Applications of pseudo-analysis

Endre Pap
Institute of Mathematics and Informatics, 21 000 Novi Sad
Trg Dositeja Obradovića 4, Yugoslavia
e-mail: pape@eunet.yu, pap@im.ns.ac.yu

Abstract

There are presented some recent results and applications of pseudo-analysis, analogs in measure theory, integration, integral operators, convolution, Laplace transform. There are presented many applications in different fields as optimization, nonlinear differential and difference equations, economy, game theory, risk management, etc.

Keywords: semiring, pseudo-addition, pseudo-multiplication, pseudo-convolution, pseudo-Laplace transform, option pricing, large deviation principle.

1 Introduction

We want to stress here some of the advantages of the pseudo–analysis. There is covered with one theory and so with unified methods equations (usually nonlinear) from many different fields (system theory, optimization, control theory, differential equations, difference equations, decision making, etc.). Pseudo-analysis is based on the semiring structure on the real interval $[a,b] \subseteq [-\infty,\infty]$, [6, 9, 10, 11, 12, 13, 14]. We can stress three main problems which usually occurs in soft computing: uncertainty, nonlinearity and optimization. Namely, instead of the usual plus-product structure of real numbers a semiring structure on extended reals with respect to some other operations (pseudo-operations) is considered. For example, max-min, max-plus, max- product or operations generated by some additive generator g are included, and specially triangular conorm-triangular norm.

We present some parts of mathematical analysis in analogy with the classical mathematical analysis as measure theory, integration, integral operators, convolution, Laplace transform, etc. Many problems in fuzzy logic, fuzzy sets, neural nets, fuzzy-neural nets, multicriteria decision making, etc. can be treated by this mathematical tool. There are also many applications in different fields as optimization, nonlinear differential and difference equations, economy, game theory, risk management, [3, 4, 6, 9, 14, 16, 19].

2 Pseudo-operations

2.1 Semirings

Let [a,b] be a closed (in some cases semiclosed) subinterval of $[-\infty,+\infty]$. We consider here a total order $\leq$ on $[a,b]$ (although it can be taken in the general case a partial order). The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a,b] \times [a,b] \to [a,b]$ which is commutative, nondecreasing, associative and has a zero element, denoted by $\mathbf{0}$.

Let $[a,b]_+ = \{x : x \in [a,b], \ \ x \geq \mathbf{0}\}$.

The operation $\odot$ (*pseudo-multiplication*) is a function $\odot : [a,b] \times [a,b] \to [a,b]$ which is commutative, positively nondecreasing, i.e. $x \leq y$ implies $x \odot z \leq y \odot z, \ z \in [a,b]_+$, associative and for which there exist a unit element $\mathbf{1} \in [a,b]$, i.e., for each $x \in [a,b]$ we have $\mathbf{1} \odot x = x$.

We suppose, further, $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e.

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a,b],\oplus,\odot)$ is called a *semiring*. It can be considered a general algebraic structure $(P,\oplus,\odot)$ on an arbitrary set P endowed with the operations $\oplus$ and $\odot$ which satisfy the previously given conditions ([5]). In this paper we will consider only very special semirings ([6, 9]) with the following continuous operations:

Case I) $x\oplus y=\min\{x,y\}, \quad x\odot y=x+y$, on the interval $]-\infty,+\infty]$. We have $\mathbf{0}=+\infty$ and $\mathbf{1}=0$.

Case II) Semirings with pseudo-operations defined by monotone and continuous generator g ([9, 11]). In this case we will consider only strict pseudo - addition, i.e., such that the function $\oplus$ is continuous and strictly increasing in $(a,b)\times(a,b)$.

By representation theorem ([2, 11]) for each strict pseudo-addition $\oplus$ there exists a monotone function g (generator for $\oplus$) , $g:[a,b]\to[-\infty,\infty]$ (or with values in $[0,\infty]$) such $g(\mathbf{0})=0$ and

$$u\oplus v=g^{-1}(g(u)+g(v)).$$

Using a generator g of strict pseudo-addition $\oplus$, we can define pseudo-multiplication $\odot$:

$$u\odot v=g^{-1}(g(u)g(v)).$$

This is the only way to define pseudo-multiplication $\odot$, which is distributive with respect to $\oplus$ generated by the function g.

Case III Let $\oplus=\max$ and $\odot=\min$ on the interval $[-\infty,+\infty]$.

Example 1 The special cases on interval $[0,1]$, see [1, 2].

(i) A *triangular norm* (*t-norm* for short) is a binary operation $T:[0,1]^2\to[0,1]$ which is commutative, associative, monotone and 1 is the neutral element.

If T is a t-norm, then its *dual t-conorm* $S:[0,1]^2\to[0,1]$ is given by

$$S(x,y)=1-T(1-x,1-y).$$

(ii) A uninorm U (respectively t-norm T) is a commutative, associative, monotone binary operation on the unit interval $[0,1]$ and for some $0<e<1$ we have $U(x,e)=x$ (respectively $T(x,1)=x$) for all $x\in[0,1]$.

2.2 Semiring of functions

An idempotent semigroup (semiring, e.g., cases I and III form the preceding part) M is called an *idempotent metric semigroup (semiring)* if it is endowed with a metric $\rho\colon M\times M\to\mathbb{R}$ such that the operation $\oplus$ is (respectively, the operations $\oplus$ and $\odot$ are) uniformly continuous on any order-bounded set in the topology induced by ρ and any order-bounded set is bounded in the metric. Let X be a set, and let $M=(M,\oplus,\rho)$ be an idempotent metric semigroup. The set $B(X,M)$ of bounded mappings $X\to M$,i.e., mappings with order-bounded range, is an idempotent metric semigroup with respect to the pointwise addition $(\varphi\oplus\psi)(x)=\varphi(x)\oplus\psi(x)$, the corresponding partial order, and the uniform metric $\rho(\varphi,\psi)=\sup_x\rho(\varphi(x),\psi(x))$. If $P=(P,\oplus,\odot,\rho)$ is a semiring, then $B(X,S)$ has the structure of an P*-semimodule*, i.e., the multiplication by elements of P is defined on $B(X,P)$ by $(a\odot\varphi)(x)=a\odot\varphi(x)$. This P-semimodule will also be referred to as the space of (bounded) P-valued functions on X. If X is a topological space, then by $C(X,A)$ we denote the subsemimodule of continuous functions in $B(X,A)$. If X is finite, $X=\{x_1,\dots,x_n\}$, $n\in\mathbb{N}$, then the semimodules $C(X,A)$ and $B(X,A)$ coincide and can be identified with the semimodule $P^n=\{(a_1,\dots,a_n):a_j\in A\}$. Any vector $a\in P^n$ can be uniquely represented as a linear combination $a=\bigoplus_{j=1}^n a_j\odot e_j$, where $\{e_j,\ j=1,\dots,n\}$ is the standard basis of P^n (the jth coordinate of e_j is equal to $\mathbf{1}$, and the other coordinates are equal to $\mathbf{0}$). As in the classical linear algebra, we can readily prove that the semimodule of continuous homomorphisms $m\colon P^n\to P$ (in what follows such homomorphisms are called *linear functionals* on P^n) is isomorphic to P^n itself. Similarly, any endomorphism $H\colon P^n\to P^n$ (a linear operator on P^n) is determined by an P-valued $n\times n$ matrix, see [4, 6].

2.3 Fuzzy neural networks and parallel processing.

Fuzzy neural networks give more information with respect to the classical neural networks, which are more or less Black Boxes. Since the t-norm $T_{\mathbf{M}}$ can be obtained as a limit of a family of continuous Archimedean t-norms (see [2]), taking enough big value for this parameter we obtain satisfactory approximation. In the Archimedean case (t–norm or t–conorm) we have an additive generator h. For example, if the activation function of the neuron u_j is given by by a differentiable function h which is induced by a t–norm or t–conorm , then for activation a_{u_i} by neuron u_i we have

$$\frac{\delta net_{u_j}^{(p)}}{\delta a_{u_i}^{(p)}} = \frac{h'(a_{u_i}^{(p)}}{h'(h^{(-1)}(\sum_k h(a_{u_k}^{(p)})))},$$

where the sum goes through all neurons u_k which have connection with the neuron u_j.

Pseudo-analysis is applied also on the following difference equation for given $\alpha, \beta \in [a, b]$

$$c_{m,n}^{k+1} = \alpha \odot c_{m-1,n}^{k} \oplus \beta \odot c_{m,n-1}^{k}, \; k = 0, 1, 2, ...; \;\; m, n = 0, \pm 1, \pm 2, ...$$

with the initial condition $c_{m,n}^0 = \mathbf{1}$ for $n = 0, \; m \geq 0$ and $m = 0, \; n \geq 0$ and $c_{m,n}^0 = \mathbf{0}$ otherwise. For more details see [9].

3 Measures and integrals

Let X be a non-empty set. Let Σ be a $\sigma-$algebra of subsets of X.

A set function $m : \Sigma \to [a, b]_+$ (or semiclosed interval) is a $\oplus-$decomposable measure if there hold $m(\varnothing) = 0$ (if $\oplus$ is not idempotent) ; $m(A \cup B) = m(A) \oplus m(B)$ for $A, B \in \Sigma$ such that $A \cap B = \varnothing$. In the case when $\oplus$ is idempotent, it is possible that m is not defined on an empty set. A $\oplus-$decomposable measure m is $\sigma - \oplus$-decomposable if

$$m(\bigcup_{i=1}^{\infty} A_i) = \bigoplus_{i=1}^{\infty} m(A_i)$$

hold for any sequence (A_i) of pairwise disjoint sets from Σ. Further on, we shall suppose that m be a $\sigma - \oplus-$decomposable measure and $\preceq$ is total order on $[a, b]$.

Let $([a, b], \oplus, \odot)$ be a semiring. $([a, b], \oplus)$ and $([a, b], \odot)$ are complete lattice ordered semigroups. Let interval $[a, b]$ be endowed with a metric d compatible with sup and inf ($\limsup x_n = x$ and $\liminf x_n = x$ imply $\lim\limits_{n\to\infty} d(x_n, x) = 0$) and let the metric d satisfies at least one of the following conditions:

(a) $d(x \oplus y, \; x' \oplus y') \leq d(x, x') + d(y, y')$

(b) $d(x \oplus y, \; x' \oplus y') \leq \max\{d(x, x'), \; d(y, y')\}$.

Both conditions imply:

$$d(x_n, y_n) \to 0 \quad \Rightarrow \quad d(x_n \oplus z, \; y_n \oplus z) \to 0$$

We suppose that metric d is also monotonic, i.e.,

$$x \preceq z \preceq y \quad \Rightarrow \quad d(x, y) \geq \max\{d(y, z), d(x, z)\}.$$

Example 2 *Metric with property* (b) *on the semiring* $([-\infty, +\infty[, \max, +)$ *is*

$$d_1(x, y) = \mid e^{-x} - e^{-y} \mid$$

Metric with property (b) *on the semiring* $([-\infty,+\infty],\max,\min)$ *is*

$$d_2(x,y) = \mid \arctan x - \arctan y \mid .$$

Both metric are monotonic.

The pseudo-characteristic function of a set A is:

$$\mathbf{1}_A(x) = \begin{cases} \mathbf{1} & \text{for } x \in A, \\ \mathbf{0} & \text{for } x \notin A, \end{cases}$$

where $\mathbf{0}$ is zero element for $\oplus$ and $\mathbf{1}$ is unit element for $\odot$. For functions defined on X and values in $[a,b]$ we transfer pointwise the operations $\oplus$ and $\odot$.

A mapping $s : X \to [a,b]$ is a *simple function* if it has the following representation $s = \bigoplus_{i=1}^{n} a_i \odot \mathbf{1}_{A_i}$, where $a_i \in [a,b], A_i \in \Sigma$ and if $\oplus$ is not idempotent then sets A_i are disjoint. An *elementary (measurable) function* is mapping $e : X \to [a,b]$ that has the following representation

$$e = \bigoplus_{i=1}^{\infty} a_i \odot \mathbf{1}_{A_i}, \tag{1}$$

for $a_i \in [a,b], A_i \in \Sigma, x \in X$ and if $\oplus$ is not idempotent sets A_i are disjoint, when the right-hand side of equality (1) exist.

Let $f : X \to [a,b]$ be measurable if pseudo-addition is idempotent, and if not, let f be measurable such that for each positive real number ε exists a monotone ε-net in $f(X)$.

The construction of pseudo-integral is similar to the construction of the Lebesgue integral.

Definition 3 *The* pseudo-integral *of a simple function s (elementary function e) with respect to the $\sigma - \oplus -$decomposable measure m is:*

$$\int_X^{\oplus} s \odot dm = \bigoplus_{i=1}^{n} a_i \odot m(A_i), \qquad \left(\int_X^{\oplus} e \odot dm = \bigoplus_{i=1}^{\infty} a_i \odot m(A_i) \right).$$

The pseudo-integral *of a bounded measurable function $f : X \to [a,b]$, for which if $\oplus$ is not idempotent for each $\varepsilon > 0$ there exists a monotone $\varepsilon-$ net in $f(X)$, is defined by:*

$$\int_X^{\oplus} f \odot dm = \lim_{n\to\infty} \int_X^{\oplus} \varphi_n \odot dm,$$

where $\{\varphi_n\}_{i\in\mathbf{N}}$ is the sequence of elementary functions from theorem above.
The pseudo-integral over A, when A is an arbitrary subset of X, is given by:

$$\int_A^{\oplus} f \odot dm = \int_X^{\oplus} \left(\mathbf{1}_A \odot f \right) \odot dm.$$

Let $(G,+)$, $G \subset \mathbb{R}^n$, where $+$ is the coordinatewise addition.

Definition 4 *The semiring $B(G,[a,b])$ consists in cases I) and III) (at least pseudo-addition is idempotent) of the bounded (with respect to the order in $[a,b]$) functions, and in the case II) (pseudo-addition has been represented by its additive generator g) of functions $f : G \to [a,b]$ with property $g(|f|) \in L_1(G)$ (the space $L_1(G)$ consists of Lebesgue integrable functions which satisfy the condition $\int_G |f(x)|\, dx < +\infty$).*

All previous considerations can be transferred to the case $[a,b] \subset \mathbb{R}^n$, taking care that the order in $\mathbb{R}^n$ is a partial order.

4 Pseudo-convolution

4.1 Basic definitions and properties

Let G be subset of $\mathbb{R}^n$ and $*$ a commutative binary operation on G such that $(G, *)$ is a cancellative semigroup with unit element e and $G_+ = \{x | x \in G, x \geq e\}$ is a subsemigroup of G. All considerations can be managed also for a general topological group G. We shall consider functions whose domain will be G. We have by [14, 15].

Definition 5 *The* pseudo-convolution of the first type *of two functions $f : G \to [a, b]$ and $h : G \to [a, b]$ with respect to a $\sigma - \oplus$-decomposable measure m and $x \in G_+$ is given in the following way*

$$f \star h(x) = \int_{G_+^x}^{\oplus} f(u) \odot dm_h(v),$$

where $G_+^x = \{(u, v) | u * v = x, v \in G_+, u \in G_+\}$, $m_h = m$ in the case of sup*-decomposable measure $m(A) = \sup_{x \in A} h(x)$, in the case of* inf*-decomposable measure $m(A) = \inf_{x \in A} h(x)$, and $dm_h = h \odot dm$ in the case of $\oplus$-decomposable measure m, where $\oplus$ has an additive generator g and $g \circ m$ is the Lebesgue measure and $f \in B(G, [a, b])$.*

We consider also the second type of pseudo-convolution *when $(G, *)$ is a group and the pseudo-integral is taken over whole set G:*

$$f \star h(x) = \int_{G}^{\oplus} f(x * (-t)) \odot dm_h(t).$$

where $(-t)$ is unique inverse element for t and $x \in G$.

Remark 6 When $*$ is the usual addition on $\mathbb{R}$ and $G = \mathbb{R}$, pseudo-convolutions of the first and the second type, for $x \in \mathbb{R}^+$, are

$$(f \star h)(x) = \int_{[0,x]}^{\oplus} f(x - t) \odot dm_h(t), \quad (f \star h)(x) = \int_{G}^{\oplus} f(x - t) \odot dm_h(t),$$

respectively. For both types for the case II) we shall use also the notation

$$(f \star h)(x) = \int^{\oplus} f(x - t) \odot h(t)\, dm(t).$$

Next definition considers cases when pseudo-addition $\oplus$ is an idempotent operation.

Definition 7 Pseudo-delta function *is given by*

$$\delta^{\oplus,\odot}(x) = \begin{cases} \mathbf{1} & \text{for } x = e, \\ \mathbf{0} & \text{for } x \neq e, \end{cases}$$

where $\mathbf{0}$ is zero element for $\oplus$, $\mathbf{1}$ is unit element for $\odot$ and e is zero element for $*$.

We shall give some examples of pseudo integrals and related pseudo-convolutions for I-III cases. We restrict here to the case $G \subset \mathbb{R}$.

Example 8 In this example we shall give the form of the pseudo-convolution of the first and the second type for some characteristic cases (the relevant semirings are the semirings that are mentioned in the section above) and for $* = +$ and $G = \mathbb{R}$.
Case I) For the semiring $([-\infty, \infty[, \max, +)$ the pseudo-convolution of the first type and second type of the functions f and h will be

$$(f \star h)(x) = \sup_{0 \leq t \leq x} (f(x-t) + h(t)), \quad (f \star h)(x) = \sup_{t \in \mathbb{R}} (f(x-t) + h(t)),$$

respectively. Unit element for this pseudo-convolutions is the following pseudo-delta function

$$\delta^{\max,+}(x) = \begin{cases} \mathbf{1} \ (= 0) & \text{if } x = 0, \\ \mathbf{0} \ (= -\infty) & \text{if } x \neq 0. \end{cases}$$

We can consider in analogous way the more general semiring $([a, b], \max, \odot)$, such that $\odot$ is non-idempotent pseudo-multiplication.

Case II) Pseudo-convolution of the first type in the sense of the g-integral (see [7, 8, 9]), i.e., when the pseudo-operations are represented by a generator g as $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot y = g^{-1}(g(x)g(y))$, is given in the following way

$$(f \star h)(x) \quad = \quad g^{-1} \left(\int_0^x g(f(t)) \cdot g(h(x-t)\right) dt).$$

Case III) For the semiring $([-\infty, \infty], \max, \min)$ pseudo-integral is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \sup_{x \in \mathbb{R}} (\min(f(x), h(x))),$$

where the function h defines the sup-decomposable measure m. The domain of functions will be $\mathbb{R}$ (or some subset of $\mathbb{R}$) and the domain of the semiring is $[-\infty, \infty]$ (or any subinterval). The zero element for the $\oplus$ is $-\infty$ and the unit element for the $\odot$ is $+\infty$.

The basic properties of the generalized pseudo-convolution for an idempotent pseudo-addition are given in the following theorem (see [13, 14]).

Theorem 9 *Let $\mathcal{F}$ be a class of functions f such that $f : G \to [a, b]$, where $(G, *)$ is a commutative semigroup with unit element e. Let $\odot$ be continuous (up to some distinguished points) pseudo-multiplication of the first or the second type on interval $[a, b]$.*

Then the pseudo-convolution of the first type (second type for G a commutative group) for the idempotent pseudo-addition $\Big($ cases I) and III) $\Big)$ is commutative, associative operation with the unit element $\delta^{\oplus,\odot}$.

Remark 10 When pseudo-addition $\oplus$ is max , then, we can take left continuous pseudo-multiplication instead of continuous pseudo-multiplication.

Restricting on the case I: $P = (\min, +)$, we have $\star$ of convolution on $B(G, P)$

$$(f \star h)(x) = \inf_{y \in G} (f(y) \odot h(x-y)).$$

This operation turns $B(G, P)$ into an idempotent semiring, which will be denoted by $CS(G)$ and referred to as the *convolution semiring*. Some subsemirings of $CS(G)$ are of interest in studying multicriteria optimization. Namely, let L denote the hyperplane in $\mathbb{R}^k$ determined by the equation

$$L = \left\{ (a^j) \in \mathbb{R}^k : \sum a^j = 0 \right\},$$

and let us define a function $n \in CS(L)$ by setting $n(a) = \max_j(-a^j)$. Obviously, $n \star n = n$; that is, n is a multiplicatively idempotent element of $CS(L)$. Let $CS_n(L) \subset CS(L)$ be the subsemiring of functions h such that $n \star h = h \star n = h$. It is easy to see that $CS_n(L)$ contains the function identically equal to $\mathbf{0} = \infty$ and that the other elements of $CS_n(L)$ are just the functions that take the value $\mathbf{0}$ nowhere and satisfy the inequality $h(a) - h(b) \leq n(a - b)$ for all $a, b \in L$. In particular, for each $h \in CS_n(L)$ we have

$$|h(a) - h(b)| \leq \max_j |a^j - b^j| = \|a - b\|,$$

which implies that h is differentiable almost everywhere. Closely related to this convolution semiring is the semiring in the following example.

Example 11 Pareto order ($a = (a_1, \ldots, a_n) \leq b = (b_1, \ldots, b_n)$ if and only if $a_i \leq b_i$ for all $i = 1, \ldots, n$) defines in $\mathbb{R}^n_+$ the structure of an idempotent semigroup. For any subset $M \subset \mathbb{R}^k$, by $\mathrm{Min}(M)$ we denote the set of minimal elements of the closure of M in $\mathbb{R}^k$. Let $P(\mathbb{R}^k)$ denote the class of subsets $M \subset \mathbb{R}^k$ whose elements are pairwise incomparable,

$$P(\mathbb{R}^k) = \{M \subset \mathbb{R}^k : \mathrm{Min}(M) = M\}.$$

Obviously, $P(\mathbb{R}^k)$ is a semiring with respect to the operations $M_1 \oplus M_2 = \mathrm{Min}(M_1 \cup M_2)$ and $M_1 \odot M_2 = \mathrm{Min}(M_1 + M_2)$; the neutral element $\mathbf{0}$ with respect to addition in this semiring is the empty set, and the neutral element with respect to multiplication is the set whose sole element is the zero vector in $\mathbb{R}^k$. The semiring $P(\mathbb{R}^k)$ is isomorphic to the semiring of *normal sets*, that is, closed subsets $N \subset \mathbb{R}^k$ such that $b \in N$ implies $a \in N$ for any $a \geq b$; the sum and the product of normal sets are defined as their usual union and sum, respectively. Indeed, if N is normal, then $\mathrm{Min}(N) \in P(\mathbb{R}^k)$; conversely, with each $M \in P(\mathbb{R}^k)$ we can associate the normalization $\mathrm{Norm}(M) = \{a \in \mathbb{R}^k \mid \exists\, b \in M : a \geq b\}$.

It turns out that last two semirings are closely connected, as shows the following proposition that is a specialization of a more general result given in [6].

Theorem 12 *The semirings $CS_n(L)$ and $P(\mathbb{R}^k)$ are isomorphic.*

For functions with values in semirings of type II we have the following

Theorem 13 *Pseudo-convolution of the first or the second type for g-case (case II) is commutative and associative operation while G is whole set of reals and $*$ is the usual addition on $\mathbb{R}$.*

4.2 Applications

4.2.1 Probabilistic metric spaces

We shall show that the basic notion of the theory of probabilistic metric spaces, the triangle function, is based on the pseudo-convolution of the first type.

Let $F, H \in \Delta^+$ and $u, v, x \in [0, \infty]$. Taking for triangle function $\tau = \tau_T$, where

$$\tau_T(F, H)(x) = \sup \{T(F(u), H(v)) \mid u + v = x\},$$

(pseudo-convolution of the first type with respect to $(\max, T)$ and $* = +$) for a left continuous t-norm T we obtain a special important probabilistic metric space, the Menger space. The fact that function τ_T is triangle function yields from Theorem 9, see [1].

4.2.2 Fuzzy numbers

The arithmetical operations with fuzzy numbers are based on Zadeh's extension principle (see [2]): Let T be an arbitrary but fixed t-norm and $*$ a binary operation on $\mathbb{R}$. Then the operation $*$ is extended to fuzzy numbers A and B by

$$A *_T B(z) = \sup_{x*y=z} T(A(x), B(y))$$

for $z \in \mathbb{R}$. Some usual operations with fuzzy numbers are following: Addition is obtained for $* = +$: $A \oplus_T B(z) = \sup_{x+y=z} T(A(x), B(y))$. Multiplication for $* = \cdot$: $A \odot_T B(z) = \sup_{x \cdot y=z} T(A(x), B(y))$.

4.2.3 Optimization and morphism with the probability

First, we state the problem of finding the maximum of the utility function

$$f_1(x_1) + f_2(x_2) + \cdots + f_N(x_N),$$

on the domain $D = \{(x_1, x_2, \ldots, x_N) | x_1 + x_2 + \cdots + x_N = x,\, x_i \geq 0,\ i = 1, ..., N\}$.
We can rewrite this problem in such manner that it represent generalized pseudo-convolution of the first type applied on N functions. Semiring that is used for this particular problem as a range of this pseudo-convolution is $([0, \infty[, \max, +)$.

This type of problem often occurs in the mathematical economics and operation research and it can be solved by applying the pseudo-Laplace transform, the pseudo-exchange formula and inverse of pseudo-Laplace transform.

Definition 14 *The* pseudo-character *of the group $(G, +)$, $G \subset \mathbb{R}^n$ is a continuous (with respect to the usual topology of reals) map $\xi : G \to [a, b]$, of the group $(G, +)$ into the semiring $([a, b], \oplus, \odot)$, with property $\xi(x + y) = \xi(x) \odot \xi(y), \quad x, y \in G$.*

The map $\xi \equiv \mathbf{0}$ is trivial pseudo-character.
The forms of pseudo-character in the special cases can be found in [2, 14, 15]. Interesting case for us is $([a, b], \oplus, \odot) = ([0, \infty[, \max, +)$ and then pseudo-character has the form $\xi(x, c) = c \cdot x$, for each $c \in \mathbb{R}$, where we have taken the dependence of the function ξ also with respect to the parameter c.

Definition 15 *The* pseudo-Laplace transform $\mathcal{L}^{\oplus}(f)$ *of a function $f \in B(G, [a, b])$ is defined by*

$$(\mathcal{L}^{\oplus} f)(\xi)(z) = \int_{G \cap [0, \infty[^n}^{\oplus} \xi(x, -z) \odot \, dm_f(x),$$

where ξ is the pseudo-character.

When at least pseudo-addition is idempotent operation we can consider the second type of pseudo-Laplace transform: $(\mathcal{L}^{\oplus} f)(\xi)(z) = \int_G^{\oplus} \xi(x, -z) \odot \, dm_f(x)$, i.e., pseudo-integral has been taken over the whole G.

The forms of the pseudo-Laplace transform are known for three special cases and can be found in [15]. We shall restrict to the special important case is $([0, \infty[, \max, +)$ and then pseudo-Laplace transform has the following form $(\mathcal{L}^{\oplus} f)(z) = \sup\limits_{x \geq 0}(-xz + f(x))$, and in n-dimensional case $(\mathcal{L}^{\oplus} f)(z) = \sup_{x_i \geq 0, i=1,\ldots,n}(-z_1 x_1 - \cdots - z_n x_n + f(x))$. The important result that has been proved in [15] (see [4]) is the *pseudo-exchange formula*, that transforms operation of generalized pseudo-convolution of the first or second type to pseudo-multiplication: $\mathcal{L}^{\oplus}(f_1 \star f_2) = \mathcal{L}^{\oplus}(f_1) \odot \mathcal{L}^{\oplus}(f_2)$, where functions f_1, f_2 belong to $B(G, [a, b])$.

In order to solve the problem from the beginning of this section we need the next theorem (see [15])

Theorem 16 *If $\mathcal{L}^{\oplus}(f) = F$ for semiring $([0,\infty[, \max, +)$, then there exists $(\mathcal{L}^{\oplus})^{-1}$, inverse of pseudo-Laplace transform, and it has the following form: $((\mathcal{L}^{\oplus})^{-1}(F))(x) = \inf_{z\geq 0}(xz + F(z))$.*

Now, let $f(x) = \max_D(f_1(x_1) + f_2(x_2) + \cdots + f_N(x_N))$. This is the pseudo-convolution of the first type of functions $f_1, \ldots, f_N$ with respect to $(\max, +)$. Applying the pseudo-Laplace transform (for the case $\oplus = \max$ and $\odot = +$), the pseudo-exchange formula and inverse of pseudo-Laplace transform we obtain

$$f(x) = \left(\left(\mathcal{L}^{\oplus}\right)^{-1} \sum_{i=1}^{N} \mathcal{L}^{\oplus}\left(f_i\right)\right)(x) = \min_{z\geq 0}\left(xz + \sum_{i=1}^{N} \mathcal{L}^{\oplus}\left(f_i\right)(z)\right).$$

There is an interesting correspondence principle between probability theory and stochastic processes on the one hand, and optimization theory and decision processes on the other hand (see [4]). In particular, the Markov causality principle corresponds to the Bellman optimality principle.

4.3 The Riesz type theorem

The Riesz theorem in functional analysis establishes a one-to-one correspondence between continuous linear functionals on the space of continuous real functions on a locally compact space X vanishing at infinity and regular finite Borel measures on X. Similar correspondence exists in idempotent analysis.

We restrict here our consideration to the case of the semiring $P = (\min, +)$. Proofs, generalizations and references could be found in [4].

All idempotent measures are absolutely continuous; i.e., any such measure can be represented as the idempotent integral of a density function with respect to some standard measure. Let us formulate this fact more precisely. Let $C_{\mathbf{0}}(X, P)$ denote the space of continuous functions $f\colon X \to P$ on a locally compact normal space X vanishing at infinity,i.e., such that for any $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $\rho(\mathbf{0}, f(x)) < \varepsilon$ for all $x \in X \setminus K$. The topology on $C_{\mathbf{0}}(X, P)$ is defined by the uniform metric $\rho(f, g) = \sup_X \rho(f(x), g(x))$. The space $C_{\mathbf{0}}(X, P)$ is an idempotent semimodule. If X is a compact set, then the semimodule $C_{\mathbf{0}}(X, P)$ coincides with the semimodule $C(X, P)$ of all continuous functions from X to A. The homomorphisms $C_{\mathbf{0}}(X, P) \to P$ will be called *pseudo linear functionals* on $C_{\mathbf{0}}(X, P)$. The set of pseudo linear functionals will be denoted by $C_{\mathbf{0}}^*(X, P)$ and called the *dual semimodule* of $C_{\mathbf{0}}(X, P)$.

Theorem 17 *For any $m \in C_{\mathbf{0}}^*(X, P)$ there exists a unique lower semicontinuous and bounded below function $f\colon X \to P$ such that*

$$m(h) = \inf_x f(x) \odot h(x) \qquad \forall h \in C_{\mathbf{0}}(X, P). \tag{2}$$

Conversely, any function $f\colon X \to P$ bounded below defines an element $m \in C_{\mathbf{0}}^*(X, P)$ by (2). At last, the functionals m_{f_1} and m_{f_2} coincide if and only if the functions f_1 and f_2 have the same lower semicontinuous closures; that is, $\mathrm{Cl}\, f_1 = \mathrm{Cl}\, f_2$, where

$$(\mathrm{Cl}\, f)(x) = \sup\{\psi(x) |\ \psi \leq f,\ \psi \in C(X, P)\}.$$

One can develop the concept of weak convergence and the corresponding theory of generalized functions. For $X = \mathbb{R}^n$, simple delta-shaped sequences can be constructed of smooth convex functions; for example, $\delta_y^{\min,+}(x)$ is the weak limit of the sequence $f_n(x) = n(x-y)^2$. Thus, by virtue of the preceding, each linear functional (or operator) on $C_{\mathbf{0}}(\mathbb{R}^n)$ is uniquely determined by its values on smooth convex functions.

Let $C_{\mathbf{0}}(\mathbb{R}^n)$ be the space of continuous functions $f : \mathbb{R}^n \to P$ (P is of type I (i), III) with the property that for each $\varepsilon > 0$ there exists a compact subset $K \subset \mathbb{R}^n$ such that $d(\mathbf{0}, \inf_{x\in\mathbb{R}^n\setminus K} f(x)) < \varepsilon$, with the metric $D(f,g) = \sup_x d(f(x), g(x))$. Let $C_{\mathbf{0}}^{cs}(\mathbb{R}^n)$ be the subspace of $C_{\mathbf{0}}(\mathbb{R}^n)$ of functions f with compact support $\text{supp}_{\mathbf{0}} = \overline{\{x|\ f(x) \neq \mathbf{0}\}}$.

The dual semimodul $(\mathcal{S}_{\mathbf{0}}(\mathbb{R}^n))^*$ is the semimodul of continuous pseudo-linear P-valued functionals on $\mathcal{S}_{\mathbf{0}}(\mathbb{R}^n)$ (with respect to pointwise operations). Analogously the dual semimodul $(C_{\mathbf{0}}^{cs}(\mathbb{R}^n))^*$ is the semimodul of continuous pseudo-linear P-valued functionals on $C_{\mathbf{0}}^{cs}(\mathbb{R}^n)$ (with respect to pointwise operations). The following representation theorem is a consequence of Theorem 17, see [4] (it is also important in the theory of nonlinear PDE-[4, 12, 15]).

Theorem 18 *Let f be a function defined on $\mathbb{R}^n$ and with values in the semiring P of type I) (i) or III) and a functional $m_f : C_{\mathbf{0}}^{cs}(\mathbb{R}^n) \to P$ is given by*

$$m_f(h) = \int^{\oplus} f \odot dm_h = \inf_x (f(x) \odot h(x)).$$

Then

1) *The mapping $f \mapsto m_f$ is a pseudo-isomorphism of the semimodule of lower semicontinuous functions onto the semimodule $(C_{\mathbf{0}}^{cs}(\mathbb{R}^n))^*$.*

2) *The space $C_{\mathbf{0}}^*(\mathbb{R}^n)$ is isometrically isomorphic with the space of bounded functions, i.e., for every $m_{f_1}, m_{f_2} \in C_{\mathbf{0}}^*(\mathbb{R}^n)$ we have*

$$\sup_x d(f_1(x), f_2(x)) = \sup\{d(m_{f_1}(h), m_{f_2}(h)) : h \in C_{\mathbf{0}}(\mathbb{R}^n), D(h, \mathbf{0}) \leq 1\}.$$

3) *The functionals m_{f_1} and m_{f_2} are equal if and only if $Clf_1 = Clf_2$, where*

$$Clf(x) = \sup\{\psi(x) : \psi \in C(\mathbb{R}^n), \psi \leq f\}.$$

We remark that Theorem 18 is not valid for the semimodule $C(\mathbb{R}, P)$ of bounded continuous functions defined on $\mathbb{R}$.

5 Some recent results

5.1 Option pricing

The famous Black-Sholes and Cox-Ross-Rubinstein (1979) formulas are basic results in the modern theory of option pricing in financial mathematics. They are usually deduced by means of stochastic analysis; various generalizations of these formulas were proposed using more sophisticated stochastic models for common stocks pricing evolution. The systematic deterministic approach to the option pricing leads to a different type of generalizations of Black-Sholes and Cox-Ross-Rubinstein formulas characterized by more rough assumptions on common stocks evolution (which are therefore easier to verify). This approach reduces the analysis of the option pricing to the study of certain homogeneous nonexpansive maps, which however, unlike the situations described in previous subsections, are "strongly" infinite dimensional: they act on the spaces of functions defined on sets, which are not (even locally) compact.

In the paper of Kolokoltsov [3] it was shown what type of generalizations of the standard Cox-Ross-Rubinstein and Black-Sholes formulas can be obtained using the deterministic (actually game-theoretic) approach to option pricing and what class of homogeneous nonexpansive maps appear in these formulas, considering first a simplest model of financial market with only two securities in discrete time, then its generalization to the case of several common stocks, and then the continuous limit. One of the objective was to show that the infinite dimensional generalization of the theory of homogeneous nonexpansive maps (which does not exists at the moment) would have direct applications to the analysis of derivative securities pricing. On the other hand, this approach, which uses neither martingales nor stochastic equations, makes the whole apparatus of the standard game theory appropriate for the study of option pricing.

5.2 Non-commutative and non-associative pseudo-operations

There were obtained in [16, 17] a relaxation of the properties of pseudo-addition and pseudo-multiplication and applications of obtained results on nonlinear PDE.

Definition 19 *We call real operations $\oplus$ and $\odot$ generalized pseudo-addition and generalized pseudo-multiplication (from the right), respectively, if they satisfy the following conditions:*

(i) $\oplus$ *and* $\odot$ *are functions from* $C^2(\mathbb{R}^2)$,

(ii) the equation $t \oplus t = z$ *for given* z *is uniquely solvable,*

(iii) $\odot$ *is right distributive over* $\oplus$:

$$(D_r) \quad (x \oplus y) \odot z = (x \odot z) \oplus (y \odot z).$$

Changing in the previous definition in (iii) that $\odot$ is left distributive over $\oplus$:

$$(D_l) \quad z \odot (x \oplus y) = (z \odot x) \oplus (z \odot y),$$

we obtain generalized pseudo-addition and generalized pseudo-multiplication (from the left), respectively. The corresponding measure and integral were introduced in [18].

5.3 Large deviation principle

Let Ω be a topological space and Σ be the algebra of its Borel sets. One says that a family of probabilities (P_ε), $\varepsilon > 0$, on (Ω, Σ) obeys the *large deviation principle* if there exists a *rate function* $I : \Omega \to [0, \infty]$ such that

1) I is lower semi-continuous and $\Omega_a = \{\omega \in \Omega : I(\omega) \leq a\}$ is a compact set for any $a < \infty$,

2) $-\limsup_{\varepsilon \to 0} \varepsilon \log P_\varepsilon(C) \geq \inf_{\omega \in C} I(\omega)$ for each closed set $C \subset \Omega$,

3) $-\liminf_{\varepsilon \to 0} \varepsilon \log P_\varepsilon(U) \leq \inf_{\omega \in U} I(\omega)$ for each open set $U \subset \Omega$.

Obviously $m(A) = \inf_{\omega \in A} I(\omega)$ is then a positive sigma-additive with respect to the operation $\oplus$ = min function on Σ. Therefore, it is naturally to generalize the previous definition in the following way [19]. For any Borel set A let

$$P^{out} = \limsup_{\varepsilon \to 0} \varepsilon \log P_\varepsilon(A), \quad P^{in} = \liminf_{\varepsilon \to 0} \varepsilon \log P_\varepsilon(A).$$

One says that (P_ε) obeys *the weak large deviation principle*, if there exists a positive idempotent measure m on (Ω, Σ) such that

1) there exists a sequence (Ω_n) of compact subsets of Ω such that $m(\Omega_n^c) \to \mathbf{0} = +\infty$ as $n \to \infty$, where C^c stands for the complimentary set of C,

2) $m(C) \leq -P^{out}(C)$ for each closed $C \subset \Omega$,

2) $m(U) \geq -P^{in}(U)$ for each open $U \subset \Omega$.

Using Theorem 17 and its generalizations one can prove (see details in [19]) that the large deviation principle and its weak version are actually equivalent for some (rather general) "good" spaces Ω. One can obtain also an interesting correspondence between the tightness conditions for probability and idempotent measures.

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