Compositional Rule of Inference with Distance-based Operators

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Abstract: In the approximate fuzzy reasoning the covering over of fuzzy rule base input and rule premise of a rule determines the importance of that fuzzy rule and the rule output as well. An axiom system has been created, describing the relationship between the fuzzy rule base system, rule input and rule output. By using distance-based operators a novel reasoning method appears by the compositional rule of inference, which is based on similarity measures of fuzzy sets, and it will be shown that this new system satisfies the mentioned axiom system. [1]

Keywords: compositional rule of inference, fuzzy rule base system, distance based operators, similarity measures

1 Introduction

The concept of approximate reasoning in the known framework of the linguistic information was introduced by Zadeh [13]. The system state is described by a fuzzy rule base system, and the relationship between fuzzy rule base system, system output and system input is modelled by compositional rule of inference.

The first successful practical applications of fuzzy sets were realized by means of the Mamdani inference [10], but the Mamdani's approach is not fully coherent with the paradigm of approximate rasoning [3], [9].

In many applications where the expressing of quantitatively overlapping of two sets is needed, similarity measures are used. The restriction to crisp subsets of a finite universe X with cardinality $\#X = n>0$ was given in [1],[2]. It raises the

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following questions: what are the equivalents of those measures in fuzzy set theory, and which fields for their application can we find.

In the fuzzy rule based control theory and usually in the approximate reasoning, as well as in the covering over of fuzzy rule base input and rule premise of a rule determine the importance of that fuzzy rule and the rule output, too. The practical realization of that notion usually depends on the application. A very thorough overview of mathematical background of that principle can be found in [4], [6], [7].

The Mamdani type controller is based on Generalized Modus Ponens (GMP) inference rule, and the rule output is given with a fuzzy set, which is derived from rule consequence, as a cut of them. This cut is the generalized degree of firing level of the rule, considering actual rule base input, and usually it is the supremum of the minimum of the rule premise and rule input (calculating with their membership functions, of course). The firing level depends on the covering over of the rule base input and rule premise, but first of all it depends on the *height* of those covered membership functions. Engineering applications are satisfied with the minimum operator, but from a mathematical point of view it is interesting to study the behavior of other t-norms in inference mechanism. The using of distance based operators in fuzzy control theory (FLC) was described in [11],[12].

In fact the uninorms offer new possibilities in fuzzy approximate reasoning, because the low level of covering over of rule premise and rule input has measurable influence on rule output as well. In some applications the meaning of that novel t-norms, has practical importance.

In approximate reasoning in a fuzzy control system it can be seen that for distance based operators and other uninorms the height of the firing level is not a real measure of the covering over of the rule premise and rule input any more, we get a normal fuzzy set as uninorm of them. We need a new height-independent measure, and it opens new possibilities and also new problems in approximate reasoning. One of the possibilities is the degree of coincidence, *DOC*, used in the simulation system in [11], and defined by

$$Doc = \frac{\int_X T_e(A, A')dx}{\int_X \max(A, A')dx}$$

where T_e is the minimum distance based minimum operator.

The modified Mamdani’s approach does not rely on the compositional rule inference any more, but still satisfies the basic conditions supposed for the approximate reasoning for a fuzzy rule base system (section 4).

2. Basic Notations

Let be $X \subseteq R$, and $X \neq \emptyset$. A fuzzy subset of A is represented by its membership function $\mu_A : X \to [0,1]$ where the value μ_A is interpreted as the degree to which the value x is contained in A. The set of all fuzzy subsets on X is called set of fuzzy sets on X, and denoted by $F(X)$.

Let be $A \in F(X)$.

The height of fuzzy set A is $\text{height}(A) = \sup_X(\mu_A(x))$,

the support of fuzzy set A is $\text{supp}(A) = \{x | x \in X, \mu_A(x) > 0\}$,

the kernel of fuzzy set A is $\ker(A) = \{x | x \in X, \mu_A(x) = 1\}$.

Fuzzy set is normal iff $\ker(A) \neq \emptyset$.

Let be $A,B \in F(X)$.

A and B are equal ($A=B$), if $\mu_A(x) = \mu_B(x), (\forall x \in X)$,

A is subset of B, ($A<B$ or $A \subset B$), (i.e. B is superset of A), if $\mu_A(x) < \mu_B(x), (\forall x \in X)$.

The convex hull of set $\{B_1(x), B_2(x), ..., B_n(x)\}$ of fuzzy subsets on the same universe X is the smallest convex fuzzy subset $C(x)$ satisfying $B_i(x) \leq C(x)$ for all $i = 1,2,...n$.

A function $T : [0,1]^2 \to [0,1]$ is called *triangular norm (t-norm)* if and only if fulfils the following properties for all $x, y, z \in [0,1]$

(T1) $T(x,y) = T(y,x)$, i.e. the t-norm is commutative

(T2) $T(T(x,y),z) = T(x,T(y,z))$, i.e. the t-norm is associative

(T3) $x \leq y \Rightarrow T(x,z) \leq T(y,z)$, i.e. the t-norm is monotone

(T4) $T(x,1) = x$, i.e., it exist neutral element, which is 1.

If for two t-norms T_1 and T_2 we have $T_1(x,y) \leq T_2(x,y)$ for $\forall (x,y) \in [0,1]^2$, then we say, that T_1 is *weaker than* T_2, or T_2 is *stronger than* T_1. We denote this relation with $T_1 \leq T_2$.

If $T_1 \leq T_2$ and $T_1 \neq T_2$, we shall write $T_1 < T_2$, i.e. in this case, if $T_1 \leq T_2$ and $\left(\exists (x_0, y_0) \in [0,1]^2\right)\left(T_1(x_0, y_0) < T_2(x_0, y_0)\right)$.

A function $S : [0,1]^2 \to [0,1]$ is called *triangular conorm (t-conorm)* if and only if fulfils the following properties for all $x, y, z \in [0,1]$

S1. $S(x,y) = S(y,x)$, i.e. the t-conorm is commutative,

S2. $S(S(x,y),z) = S(x,S(y,z))$, i.e. the t-conorm is associative,

S3. $x \le y \Rightarrow S(x,z) \le S(y,z)$, i.e. the t-conorm is monotone,

S4. $S(x,0) = x$, i.e., it exist neutral element, which is 1.

Both the neutral element 1 of a t-norm and the neutral element 0 of a t-conorm are boundary points of the unit interval. However, there are many important operations whose neutral element is an interior point of the underlying set. The fact that the first three axioms (T1)-(T3) for t-norms coincide with (S1)-(S3) for t-conorms, i.e., the only axiomatic difference lies in the location of the neutral element, has led to the introduction of a new class of binary operations closely related to t-norms and t-conorms. [5]

A *uninorm* is a binary operation U on the unit interval, i.e., a function $U : [0,1]^2 \to [0,1]$ which satisfies the following properties for all $x, y, z \in [0,1]$

(U1) $U(x,y) = U(y,x)$, i.e. the uninorm is commutative

(U2) $U(U(x,y),z) = U(x,U(y,z))$, i.e. the uninorm is associative

(U3) $x \le y \Rightarrow U(x,z) \le U(y,z)$, i.e. the uninorm monotone

(U4) $U(e,x) = x$, i.e., it exist neutral element, which is $e \in [0,1]$.

The distance-based operators can be expressed by means of the min and max operators as follows:

th*e maximum distance minimum operator with respect* to $e \in [0,1]$ is defined as

$$max_e^{min} = \begin{cases} max(x,y), & \text{if } y > 2e - x \\ min(x,y), & \text{if } y < 2e - x \\ min(x,y), & \text{if } y = 2e\text{-}x \end{cases}$$

the *minimum distance minimum operator* with respect to $e \in [0,1]$ is defined as

$$min_e^{min} = \begin{cases} min(x,y), & \text{if } y > 2e - x \\ max(x,y), & \text{if } y < 2e - x \\ min(x,y), & \text{if } y = 2e\text{-}x \end{cases}$$

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The distance-based operators have the following properties

max_e^{min} *and* max_e^{max} are uninorms,

the dual operator of the uninorm max_e^{min} *is* max_{1-e}^{max}, *and*

the dual operator of the uninorm max_e^{max} *is* max_{1-e}^{min}.

3. Approximate reasoning

In the theory of approximate reasoning introduced by Zadeh in 1979, the knowledge of system behaviour and system control can be stated in the form of if-then rules. In Mamadani-based sources it was suggested to represent an

if *x* is *A* then *y* is *B*

simply as a connection (for example as a t-norm, *T*(*A*,*B*) or as *min*) between the so called rule premise: *x* is *A* and rule consequence: *y* is *B*. Let *x* be from universe *X*, *y* from universe *Y*, and let *x* and *y* be linguistic variables. Fuzzy set *A* on $X \subset \Re$ finite universe is characterized by its membership function μ_A: $x \to [0,1]$, and fuzzy set *B* on *Y* universe is characterized by its membership function μ_B: $y \to [0,1]$. The most significant differences between the models of FLC-s lie in the definition of this connection, relation or implication.

The strict modus ponens is replaced with the expectation: let be $B' \supset B$, where B' is a cut of B. That is the Generalized Modus Ponens (GMP), in which the main point is, that the inference y is B' is obtained when the propositions are:

- the i^{th} rule from the rule system of n rules: if x is A_i then y is B_i

- and the system input x is A'.

GMP sees the real influences of the implication or connection choice on the inference mechanisms in fuzzy systems. Usually the general rule consequence for one rule from a rule system is obtained by

$B'(y)=\sup_{x\in X}(T(A'(x),Imp(A(x),B(y)))$

The connection Imp(A,B) is generally defined, and it can be some type of t-norm, too.

In engineering applications the Mamdani implication is widely used. The Mamadani GMP with Mamdani implication inference rule says, that the membership function of the consequence B' is defined by

$B'(y)=\sup_{x\in X}(min(A'(x),min(A(x),B(y)))$

or generally

$B'(y)=\sup_{x\in X}(T(A'(x),T(A(x),B(y)))$

where T is a t-norm.

Using the t-norm properties, from the above expression

$B'(y)=T(\sup_{x\in X} (T(A'(x),A(x))),B(y))$.

Generally speaking, the consequence (rule output) is given with a fuzzy set $B'(y)$, which is derived from rule consequence $B(y)$, as a cut of the $B(y)$. This cut, $\sup_{x\in X}$ $(T(A'(x),A(x)))$, is the generalized degree of firing level of the rule, considering actual rule base input $A'(x)$, and usually depends on the covering over $A(x)$ and $A'(x)$. But first of all it depends on the *sup* of the membership function of $T(A'(x),A(x))$.

The FLC rule base output is constructed as a crisp value calculated with a defuzzyfication model, from rule base output. Rule base output is an aggregation of all rule consequences $B_i'(y)$ from the rule base. As aggregation operator, t-conorm is usually used.

$B'_{out}(y) = S(B_n', S(B_{n-1}', S(\ldots, S(B_2', B_1')))).$

4 The axioms of inference mechanism

Let *RB* a fuzzy rule base system, with rule premises x is A_i and rule consequences: y is B_i. Let x be from universe X, y from universe Y, and let x and y be linguistic variables. Fuzzy set A_i on $X \subset \Re$ finite universe is characterized by its membership function μ_{Ai}: $x \rightarrow [0,1]$, and fuzzy set B on Y universe is characterized by its membership function μ_{Bi}: $y \rightarrow [0,1]$. Let x is A' be the system input, where A' is characterized by its membership function $\mu_{A'}$: $x \rightarrow [0,1]$.

Applying the generalized compositional rule of inference to given components the *i-th* rule output with respect to the given *RB* and given system input A' is y is B_i' given by the expression

$$B_i'(y) = \sup_{x \in X} T\left(A'(x), \mathrm{Imp}\left(A_i(x), B_i(y)\right)\right),$$

where, on a general level, Imp is the relationship between rule base premise and rule base concequence, satisfying the following conditions:

(out1) If the input coincides with one of the premises, then the resulting output coincides with the corresponding consequence, i.e.,

$$(\exists i \in \{1,2,\ldots n\})(A' \geq A_i) \text{ then } B_i' = B_i .$$

(out2) For each normal input A' the output is not contained in all consequences, i.e.,

$$(\exists i \in \{1,2,\ldots n\})(B_i' < B_i) .$$

(out3) The rule output belongs to the convex hull of B_i, ($i \in I$), where $I = \{i | 1 \leq i \leq n, Supp(A') \cap Supp(A_i) \neq 0\}$.

In [9] we can find an axiom system on the same principle.

5 Approximate rasoning with degree og coincidence

Although the minimum plays an exceptional role in fuzzy control theory, there are situations requiring new models. In system control one would intuitively expect: to make the powerful coincidence between fuzzy sets stronger, and the weak coincidence even weaker. The distance-based operators group satisfy these properties, but the covering over $A(x)$ and $A'(x)$ are not really reflected by the *sup* of the membership function of the $min_e^{max}(A_i(x),A'(x))$.

Hence, and because of the non-continuous property of distance-based operators, it was unreasonable to use the classical degree of firing, to give expression to the coincidence of the rule premise (fuzzy set A), and system input (fuzzy set A'), therefore a Degree of Coincidence (*Doc*) for those fuzzy sets has been initiated. This is actually the proportion of area under membership function of the distance-based intersection of those fuzzy sets, and the area under membership function of their union (using *max* as the fuzzy union).

$$Doc_i = \frac{\int_X min_e^{max}\left(A_i(x), A'(x)\right)dx}{\int_X max\left(A_i(x), A'(x)\right)dx}$$

This definition has two advantages:

- it consider the width of coincidence of A_i and A', and not only the "height", the *sup*, and

- the rule output is weighted with a measure of coincidence of A_i and A' in each rule .

5.1 Similarity measures as degree of coincidence

Based on definition of similarity measures from [1] and [2], we can give a generalization of this reason. The next are acceptable:

5.1.1 The Jackard measure

For common sets $R_5 = \frac{\# A \cap B}{\# A \cup B}$, and

For fuzzy sets: $Doc_{R5i} = \frac{\int_X min\left(\mu_{A_i}(x), \mu_{A'}(x)\right)dx}{\int_X max\left(\mu_{A_i}(x), \mu_{A'}(x)\right)dx}$.

5.1.2 The modified cardinality measure

For common sets $R_6 = \dfrac{\# A \cap B}{n}$

For fuzzy sets: $Doc_{R6i} = \dfrac{\int_X min\left(\mu_{Ai}(x), \mu_{A'}(x)\right)dx}{\int_X \max_{forallrules}\left(\mu_{A_i}(x)\right)dx}$.

5.1.3 Similarity measures with distance based operators

$$Doc_{R5_{disbased}}(i) = \frac{\int_X max_e^{min}\left(\mu_{Ai}(x), \mu_{A'}(x)\right)dx}{\int_X max_{1-e}^{max}\left(\mu_{Ai}(x), \mu_{A'}(x)\right)dx}$$

$$Doc_{R6dibased}(i) = \frac{\int_X max_e^{min}\left(\mu_{Ai}(x), \mu_{A'}(x)\right)dx}{\int_X \underset{forallrules}{max_{1-e}^{max}}\left(\mu_{A_i}(x)\right)dx}$$

5.2. How to get the rule output?

The rule output can be the cut of the rule consequence, i.e.,

$B_i'(y)=T(\sup_{x\in X}(T(A'(x),A_i(x))),B_i(y))$, or in this case

$B_i'(y) = min\left(Doc_{similarity}(i), B_i(y)\right)$, where

$Doc_{similarity}(i) \in \left\{Doc_{R5}, Doc_{R6}, Doc_{R5disbased}, Doc_{R6disbased}\right\}$ (1)

Theorem. The rule output (1) and the rule base output $B'_{out}(y) = S(B_n', S(B_{n-1}', S(...., S(B_2', B_1'))))$ satiesfy the axioms (out1)-(out3).

6 Conclusions

Despite the fact, that Mamdani's approach is not entirely based on compositional rule of inference, it is nevertheless very effective in fuzzy approximate reasonng. Because of this it is poosible to apply several t-norms, or, as in this case, uninorms. This leads to further tasks and problems. The problem of the measurement of covering over of the rule premise and rule input is partly solved with the degree of coincidence. But in any case there must be a system of conditions that is to be satisfied by the new model of inference mechanism in fuzzy systems.

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