Absorbing-Norms

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Abstract: It is discussed that the monotony of an operation is a necessary requirement only in that case if there exists a neutral element, hence we omit the axiom of monotony in case of operations without neutral element. Absorbing-norms are not necesseraly monotone operations with an absorbing element. The papers summariezes the definition and the structure of absorbing-norms. Distance-based operations are examles that such kind of operations can be obtained from a practical approach.

Keywords: Fuzzy connectives, agregation operations, t-norms, uninorms.

1 Introduction

The applications of Computational Intelligent techniques strongly relay on the integration of membership values representing uncertain information. Since the pioneering work of Lotfi A. Zadeh the basic research was oriented towards the investigation of the properties of t-operators and also to find new ones satisfying the axiom system. As a result of this a great number (of various type) of t-operators have been introduced accepting the axiom system as a fixed, unchangeable skeleton.

Recently for the generalization of t-operators the concept of uninorms was introduced by Yager and Rybalov [8], and their structure was described by Fador et. al. and De Baets [2]. They also studied the functional equations of Frank and Alsina for two classes of commutative, associative and increasing binary operators. The first one is the class of uninorms, while the second one is the class of nullnorms.

In this paper the generalization of nullnorms is discussed. The idea is based on the distance-based operators introduced by Rudas [7]. As it is summarized in Chapter

6.2 the maximum distance operators are uninorms, while the minimum distance operators have absorbing elements but they are not monotone mappings. This property suggested the examination of axiom system of t-operators from the point of view of monotony. It is shown that there is strong relation between monotony and the existence of the neutral element, i. e. if there exists a neutral element then the operation should be non-decreasing. However this result suggests that if an operator has an absorbing element instead of neutral element than the monotony is not a “natural” requirement, and it can be omitted from the definition.

After the brief summary of uninorms and nullnorms the definition and the basic properties some absorbing norms are defined. It is shown that suitable pairs formed from uninorms and absorbing norms satisfy the absorption and distributivity laws. Finally the structure and construction of absorbing norms are discussed.

In the second part of the paper the distance-based operators as typical examples of absorbing and uninorms are summarized.

2 Uninorms and Nullnorms

Uninorms are such kind of generations of t-norms and t-conorms where the neutral element can be any number from the unit interval. The class of uninorms seems to play an important role both in theory and application [10], [2], [4].

Definition 1 [10] A *uninorm* U is a commutative, associative and increasing binary operator with a neutral element $e \in [0,1]$, i.e. $U(x,e) = x, \forall x \in [0,1]$.

The neutral element e is clearly unique. The case e = 1 leads to t-conorm and the case e = 0 leads to t-norm.

The first uninorms were given by Yager and Rybalov [10]

$$U_c(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in [e,1]^2 \\ \min(x,y) & \text{elsewhere} \end{cases} \quad (1)$$

$$U_d(x,y) = \begin{cases} \min(x,y), & \text{if } (x,y) \in [0,e]^2 \\ \max(x,y), & \text{elsewhere} \end{cases} \quad (2)$$

U_c is a conjunctive right-continuous uninorm and U_d is a disjunctive left-continuous uninorm.

Regarding the duality of uninorms Yager and Rybalov have proved the following theorem [10].

Theorem 1 *Assume U is a uninorm with identity element e, then* $\overline{U}(x,y)=1-U(1-x,1-y)$ *is also a uninorm with neutral element* $1-e$.

Definition 2 [2] A mapping $U:[0,1]\times[0,1]\to[0,1]$ *nullnorm, if there exists an absorbing element* $a\in[0,1]$, i. e., $V(x,a)=a, \forall x\in[0,1]$, *V* is *commutative*, *V* is *associative, non-decreasing and satisfies*

$$V(x,0)=x \text{ for all } x\in[0,a] \tag{3}$$

$$V(x,1)=x \text{ for all } x\in[a,1] \tag{4}$$

The Frank equation was studied by Calvo, De Baets, and Fodor in case of uninorms and nullnorms, and they found the followings.

3 Monotony

From an algebraic point of view t-norms and t-conorms are semigroup operations on the domain $[0,1]\times[0,1]$ with the neutral element 1 and 0, respectively. By investigating their axiom skeletons the axiom of monotony is a revealing property, i. e. it is not a usual axiom in defining algebraic structures. Hence natural questions can be arisen: Why monotony is included in the axiom system? Is it independent from the other axioms?

Most books on fuzzy logic explain it as a natural requirement: "a decrease in the degree of membership in the fuzzy sets cannot produce an increase in the degree of the membership in their intersection and union." This seems to be a logical explanation but not an exact answer.

As it is shown in the followings there is strong relation between monotony and the existence of the neutral element, i. e. if there exists a neutral element then the monotony is a necessary property. The discussion is based on the work L. Fuchs [5].

Definition 3 Let *f* be an operation of *A*. *f* satisfies the *monotony low* with *monotony domain C*, where *C* is non-empty and $C\subseteq A$, if

1. $f(x_1,x_2,...,x_n)\in C$ whenever $x_1,x_2,...,x_n\in C$,
2. for each *i* (*i* = 1,...,*n*), *f* is either monotone decreasing or monotone increasing or both in the variable $x_i\in C$.

If none of the x_i variables are both monotone increasing and decreasing then *f* is called *non-degenerate*.

There is a strong relation between monotony and the existence of neutral element.

Proposition 2 *Let f be a binary operation with a neutral element e. Then f is monotone increasing in both variables in any monotony domain containing e.*

Proof. If $x_1 < x_2$ then $f(x_1, e) = f(e, x_1) = x_1 < x_2 = f(x_2, e) = f(e, x_2)$. ■

Corollary 1 *In the axiom systems of t-norms and t-conorms the axiom of monotony is not independent from the others, so it can be omitted.*

Corollary 2 *If f has no neutral element the property of monotony does not implied.*

The following proposition shows the relation between associative property and monotony.

Proposition 3 *Let f is an associative binary operation defined in A. If* $g(x, y, z) = f(f(x, y), z)$ *is non-degenerate then f is monotone increasing in both variables* x_1 *and* x_2 *in any monotony domain C.*

Proof. On the contrary let us suppose that $f(x, y)$ is monotone decreasing in *x*.

If $x_1 \le x_2$ then $f(x_1, y) \ge f(x_2, y)$ and
$g(x_1, y, z) = f(f(x_1, y), z) \le g(x_2, y, z) = f(f(x_2, y), z)$,

hence *g* is monotone increasing in *x*.

On the other hand *f* is associative so $g(x, y, z) = f(f(x, y), z) = f(x, f(y, z))$ so by the assumption *g* should be monotone decreasing in *x*, which is a contradiction.

The same argument applies for *y*. ■

4 Absorbing-Norms

Definition 4 Let *A* be a mapping $A : [0,1] \times [0,1] \to [0,1]$. *A* is an *absorbing-norm,* if for all $x, y, z \in [0,1]$ satisfies the following axioms:

A1.a There exists *an absorbing element* $a \in [0,1]$, i. e., $A(x, a) = a, \forall x \in [0,1]$.
A1.b $A(x,y) = A(y,x)$ that is, *A* is *commutative,*
A1.c $A(A(x,y),z) = A(x,A(y,z))$ that is, *A* is *associative,*

It is clear that *a* is an idempotent element $A(a, a) = a$, hence the absorbing element is unique. If there would exist at least two absorbing elements

$a_1, a_2, a_1, \neq a_2$ for which $A(a_1, a_2) = a_1$, and $A(a_1, a_2) = a_2$, so thus $a_1, = a_2$

T-operators are special absorbing-operators, namely for any t-norm *T*, $T(0, x) = 0, \forall x \in [0,1]$ and for any t-conorm *S*, $S(1, x) = 1, \forall x \in [0,1]$.

As a direct consequence of the definition we have

if $x \leq a$ then $A(x, a) = a = \max(x, a)$, if $x \geq a$ then $A(x, a) = a = \min(x, a)$.

These properties provide the background to define some simple absorbing-norms.

Theorem 1 *The trivial absorbing-norm* $A_T : [0,1] \times [0,1] \to [0,1]$ *with absorbing element a is*

$$A_T : (x, y) \to a, \forall (x, y) \in [0,1] \times [0,1]. \qquad (5)$$

Proof. The statement is obvious from the definition. ■

Theorem 2 The mapping $A_{\min} : [0,1] \times [0,1] \to [0,1]$ defined as

$$A_{\min}(x, y) = \begin{cases} \max(x, y), & \text{if } (x, y) \in [0, a] \times [0, a] \\ \min(x, y) & \text{elsewhere} \end{cases} \qquad (6)$$

and the mapping $A_{\max} : [0,1] \times [0,1] \to [0,1]$ *defined as*

$$A_{\max}(x, y) = \begin{cases} \min(x, y), & \text{if } (x, y) \in [a, 1] \times [a, 1] \\ \max(x, y), & \text{elsewhere} \end{cases} \qquad (7)$$

are absorbing-norms with absorbing element a.

Proof.

Commutativity. It follows from the definitions of the operators.

Assocoativity. We have to prove that $A_{\min}(A_{\min}(x, y), z) = A_{\min}(x, A_{\min}(y, z))$.

Without loss of generality we can assume that $x \geq y \geq z$.

Suppose first that $a \leq z$. Then

$A_{\min}(A_{\min}(x, y), z) = A_{\min}(y, z) = z$ and $A_{\min}(x, A_{\min}(y, z)) = A_{\min}(x, z) = z$.

Suppose now $a \geq x$. In this case $A_{\min}(A_{\min}(x, y), z) = A_{\min}(x, z) = x$ and $A_{\min}(x, A_{\min}(y, z)) = A_{\min}(x, y) = x$.

Suppose $x \geq a \geq y \geq z$.

$A_{\min}(A_{\min}(x,y),z) = A_{\min}(y,z) = z$ and $A_{\min}(x, A_{\min}(y,z)) = A_{\min}(x,z) = z$.

Suppose $x \ge y \ge a \ge z$.

$A_{\min}(A_{\min}(x,y),z) = A_{\min}(y,z) = z$ and $A_{\min}(x, A_{\min}(y,z)) = A_{\min}(x,z) = z$.

We have to prove that $A_{\max}(A_{\max}(x,y),z) = A_{\max}(x, A_{\max}(y,z))$.

Assume again $x \ge y \ge z$. Since $A_{\min}$ and $A_{\max}$ are equal to each other in the domains $[0,a]\times[0,a]$ and $[a,1]\times[a,1]$ the cases $a \le z$ and $a \ge x$ hold for $A_{\max}$, too.

Suppose now that $x \ge a \ge y \ge z$. $A_{\max}(A_{\max}(x,y),z) = A_{\max}(x,z) = x$ and $A_{\max}(x, A_{\max}(y,z)) = A_{\max}(x,y) = x$.

Suppose $x \ge y \ge a \ge z$. $A_{\max}(A_{\max}(x,y),z) = A_{miax}(y,z) = y$ and $A_{\max}(x, A_{\max}(y,z)) = A_{\max}(x,y) = y$.

Absorbing element.

If $x \le a$ then $A_{\min}(x,a) = \max(x,a) = a$, If $x \ge a$ then $A_{\min}(x,a) = \min(x,a) = a$, and the same are true for $A_{\max}$. ■

With the combination of $A_{\min}$, $A_{\max}$ .and A_T further absorbing-norms can be defined.

Theorem 3 *The mapping* $A^a_{\min} : [0,1]\times[0,1] \to [0,1]$ *defined as*

$$A^a_{\min}(x,y) = \begin{cases} a, & \text{if } (x,y) \in [0,a]\times[0,a] \\ \min(x,y), & \text{elsewhere} \end{cases} \tag{8}$$

and the mapping $A^a_{\max} : [0,1]\times[0,1] \to [0,1]$ *defined as*

$$A^a_{\max}(x,y) = \begin{cases} a, & \text{if } (x,y) \in [a,1]\times[a,1] \\ \max(x,y), & \text{elsewhere} \end{cases} \tag{9}$$

are absorbing-norms with absorbing element a.

Proof.

Commutativity. It follows from the definitions of the operators.

Assocoativity.

We have to prove that $A^a_{\min}(A^a_{\min}(x,y),z) = A^a_{\min}(x, A^a_{\min}(y,z))$.

Wihout loss of generality we can assume that $x \geq y \geq z$.

Suppose first that $a \leq z$. Then $A^a_{\min}\left(A^a_{\min}(x,y),z\right) = A^a_{\min}(y,z) = z$ and $A^a_{\min}\left(x, A^a_{\min}(y,z)\right) = A^a_{\min}(x,z) = z$.

Suppose now $a \geq x$. In this case $A^a_{\min}\left(A^a_{\min}(x,y),z\right) = A^a_{\min}(a,z) = a$ and $A^a_{\min}\left(x, A^a_{\min}(y,z)\right) = A^a_{\min}(x,a) = a$.

Suppose $x \geq a \geq y \geq z$. $A^a_{\min}\left(A^a_{\min}(x,y),z\right) = A^a_{\min}(y,z) = a$ and $A^a_{\min}\left(x, A^a_{\min}(y,z)\right) = A^a_{\min}(x,a) = a$.

Suppose $x \geq y \geq a \geq z$. $A^a_{\min}\left(A^a_{\min}(x,y),z\right) = A^a_{\min}(y,z) = z$ and $A^a_{\min}\left(x, A^a_{\min}(y,z)\right) = A^a_{\min}(x,z) = z$.

We have to prove that. $A^a_{\max}\left(A^a_{\max}(x,y),z\right) = A^a_{\max}\left(x, A^a_{\max}(y,z)\right)$

Assume again $x \geq y \geq z$ and suppose first that $a \leq z$. Then $A^a_{\max}\left(A^a_{\max}(x,y),z\right) = A^a_{\max}(a,z) = a$ and $A^a_{\max}\left(x, A^a_{\max}(y,z)\right) = A^a_{\max}(x,a) = a$.

Suppose now $a \geq x$. In this case $A^a_{\max}\left(A^a_{\max}(x,y),z\right) = A^a_{\max}(x,z) = x$ and $A^a_{\max}\left(x, A^a_{\max}(y,z)\right) = A^a_{\max}(x,y) = x$.

Suppose now that $x \geq a \geq y \geq z$. $A^a_{\max}\left(A^a_{\max}(x,y),z\right) = A^a_{\max}(x,z) = x$ and $A^a_{\max}\left(x, A^a_{\max}(y,z)\right) = A^a_{\max}(x,y) = x$.

Suppose $x \geq y \geq a \geq z$. $A^a_{\max}\left(A^a_{\max}(x,y),z\right) = A^a_{\max}(a,z) = a$ and $A^a_{\max}\left(x, A^a_{\max}(y,z)\right) = A^a_{\max}(x,y) = a$.

Absorbingl element.

If $x \leq a$ then $A^a_{\min}(x,a) = a$, and

if $x \geq a$ then $A^a_{\min}(x,a) = \min(x,a) = a$.

If $x \leq a$ then $A^a_{\max}(x,a) = \max(x,a) = a$, and if $x \geq a$ then $A^a_{\max}(x,a) = a$. ■

Theorem 4 *Assume that A is an absorbing-norm with absorbing element a. The dual operator of A defined as* $\overline{A}(x,y) = 1 - A(1-x, 1-y)$ *is an absorbing-norm with absorbing element* 1-*a.*

Proof.

Commutativity: follows from the commutativity of *A*.

Associativity:

$$\overline{A}\left(x,\overline{A}(y,z)\right)=\overline{A}(x,1-A(1-y,1-z))=1-A(1-x,A(1-y,1-z))=1-A(A(1-x,1-y),1-z)=$$
$$=1-A\left(1-\overline{A}(x,y),1-z\right)=\overline{A}\left(\overline{A}(x,y),z\right)$$

Absorbing element:

$\overline{A}(x,1-a)=1-A(1-x,a)=1-a$. ■

Let us define a kind of complements of $A_{\min}$ and $A_{\max}$ .by replacing in the definitions the operator min with max and the max with min as follows.

Definition 5

$$\overline{(A_{\min})}^{\max}(x,y)=\begin{cases}\min(x,y), & \text{if } (x,y)\in[0,a]\times[0,a]\\ \max(x,y) & \text{elsewhere}\end{cases} \tag{10}$$

$$\overline{(A_{\max})}^{\max}(x,y)=\begin{cases}\max(x,y), & \text{if } (x,y)\in[a,1]\times[a,1]\\ \min(x,y), & \text{elsewhere}\end{cases} \tag{11}$$

We have received the first uninorms given by Yager and Rybalov

$$U_c(x,y)=\overline{(A_{\max})}^{\max}(x,y), \tag{12}$$

$$U_d(x,y)=\overline{(A_{\min})}^{\max}(x,y). \tag{13}$$

Due to the constructions of these operators for the pairs $(A_{\min},U_d)$ and $(A_{\max},U_c)$ the laws of absorption and distributivity are fulfilled.

Theorem 5 *For the pairs* $(A_{\min},U_d)$ *and* $(A_{\max},U_c)$ *the following hold*

1. Absorption laws

$$A_{\min}(U_d(x,z),x)=x \text{ for all } x\in[0,1] , \tag{14}$$

$$U_d(A_{\min}(x,z),x)=x \text{ for all } x\in[0,1] , \tag{15}$$

$$A_{\max}(U_c(x,z),x)=x \text{ for all } x\in[0,1] , \tag{16}$$

$$U_c(A_{\max}(x,z),x)=x \text{ for all } x\in[0,1] . \tag{17}$$

2. Laws of distributivity

$$A_{\min}(x,U_d(y,z)) = U_d\ (A_{\min}(x,y),A_{\min}(x,z)) \text{ for all } x\in[0,1]\ , \tag{18}$$

$$U_d(x,A_{\min}(y,z)) = A_{\min}\ (U_d(x,y),U_d(x,z)) \text{ for all } x\in[0,1]\ , \tag{19}$$

$$A_{\max}(x,U_c(y,z)) = U_c\ (A_{\max}(x,y),A_{\max}(x,z)) \text{ for all } x\in[0,1]\ , \tag{20}$$

$$U_c(x,A_{\max}(y,z)) = A_{\max}\ (U_c(x,y),U_c(x,z)) \text{ for all } x\in[0,1]\ . \tag{21}$$

Proof. In each disjunctive sub-domain of the unit square the pairs are defined as min and max or max and min operators for which these properties hold. ■

5 The structure of absorbing-norms

Like uninorms the structure of absorbing-norms is closely related to t-norms and t-conorms on the domains $[0,a]\times[0,a]$ and $[a,1]\times[a,1]$.

Following the construction given by Fodor, Yager and Rybalov [4] for uninorms any t-norm T can be transformed to an absorbing-norm on $[a,1]\times[a,1]$ in the following manner.

Definition 6 Let T be any t-norm and define

$$T_A(x,y) = a + (1-a)T\left(\frac{x-a}{1-a},\frac{y-a}{1-a}\right) \quad a \le x,y \le 1. \tag{22}$$

It is easy to see that T_A has the properties of t-norms, and it is also an absorbing norm with absorbing element *a*.

In a similar manner any t-conorm can be transformed to an absorbing-norm.

Definition 7 Let S be any t-conorm and define

$$S_A(x,y) = aS\left(\frac{x}{a},\frac{y}{a}\right) \quad \text{if } 0 \le x,y \le a \tag{23}$$

S_A has the properties of t-conorms, and it is also an absorbing norm with absorbing element *a*.

Theorem 6 *Let be S and T a t-conorm and a t-norm, respectively. The mapping* $A_{\min}^{ST} : [0,1]\times[0,1]\to[0,1]$

$$A_{\min}^{ST}(x,y)=\begin{cases} S_A(x,y), & \text{if } (x,y)\in[0,a]\times[0,a] \\ T_A(x,y), & \text{if } (x,y)\in[a,1]\times[a,1] \\ \min(x,y), & \text{elsewhere} \end{cases} \quad (24)$$

and the mapping $A_{\max}^{ST}:[0,1]\times[0,1]\to[0,1]$

$$A_{\max}^{ST}(x,y)=\begin{cases} S_A(x,y), & \text{if } (x,y)\in[0,a]\times[0,a] \\ T_A(x,y), & \text{if } (x,y)\in[a,1]\times[a,1] \\ \max(x,y), & \text{elsewhere} \end{cases} \quad (25)$$

are absorbing-norms with absorbing element a.

Proof.

Commutativity. It follows from the definitions of the operators.

Assocoativity.

On the domains $[0,a]\times[0,a]$ and $[a,1]\times[a,1]$ $A_{\min}^{ST}$ and $A_{\max}^{ST}$ inherit the properties of S_A and T_A, so associativity is fulfilled.

Consider the rest of the unit square. The following inequalities hold:

$$S_A(x,y)\le S_A(x,a)=a \text{ if } (\mathrm{x},\mathrm{y})\in[0,a]\times[0,a] \quad (26)$$

$$a=T_A(x,a)\le T_A(x,y) \text{ if } (\mathrm{x},\mathrm{y})\in[a,1]\times[a,1] \quad (27)$$

a) First we have to prove that $A_{\min}^{ST}\left(A_{\min}^{ST}(x,y),z\right)=A_{\min}^{ST}\left(x,A_{\min}^{ST}(y,z)\right)$.

Suppose $x\ge a\ge y\ge z$. $A_{\min}^{ST}\left(A_{\min}^{ST}(x,y),z\right)=A_{\min}^{ST}(y,z)=S_A(y,z)$.

$A_{\min}^{ST}\left(x,A_{\min}^{ST}(y,z)\right)=A_{\min}^{ST}(x,S_A(y,z))=S_A(y,z)$ since $S_A(y,z)\le a$ and $S_A(y,z)\le x$.

Suppose $x\ge y\ge a\ge z$. $A_{\min}^{ST}\left(A_{\min}^{ST}(x,y),z\right)=A_{\min}^{ST}(T_A(x,y),z)=z$ since $T_A(x,y)\ge a$ and $A_{\min}^{ST}\left(x,A_{\min}^{ST}(y,z)\right)=A_{\min}^{ST}(x,z)=z$.

b) $A_{\max}^{a}\left(A_{\max}^{a}(x,y),z\right)=A_{\max}^{a}\left(x,A_{\max}^{a}(y,z)\right)$ should be proved.

Assume $x\ge a\ge y\ge z$. $A_{\max}^{ST}\left(A_{\max}^{ST}(x,y),z\right)=A_{\max}^{ST}(x,z)=x$

and $A_{\max}^{ST}\left(x,A_{\max}^{ST}(y,z)\right)=A_{\max}^{ST}(x,S_A(y,z))=x$ since $S_A(y,z)\le a$.

Suppose now $x \geq y \geq a \geq z$. $A_{\max}^{ST}\left(A_{\max}^{ST}(x,y),z\right) = A_{\max}^{ST}\left(T_A(x,y),z\right) = T_A(x,y)$ since $T_A(x,y) \geq a$ and $A_{\max}^{ST}\left(x, A_{\max}^{ST}(y,z)\right) = A_{\max}^{ST}(x,y) = T_A(x,y)$.

2) *Absorbing element*. It is satisfied by the assumptions of the theorem. ■

T_A and S_A are called the *underlying t-norm and t-conorm* of the absorbing-norms, respectively.

It is simple to prove that if *A* is a nullnorm then by the inversion of formulas (22), (23) t-norm and t-conorm are obtained.

Proposition 4 *If A is a given nullnorm, then*

$$S(x,y) = \frac{1}{a} A(ax, ay) \quad \text{if} \quad 0 \leq x, y \leq a, \tag{28}$$

$$T(x,y) = \frac{A(a + (1-a)x, \mathrm{a} + (1\text{-}\mathrm{a})y) - a}{1-a} \quad a \leq x, y \leq 1. \tag{29}$$

are t-norm and t-conorm, respectively.

As the structure of the defined absorbing-norms show, they are constructed from t-norms and t-conorms, so from the point of view of monotony, these are partially monotone mappings on the unit square, i. e. the monotony is fulfilled on the sub-domains of $[0,1]\times[0,1]$. Hence for further investigation it can be assumed that *A* is non-decreasing on the sub-domains $[0,a]\times[a,1]$ and $[a,1]\times[0,a]$. This implies that on these domains

$$\min(x,y) \leq A(x,y) \leq \max(x,y). \tag{30}$$

Analogously to the results of Fodor et al. [4] in uninorms it is possible to introduce the weakest and strangest absorbing-norms.

Theorem 7 *The mapping* $A_W : [0,1]\times[0,1] \to [0,1]$

$$A_W(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in [0,a]\times[0,a] \\ 0, & \text{if } (x,y) \in]a,1]\times]a,1] \\ \min(x,y), & \text{elsewhere} \end{cases} \tag{31}$$

and the mapping $A_S : [0,1]\times[0,1] \to [0,1]$

$$A_S(x,y)=\begin{cases}1, & \text{if } (x,y)\in[0,a[\times[0,a[\\ \min(x,y), & \text{if } (x,y)\in[a,1]\times[a,1]\\ \max(x,y), & \text{elsewhere}\end{cases} \tag{32}$$

are the weakest and strongest absorbing-norms, respectively, i. e. for any absorbing-norm A the following inequality holds:

$$A_W(x,y)\le A(x,y)\le A_S(x,y)\,. \tag{33}$$

Proof. By using the weakest t-norm and the strongest t-conorm in definitions given by (22) and (23) and the inequality (33) the statement follows. ■

The structure of the weakest and the strongest absorbing norms are illustrated in Figures 5, 6.

6 Distance-Based Operations

Let e be an arbitrary element of the closed unit interval [0,1] and denote by $d(x,y)$ the distance of two elements x and y of [0,1]. The idea of definitions of distance-based operators is generated from the reformulation of the definition of the min and max operators as follows

$$\min(x,y)=\begin{cases}x, & \text{if } d(x,0)\le d(y,0)\\ y, & \text{if } d(x,0)> d(y,0)\end{cases},\quad \max(x,y)=\begin{cases}x, & \text{if } d(x,0)\ge d(y,0)\\ y, & \text{if } d(x,0)< d(y,0)\end{cases}$$

Definition 8 The *maximum distance minimum operator with respect to* $e\in[0,1]$ is defined as

$$\max_e^{\min}(x,y)=\begin{cases}x, & \text{if } d(x,e)>d(y,e)\\ y, & \text{if } d(x,e)<d(y,e)\\ \min(x,y), & \text{if } d(x,e)=d(y,e)\end{cases}. \tag{34}$$

Definition 9 The *maximum distance maximum operator with respect to* $e\in[0,1]$ is defined as

$$\max_e^{\max}(x,y)=\begin{cases}x, & \text{if } d(x,e)>d(y,e)\\ y, & \text{if } d(x,e)<d(y,e)\\ \max(x,y), & \text{if } d(x,e)=d(y,e)\end{cases}. \tag{35}$$

Definition 10 The *minimum distance minimum operator with respect to* $e \in [0,1]$ is defined as

$$\min_e^{\min}(x,y) = \begin{cases} x, & \text{if } d(x,e) < d(y,e) \\ y, & \text{if } d(x,e) > d(y,e) \\ \min(x,y), & \text{if } d(x,e) = d(y,e) \end{cases}. \tag{36}$$

Definition 11 The *minimum distance maximum operator with respect to* $e \in [0,1]$ is defined as

$$\min_e^{\max}(x,y) = \begin{cases} x, & \text{if } d(x,e) < d(y,e) \\ y, & \text{if } d(x,e) > d(y,e) \\ \max(x,y), & \text{if } d(x,e) = d(y,e) \end{cases}. \tag{37}$$

6.1 The structure of distance-based operators

It can be proved by simple computation that the distance-based evolutionary operators can be expressed by means of the min and max operators as follows.

$$\max_e^{\min} = \begin{cases} \max(x,y), & \text{if } y > 2e - x \\ \min(x,y), & \text{if } y < 2e - x \\ \min(x,y), & \text{if } y = 2e\text{-}x \end{cases} \tag{38}$$

$$\min_e^{\min} = \begin{cases} \min(x,y), & \text{if } y > 2e - x \\ \max(x,y), & \text{if } y < 2e - x \\ \min(x,y), & \text{if } y = 2e\text{-}x \end{cases} \tag{39}$$

$$\max_e^{\max} = \begin{cases} \max(x,y), & \text{if } y > 2e - x \\ \min(x,y), & \text{if } y < 2e - x \\ \max(x,y), & \text{if } y = 2e\text{-}x \end{cases} \tag{40}$$

$$\min_e^{\max} = \begin{cases} \min(x,y), & \text{if } y > 2e - x \\ \max(x,y), & \text{if } y < 2e - x \\ \max(x,y), & \text{if } y = 2e\text{-}x \end{cases} \tag{41}$$

6.2 Properties of Distance-based Operators

Theorem 8 *The distance-based operators have the following properties* (Rudas [7])

$\max_e^{\min}$

- $\max_e^{\min}(\mathrm{x},\mathrm{x}) = \mathrm{x}, \forall x \in [0,1]$ **,** *that is* $\max_e^{\min}$ *is idempotent,*
- $\max_e^{\min}(\mathrm{e},x) = x$ *that is, e is the neutral element,*
- $\max_e^{\min}$ *is commutative and associative,*
- $\max_e^{\min}$ *is left continuous,*
- $\max_e^{\min}$ *is increasing on each place of* $[0,1]\times[0,1]$.

$\max_e^{\max}$

- $\max_e^{\max}(\mathrm{x},\mathrm{x}) = \mathrm{x}, \forall x \in [0,1]$ **,** *that is* $\max_e^{\max}$ *is idempotent,*
- $\max_e^{\max}(\mathrm{e},x) = x$ *that is, e is the neutral element,*
- $\max_e^{\max}$ *is commutative and associative,*
- $\max_e^{\max}$ *is right continuous***,**
- $\max_e^{\max}$ *is increasing on each place of* $[0,1]\times[0,1]$.

$\min_e^{\min}$

- $\min_e^{\min}(\mathrm{x},\mathrm{x}) = \mathrm{x}, \forall x \in [0,1]$ **,** *that is* $\min_e^{\min}$ *is idempotent,*
- $\min_e^{\min}(\mathrm{e},x) = e$ *that is, e is an absorbing element,*
- $\min_e^{\min}$ *is right continuous,*
- $\min_e^{\min}$ *is commutative and associative.*

$\min_e^{\max}$

- $\min_e^{\max}(\mathrm{x},\mathrm{x}) = \mathrm{x}, \forall x \in [0,1]$ **,** *that is* $\min_e^{\max}$ *is idempotent,*
- $\min_e^{\max}(\mathrm{e},x) = \mathrm{e}$ *that is, e is the absorbing element,*
- $\min_e^{\max}$ *is left continuous,*

- $\min_e^{\max}$ *is commutative and associative.*

Corollary 3

- $\max_e^{\min}$ and $\max_e^{\max}$ are uninorms,
- $\min_e^{\min}$ and $\min_e^{\max}$ are absorbing-norms.

Corollary 4

a) *The dual operators of the uninorm* $\max_e^{\min}$ *is* $\max_{1-e}^{\max}$ *, and*

b) *the dual operators of the uninorm* $\max_e^{\max}$ *is* $\max_{1-e}^{\min}$ **.**

If $e = 0.5$ leads to the entropy based operators introduced by Rudas and Kaynak [8]

Proposition 5 *The pairs* ($\max_e^{\min}$, $\min_e^{\max}$) *and* ($\min_e^{\min}$, $\max_e^{\max}$) *satisfy the absorption laws*

$$\min_e^{\max}\left(\max_e^{\min}(x,y),x\right)=x,\ \forall x\in[0,1], \quad (42)$$

$$\max_e^{\min}\left(\min_e^{\max}(x,y),x\right)=x,\ \forall x\in[0,1], \quad (43)$$

$$\max_e^{\max}\left(\min_e^{\min}(x,y),x\right)=x,\ \forall x\in[0,1], \quad (44)$$

$$\min_e^{\min}\left(\max_e^{\max}(x,y),x\right)=x,\ \forall x\in[0,1]. \quad (45)$$

6.3 Distance-based Operators as Parametric Evolutionary Operators

The min and max operators ss special cases of distance-based operators can be obtained depending on *e* as follows:

a) if *e* = 0 then

$$\max_0^{\min}(x,y)=\max(x,y), \quad (46)$$

$$\max_0^{\max}(x,y)=\max(x,y), \quad (47)$$

$$\min_0^{\min}(x,y)=\min(x,y), \quad (48)$$

$$\min_0^{\max}(x,y)=\min(x,y), \quad (49)$$

b) if $e = 1$ then

$$\max_1^{\min}(x, y) = \min(x, y), \tag{50}$$

$$\max_1^{\max}(x, y) = \min(x, y), \tag{51}$$

$$\min_1^{\min}(x, y) = \max(x, y), \tag{52}$$

$$\min_1^{\max}(x, y) = \max(x, y). \tag{53}$$

This means that the distance-based operators form a *parametric family* with parameter *e*. They are also *evolutionary types* in the sense that if for example in case of $\max_e^{\min}$ while *e* is increasing starting from zero till $e = 1$ the max operator is developing into the min operator.

Conclusions

In this paper a new type of operation, called Absorbing-norm is introduced. It is a non necesseraly monotone operations with an absorbing element. It is shown that absorbing-norsm are kind of complements of uninorms and together fulfill the laws of absorption and distributivity. Distance-based evolutionary operators are given as examples of absorbing and uninorms.

Acknowledgment

The author gratefully acknowledges the support by the Hungarian National Research Fund OTKA in the projects T 34651 and T 034212.

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