage that not any quasi-triangular fuzzy number has opposite (inverse), but only the ones with spread zero, can be solved if quasi-triangular fuzzy numbers set is included isomorphically in an extended set and this extended set with addition operation forms a group. In section 3 this group is constructed using the addition operation over the class of quasi-triangular fuzzy numbers. The addition operation over the class of fuzzy numbers or fuzzy quantities was investigated and discussed e. g. in [2], [6], [7], [8], [9]. Section 4 presents the geometrical interpretation of the quasi-triangular fuzzy number.

2 Preliminaries

This section reviews the definitions and basic propositions applied in this paper.

Definition 2.1 (Fuzzy set). Let be X a set. A mapping $\mu : X \to [0, 1]$ is called *membership function*, and the set $A = \{ (x, \mu(x)) \ / \ x \in X\}$ is called *fuzzy set* on X. The membership function of A is denoted by μ_A . The collection of all fuzzy set on X is denoted by $F(X)$.

Triangular norms were introduced by K. Menger [10] and studied first by B. Schweizer and A. Sklar [11], [12], [13] to model distances in probabilistic metric spaces. In fuzzy sets theory triangular norms are extensively used to model the logical connection *and*.

Definition 2.2 (Triangular norm). A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a *triangular norm* if it is symmetric, associative, non-decreasing in each argument and $T(x, 1) = x$, for all $x \in [0,1]$.

Definition 2.3. A triangular norm T is said to be *Archimedean* if T is continuous and $T(x, x) < x$, for all $x \in [0,1]$.

Theorem 2.1 (C.H. Ling [5]). Every Archimedean triangular norm is representable by a continuous and decreasing function $g : [0, 1] \rightarrow [0, +\infty]$ with $g(1) = 0$ and $T(x, y) = g^{[-1]}(g(x)+ g(y))$, where

$$g^{[-1]}(x) = \begin{cases} g^{-1}(x) & \text{if } 0 \le x \le g(0), \\ 0 & \text{if } x > g(0). \end{cases}$$

Let $p \in [1, +\infty]$ and $g : [0, 1] \rightarrow [0, +\infty]$ be a continuos, strictly decreasing function with boundary properties $g(1) = 0$ and $\lim_{t\to 0} g(t) = g_0 \le +\infty$.

Definition 2.4 (Quasi-triangular fuzzy number). The *set of quasi-triangular fuzzy numbers* is

$$N_g = \{A \in F(\Re) \ / \text{ there is } a \in \Re \text{ and } d > 0 \text{ such that } \mu_A(t) = g^{[-1]}\left(\frac{|t-a|}{d}\right), \text{ for all } t \in \Re\}$$
$$\cup \{A \in F(\Re) \ / \text{ there is } a \in \Re \text{ such that } \mu_A(t) = \chi_{\{a\}}(t), \text{ for all } t \in \Re\},$$

where χ_A is characteristic function of the set A. The elements of N_g will be called *quasi- triangular fuzzy numbers* generated by g with *centre a* and *spread d* and we will denote them by < *a, d* >.

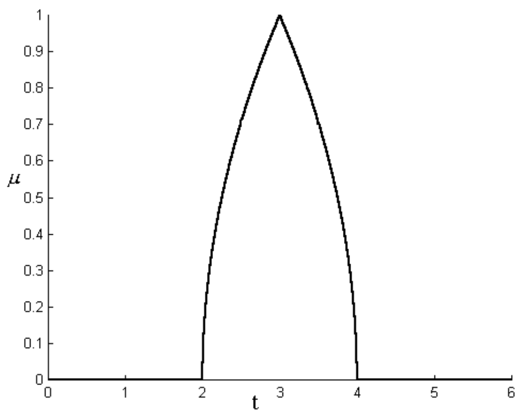

Fig.2.1. Quasi-triangular fuzzy number < 3, 1 > if $g(t) = 1 - t^2$.

Suppose A and B are fuzzy sets on $\Re$. If we are using *Generalized Zadeh's extension principle* [1] on Archimedean triangular norm T_{gp} generated by function g^p we get:

Definition 2.5 (T_{gp}-sum). If $p \in [1, +\infty)$, then T_{gp} *-sum* of $A, B \in F(\Re)$ is an fuzzy set on $\Re$ noted by A + B with membership function:

$$\mu_{A+B}(t) = \sup_{x+y=t}\left[g^{[-1]}\left(g^p(\mu_A(x)) + g^p(\mu_B(y)) \right)^{1/p} \right],$$

for all $t \in \Re$.

Definition 2.6 (T_{gp} -sum). If $p = +\infty$, then T_{gp} *-sum* of $A, B \in F(\Re)$ is an fuzzy set on $\Re$ noted by A + B with membership function:

$$\mu_{A+B}(t) = \sup_{x+y=t} \min\{\mu_A(x), \mu_B(y)\},$$

for all $t \in \Re$.

T. Keresztfalvi and M. Kovács in [3] proved the following theorems:

Theorem 2.2. Let $p \in [1, +\infty]$. If $< a, d >$ and $< b, e >$ are quasi-triangular fuzzy numbers, then

$$< a, d > + < b, e > = < a + b, (d^q + e^q)^{1/q} >,$$

where $\frac{1}{p} + \frac{1}{q} = 1.$

Theorem 2.3. $(N_g\ ,\ +\ )$ is a commutative monoid with element zero and if $p \in (1, +\infty]$, then it posses the simplification property.

3 Additive group of quasi-triangular fuzzy numbers

As follows from Theorem 2.3, the quasi-triangular fuzzy numbers do not form an additive group. This fact can complicate some theoretical considerations or applied procedures. This deficiency can be removed if the quasi-triangular fuzzy numbers set is included isomorphically in an extended set and this extended set with T_{gp}*-sum* forms an additive group. In this section we construct this group if $p > 1$.

As follows from the definition of T_{gp}*-Cartesian product,* the membership function of quasi-triangular fuzzy numbers pair ($< a, d >, < b, e >$) is

$$\mu_{(<a,d>,<b,e>)}(x, y) = T_{gp}\left(\mu_{<a,d>}(x), \mu_{<b,e>}(y)\right),$$

for all $(x, y) \in \Re \times \Re$. The set of all quasi-triangular fuzzy numbers pair we denote by Γ_{gp}.

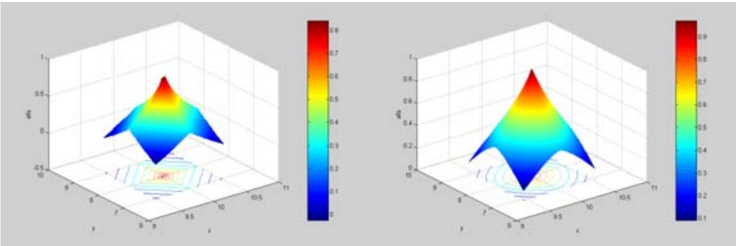

Fig. 3.1. The quasi-triangular fuzzy numbers pair $(<10, 1>, <8, 2>)$ if $g(t) = 1 - t$, $p = 1$ and $p = 1.5$ respectively.

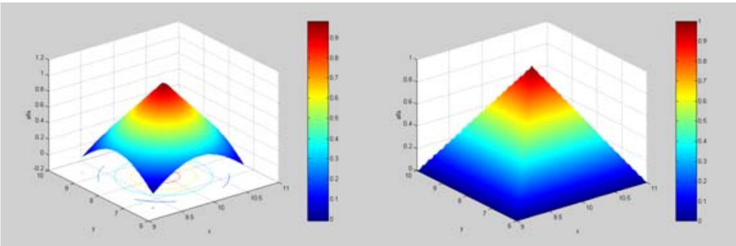

Fig. 3.2. The quasi-triangular fuzzy numbers pair $(<10, 1>, <8, 2>)$ if $g(t) = 1 - t$, $p = 2$ and $p = +\infty$ respectively.

Definition 3.1. Let $(<a_1, d_1>, <a_2, d_2>)$, $(<a_3, d_3>, <a_4, d_4>) \in \Gamma_{gp}$. Then we say that $(<a_1, d_1>, <a_2, d_2>)$ is *equivalent* to $(<a_3, d_3>, <a_4, d_4>)$, and write $(<a_1, d_1>, <a_2, d_2>) \sim (<a_3, d_3>, <a_4, d_4>)$ if

$$a_1 + a_4 = a_2 + a_3 \quad \text{and} \quad \left(d_1^{\,q} + d_4^{\,q}\right)^{1/q} = \left(d_2^{\,q} + d_3^{\,q}\right)^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1.$

It can be easily seen that $\sim$ is an equivalence relation. This relation introduces in Γ_{gp} a division on *equivalence class*. The *factor set* is

$$\Gamma_{gp}/\sim = \{\overline{(<a_1, d_1>, <a_2, d_2>)} \ / \ <a_1, d_1>, <a_2, d_2> \in \Gamma_{gp}\},$$

where

$$\overline{(<a_1,d_1>,<a_2,d_2>)} = \{(<a_3,d_3>,<a_4,d_4>) \ \ / \ <a_3,d_3>,<a_4,d_4> \in \Gamma_{gp}$$
$$\text{and } \ a_1+a_4 = a_2+a_3 \ \text{ and } \ \left(d_1{}^q + d_4{}^q\right)^{1/q} = \left(d_2{}^q + d_3{}^q\right)^{1/q}\}.$$

Definition 3.2. The addition operation in Γ_{gp} is defined by

$$\overline{(<a_1,d_1>,<a_2,d_2>)} \oplus \overline{(<a_3,d_3>,<a_4,d_4>)} =$$
$$\overline{(<a_1+a_3,\left(d_1^{\ q}+d_3^{\ q}\right)^{1/q}>,(<a_2+a_4,\left(d_2^{\ q}+d_4^{\ q}\right)^{1/q}>)},$$

for all $\overline{(<a_1,d_1>,<a_2,d_2>)}$, $\overline{(<a_3,d_3>,<a_4,d_4>)}$ $\ \Gamma_{gp}/\sim$.

Because the commutative monoid $(N_g\ ,+\ )$ possesses simplification property if $p > 1$, it follows that:

Theorem 3.1. If p > 1, then $(\,\Gamma_{gp}/\sim\ , \oplus\ )$ is an additive commutative group.

The *opposite* of $\overline{(<a_1,d_1>,<a_2,d_2>)}$ we denote by $\Theta\overline{(<a_1,d_1>,<a_2,d_2>)}$.

Proposition 3.2. If p > 1, then the function $F : N_g \to \Gamma_{gp}/\sim$,

$$F\left(<x,y>\right) = \overline{(<x,y>,<0{,}0>)}$$

is a homomorphism.

Proof. If $<x_1,y_1>,<x_2,y_2> \in N_g$ then

$$F\left(<x_1,y_1> + <x_2,y_2>\right) = F\left(<x_1+x_2,\left(y_1^{\ q}+y_2^{\ q}\right)^{1/q}>\right)$$
$$= \overline{(<x_1+x_2,\left(y_1^{\ q}+y_2^{\ q}\right)^{1/q}>,<0{,}0>)}$$
$$= \overline{(<x_1,y_1>,<0{,}0>)} \oplus \overline{(<x_2,y_2>,<0{,}0>)}$$
$$= F\left(<x_1,y_1>\right) \oplus F(<x_2,y_2>).$$

□

Theorem 3.3. $(N_g\ ,+\ )$ is isomorph to $(F(N_g\ ), \oplus\ )$.

Proof. If $<x_1,y_1>,<x_2,y_2> \in N_g$ two quasi-triangular fuzzy numbers, such that $F(<x_1,y_1>) = F(<x_2,y_2>)$, then $\overline{(<x_1,y_1>,<0{,}0>)} = \overline{(<x_2,y_2>,<0{,}0>)}$. From which it follows that $x_1 = x_2$ and $y_1 = y_2$. Consequently $(N_g\ ,+\ )$ is isomorph to $(F(N_g\ ), \oplus\ )$.

□

The consequence of Theorem 3.3 is that $\overline{(<x,y>,<0{,}0>)}$ is identical with $<x, y>$ if we consider the isomorphism in Theorem 3.3. Using this property we introduce the following notations:

Definition 3.3. We denote by $[x, y] = \overline{(<x,y>, <0,0>)}$ the quasi-triangular fuzzy number with centre x and spread y, and opposite of $[x, y]$ with $\Theta[x,y] = \overline{(<0,0>, <x,y>)}$.

Definition 3.4. If $p > 1$, then

$$\Omega_{gp} = \{[x,y] \ / <\mathrm{x},\mathrm{y}> \in N_g\} \cup \{\Theta[x,y] \ / <\mathrm{x},\mathrm{y}> \in N_g\}$$

is the *extended set of quasi-triangular fuzzy numbers*.

Theorem 3.4. If p > 1, then $\Omega_{gp} = \Gamma_{gp}/\sim$.

Proof. Let $\overline{(<x_1,y_1>, <x_2,y_2>)} \in \Gamma_{gp}/\sim$ such that $y_1 \geq y_2$. In this case we get:

$$\begin{aligned}\overline{(<x_1,y_1>, <x_2,y_2>)} &= \overline{(<x_1,y_1>, <0,0>)} \oplus \overline{(<0,0>, <x_2,y_2>)} \\ &= [x_1,y_1] \oplus (\Theta[x_2,y_2]) \\ &= \overline{(<x_1 - x_2, (y_1^q - y_2^q)^{1/q}>, <0,0>)} \\ &= [x_1 - x_2, (y_1^q - y_2^q)^{1/q}].\end{aligned}$$

Let $\overline{(<x_1,y_1>, <x_2,y_2>)} \in \Gamma_{gp}/\sim$ such that $y_1 < y_2$. In this case we get:

$$\begin{aligned}\overline{(<x_1,y_1>, <x_2,y_2>)} &= \overline{(<x_1,y_1>, <0,0>)} \oplus \overline{(<0,0>, <x_2,y_2>)} \\ &= [x_1,y_1] \oplus (\Theta[x_2,y_2]) \\ &= \overline{(<0,0>, <x_2 - x_1, (y_2^q - y_1^q)^{1/q}>)} \\ &= \Theta[x_2 - x_1, (y_2^q - y_1^q)^{1/q}].\end{aligned}$$

□

If we introduce the notation: $[x_1,y_1]\Theta[x_2,y_2] = [x_1,y_1] \oplus (\Theta[x_2,y_2])$, for all $[x_1,y_1],[x_2,y_2] \in \Omega_{\mathrm{gp}}$, from Theorem 3.4 it follows that:

Corollary 3.5. If p >1 then

(i) $[0,0]\Theta[x,y] = \Theta[x,y]$, for all $[x,y] \in \Omega_{\mathrm{gp}}$.

(ii) $\Theta(\Theta[x,y]) = [x,y]$, for all $[x,y] \in \Omega_{\mathrm{gp}}$.

(iii) $(\Theta[x_1,y_1]) \oplus (\Theta[x_2,y_2]) = \Theta([x_1,y_1] \oplus [x_2,y_2])$,

for all $[x_1,y_1],[x_2,y_2] \in \Omega_{\mathrm{gp}}$.

(iv) $[x_1, y_1]\Theta[x_2, y_2] = \begin{cases} [x_1 - x_2, (y_1{}^q - y_2{}^q)^{1/q}] & \text{if } y_1 \geq y_2, \\ \Theta[x_2 - x_1, (y_2{}^q - y_1{}^q)^{1/q}] & \text{if } y_1 < y_2, \end{cases}$

for all $[x_1, y_1], [x_2, y_2] \in \Omega_{gp}$.

4 Geometrical interpretation of quasi-triangular fuzzy numbers

In this section we show that the opposite of quasi-triangular fuzzy number is a quasi-triangular fuzzy number with centre symmetrical to the origin and spread rotated with 90^0.

Let $p \in (1, +\infty]$ be a number and $(< a_1, d_1 >, < a_2, d_2 >)$ be a quasi-triangular fuzzy numbers pair. We search for all a quasi-triangular fuzzy numbers pair $(< x_1, y_1 >, < x_2, y_2 >)$ that $(< x_1, y_1 >, < x_2, y_2 >) \in \overline{(< a_1, d_1 >, < a_2, d_2 >)}$. Immediately, it follows from the definition of relation ~, $(< x_1, y_1 >, < x_2, y_2 >) \in \overline{(< a_1, d_1 >, < a_2, d_2 >)}$ if only if the centres (x_1, x_2), (a_1, a_2) belong to the line $x_1 - x_2 = a_1 - a_2$ that is parallel with first bisector and $y_1{}^q + d_2{}^q = y_2{}^q + d_1{}^q$. The intersection point of line $x_1 - x_2 = a_1 - a_2$ with axis *Ox* is $(a_1 - a_2, 0)$ and with axis *Oy* is $(0, a_2 - a_1)$.

We denote by $c_1 = \left|d_1{}^q - d_2{}^q\right|^{1/q}$ the generalized focal length of pair $(< a_1, d_1 >, < a_2, d_2 >)$ and by $c_2 = \left|y_1{}^q - y_2{}^q\right|^{1/q}$ the generalized focal length of pair $(< x_1, y_1 >, < x_2, y_2 >)$. The large axis of $(< a_1, d_1 >, < a_2, d_2 >)$ is parallel to axis *Ox* if $d_2 < d_1$, and the large axis of $(< a_1, d_1 >, < a_2, d_2 >)$ is parallel to axis *Oy* if $d_1 < d_2$. The equality $y_1{}^q + d_2{}^q = y_2{}^q + d_1{}^q$ shows that the large axis of $(< x_1, y_1 >, < x_2, y_2 >)$ is parallel to large axis of $(< a_1, d_1 >, < a_2, d_2 >)$, and $c_1 = c_2$.

Let $[a, d] \in \Omega_{gp}$ be a fuzzy number such that $d > 0$. An element $(< a_1, d_1 >, < a_2, d_2 >)$ is in equivalence class $[a, d] \in \Omega_{gp}$ if and only if

$a = a_1 - a_2$, $d_1 > d_2$ and $d = (d_1{}^q - d_2{}^q)^{1/q}$, where a is the intersection point of line $x_1 - x_2 = a_1 - a_2$ with axis *Ox*, and *d* is the generalized focal length. Since in this case large axis of pair $(< a_1, d_1 >, < a_2, d_2 >)$ is parallel to axis *Ox* it follows that the centre of $[a, d]$ is (a , 0) and spread *d* points to axis *Ox*.

Let $\Theta[a, d] \in \Omega_{\text{gp}}$ be a fuzzy number such that d > 0. An element $(< a_1, d_1 >, < a_2, d_2 >)$ is in equivalence class $\Theta[a, d] \in \Omega_{\text{gp}}$ if and only if

$a = a_2 - a_1$, $d_2 > d_1$ and $d = (d_2^{\,q} - d_1^{\,q})^{1/q}$. Consequently, $-a$ is the intersection point of line $x_1 - x_2 = a_1 - a_2$ with axis *Ox*, and *d* is the generalized focal length. Since in this case $d_2 > d_1$, the large axis of pair $(< a_1, d_1 >, < a_2, d_2 >)$ is parallel to axis *Oy*. Consequently, the centre of $[a, d]$ is $(-a$, 0) and spread *d* is perpendicular on axis *Ox*.

Example 4.1. Let $g : [0,1] \to [0, +\infty)$, $g(x) = 1 - t^2$ be a function and $p = 2$ be a number. The membership function of quasi-triangular fuzzy number $[a, d]$ is

$$\mu(t,0) = \begin{cases} 0 & \text{if } t \le a - d, \\ \sqrt{1 - \dfrac{a}{d} + \dfrac{t}{d}} & \text{if } a - d < t \le a, \\ \sqrt{1 + \dfrac{a}{d} - \dfrac{t}{d}} & \text{if } a < t \le a + d, \\ 0 & \text{if } a + d < t, \end{cases}$$

and if $u \neq 0$, then $\mu(t, u) = 0$ for all $t \in \Re$.

The membership function of quasi – triangular fuzzy number $\Theta[a, d]$ is

$$\mu(0,u) = \begin{cases} 0 & \text{if } u \le -a - d, \\ \sqrt{1 + \dfrac{a}{d} + \dfrac{u}{d}} & \text{if } -a - d < u \le -a, \\ \sqrt{1 - \dfrac{a}{d} - \dfrac{u}{d}} & \text{if } -a < u \le -a + d, \\ 0 & \text{if } -a + d < u, \end{cases}$$

and if $t \neq 0$, then $\mu(t, u) = 0$ for all $u \in \Re$.

In Fig 4.1 it is presented that the opposite of the quasi-triangular fuzzy number is a quasi-triangular fuzzy number with centre symmetrical to the origin and spread rotated with 90^0 .

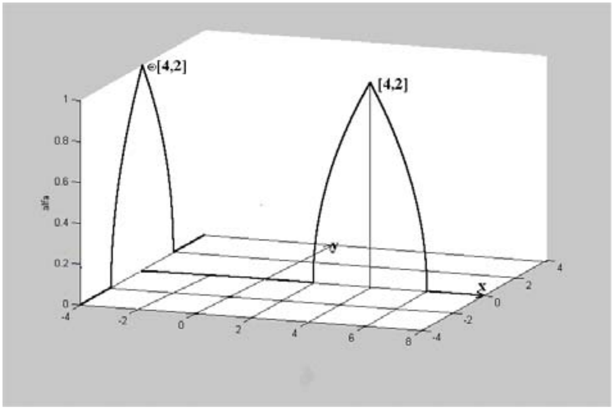

Fig 4.1. Quasi-triangular fuzzy numbers $[4,2]$ and $\Theta[4,2]$.

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