Non-additive measures

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Abstract: General non-additive measures with some related different monotone measures (some types of variations), which have some important additional properties, can be easier and more efficiently investigated. There is considered the general class of null-additive set functions. We discuss the representation through two Sugeno integrals of a comonotone-$\check{\vee}$-additive and monotone functional L, defined on the class of functions from X to $[-1, 1]$.

Key words and phrases: variation, null-additive set function, Sugeno integral.

1 Introduction

We shall consider three actual results on non-additive measures : variations, the class of null-additive set functions and a recent representation of an important functional in CPT through two Sugeno integrals.

First, we will correspond to every set function ([1, 5, 17]) special positive set functions with some additional properties. Motivated by the notion of the variation of the classical measure we introduce axiomatically the notion of the variation of the general set function and prove that it always exists, but in general case it is not unique.

Second, we consider a wide class of non-additive measures: null-additive set functions, and we present some of important results obtained for this class of set functions, see [17].

Choquet and Sugeno integrals, with respect to a fuzzy measure, are very powerful tools (as aggregation operators) in the field of the decision making. Two crucial properties of the Choquet integral are monotonicity and comonotonic additivity, see [2, 3, 6, 17]. The Cumulative Prospect Theory (CPT), introduced by Tversky and Kahneman [21], combines the Cumulative utility and the generalization of Expected utility, so called sign dependent expected utility. In this paper we consider the two Sugeno integrals representation of the functional L defined on the class of functions $f : X \to [-1, 1]$ on a finite set X.

2 Variations

We consider a general set functions m, $m : \mathcal{D} \to [-\infty, +\infty]$, with $m(\varnothing) = 0$ (extended real-valued set function), see [17], where $\mathcal{D}$ denote a family of subsets of a set X with $\varnothing \in \mathcal{D}$. m is (finite) real-valued set function if $-\infty < m(A) < +\infty$ for all $A \in \mathcal{D}$, and m is monotone if $A \subset B$ implies $m(A) \leq m(B)$ for every $A, B \in \mathcal{D}$. m is non-negative if it is finite and $m(A) \geq 0$ for all $A \in \mathcal{D}$, and $m : \mathcal{D} \to [0, +\infty]$ is positive.

We introduce for an arbitrary set function axiomatically a generalization of the variation.

Definition 1 *Let m be a set function defined on $\mathcal{D}$ with values in $\mathbb{R}$ (or $[0, +\infty]$), with $m(\varnothing) = 0$. Then variation of m is a set function $\eta : \mathcal{D} \to [0, +\infty]$ with the following properties:*

(i) For every $A \subset X$ we have

$$0 \leq \eta(A) \leq +\infty;$$

(ii) $\eta(\varnothing) = 0$;
(iii) $\mid m(A) \mid \leq \eta(A) \quad (A \in \mathcal{D})$;
(iv) η is monotone, i.e., if $B \subset A$, then $\eta(B) \leq \eta(A)$;
(v) $\eta(A) = 0$ if and only if $m(B) = 0$ for every subset B of A from $\mathcal{D}$.

We easily obtain: For every $A \subset X$ we have

$$\eta(A) \geq \sup\{\mid m(B) \mid : \; B \subset A, \; B \in \mathcal{D}\}.$$

Namely, if B is a arbitrary subset of A which belongs to $\mathcal{D}$ we have by the properties (iv) and (iii)

$$\eta(A) \geq \eta(B) \geq \mid m(B) \mid .$$

Theorem 1 *For every set function m defined on $\mathcal{D}$ and with values in $\mathbb{R}$ (or $[0, +\infty]$), with $m(\varnothing) = 0$, always exists its variation, which in general case is not uniquely determined.*

We introduce two special set functions related to a given set function m.

Definition 2 *For an arbitrary but fixed subset A of X and a set function m we define the disjoint variation $\overline{m}$ by*

$$\overline{m}(A) = \sup_I \sum_{i \in I} \mid m(D_i) \mid, \tag{1}$$

where the supremum is taken over all finite families $\{D_i\}_{i \in I}$ of pairwise disjoint sets of $\mathcal{D}$ such that $D_i \subset A \quad (i \in I)$.

Definition 3 *For an arbitrary but fixed $A \in \mathcal{D}$ and a set function m we define the chain variation $\mid m \mid$ by*

$$\mid m \mid (A) = \sup\{\sum_{i=1}^{n} \mid m(A_i) - m(A_{i-1}) \mid :$$

$$\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = A, A_i \in \mathcal{D}, i = 1, \ldots, n\}. \qquad (2)$$

We remark that the supremum in the previous definition is taken over all finite chains between $\varnothing$ and A.

Open problem: *Find all variations of a given arbitrary set function m.*

We shall give a partial answer on this problem, when we require some additional properties of the variation.

Theorem 2 *Let m be a set function defined on Σ with values in $\mathbb{R}$ (or $[0, +\infty]$), with $m(\varnothing) = 0$. Then $\overline{m}$ given by (1) is the smallest variation of m (defined on $\mathcal{P}(X)$) which is superadditive.*

3 Null-additive set functions

Let $\mathcal{R}$ be a ring of subsets of a given set X.

Definition 4 *A set function $m, m : \mathcal{R} \to [0, \infty]$ with $m(\varnothing) = 0$ is called null-additive, if we have*

$$m(A \cup B) = m(A)$$

whenever $A, B \in \mathcal{R}, A \cap B = \varnothing$ and $m(B) = 0$.

There were proved many results in the analogy of the classical measure theory. For the properties of null-additive set functions see [17, 23].

Example 1 Let S be a triangular conorm (t-conorm), i.e., a binary operation on $[0, 1]$ such that it is associative, commutative and monotone with a neutral element 0, see [13]. Some examples of t-conorms are $S_{\mathbf{M}} = \max$, $S_{\mathbf{L}}(x, y) = \min(1, x + y)$, $S_{\lambda}^{\mathbf{SW}}(x, y) = \min(x + y + \lambda xy, 1)$ for $\lambda > -1$. A set function $m : \mathcal{R} \to [0, 1]$ is called S-measure if $m(\varnothing) = 0$ and

$$m(A \cup B) = m(A) S m(B)$$

whenever $A, B \in \mathcal{R}$ and $A \cap B = \varnothing$. m is a monotone null-additive set function. For example, the Hausdorff dimension is a σ-max-measure (see [17]). Further information about these measures can be found in [13, 23].

The importance of null-additive positive monotone measures are stressed also by many results, see [17, 23]. We shall only illustrate this on few cases. Lebesgue decomposition type theorems are proved for null-additive and $\sigma-$null-additive set functions. It is introduced the notion of atoms for monotone set functions, and it turns out that again null-additivity gives good description of them. If the monotone null-additive set function is $\mu-$continuous with respect to a measure μ, then it can be represented by a monotone null-additive set function defined on $\mathcal{P}(\mathbb{N})$, where $\mathbb{N}$ is the set of all natural numbers. For null-additive exhaustive and order continuous set functions Saks decomposition theorem is obtained.

Let m be a positive set function defined on a $\sigma-$algebra Σ.

Definition 5 *Let m be a positive monotone set function on Σ. A set $A \in \Sigma$ is called an atom provided that if $B \subset A$ then*

$$(i)\ m(B) = 0, \text{ or } (ii)\ m(A) = m(B) \text{ and } m(A \setminus B) = 0.$$

Remark 1 For σ-finite fuzzy measure m, each its atom has always finite measure. For null-additive monotone set functions we may suppose in (ii) only $m(A \setminus B) = 0$.

Theorem 3 *Let m be a null-additive monotone set function which is exhaustive. If each set $E \in \Sigma, m(E) > 0$, contains an atom of m, then for each set $E \in \Sigma$ there exist at most countable number of atoms E_i $(i \in I)$ of m such that*

$$m(E \setminus \cup_{i \in I} E_i) = 0.$$

Each set $A \in \Sigma$ contains at most countable number of different atoms of m.

Let m and v be two null-additive set function.

Proposition 1 *If m is $v-$continuous and null-additive, then each atom A of v is also an atom of m proved $m(A) > 0$.*

If for m there is no atom, m is called non-atomic.

There can exist a "vacant" atom which is at most a countable union of null sets.

Example 2 Let $X = \mathbb{N}, \Sigma = \mathcal{P}(X), m(E) = \sup E - \inf E$ for $E \neq \emptyset$, and $m(\emptyset) = 0$. Then every two-point set $A = \{n_1, n_2\}$ is an atom, and each proper subset of A has m-null-measure. Note that the fuzzy measure space (X, Σ, m) is $\sigma-$finite, and that X is represented as a countable union of null sets.

For a monotone continuous measure space (X, Σ, m) we define a subclass $\mathcal{N}(m)$ of Σ as: $E \in \mathcal{N}(m)$ if and only if E is at most a countable union $\cup_n E_n$ where $m(E_n) = 0$ $(n \in \mathbb{N})$.

For the measure m in Example 2, $\mathcal{N}(m) = \Sigma$ holds.

From now on a monotone continuous set function m should be understood to be μ-continuous for some measure μ whenever the set N an P are mentioned. However the following obvious proposition holds independent of the measure μ.

Proposition 2 *(i) If* $m(E \cap P) = m(F \cap P) = 0$ *then* $m((E \cup F) \cap P) = 0$;

(ii) if m *is null-additive then* $m(N) = 0$ *and* $m(E \cap P) = m(E)$ *for any* $E \in \Sigma$.

By (i), such a vacant atom A as in Example 2 appears nowhere in P. The opposite implication of the statement (ii) is not true, see [17].

Define

$$m_P(E) = m(E \cap P) \quad (E \in \Sigma).$$

Let A be an atom of m with $m_P(A) > 0$. If $B \subset A$ and $m_P(B) > 0$, then $m(B \cap P) = m(A)$ and $m(A \backslash B \cap P) = 0$. Thus $m_P(A) \geq m_P(B) = m(A)$ and $m_P(A \backslash B) \leq m(A \backslash B \cap P)$ imply respectively that $m_P(B) = m_P(A)$ and $m_P(A \backslash B) = 0$. Therefore A is also an atom of m_P. If m is null-additive in addition, then $m_P(E) = m(E)$ for every E. Thus $X = P$ may be assumed as for as we concern with the atoms of null-additive set function.

Proposition 3 *Suppose that* A_1 *and* A_2 *are atoms of* m_P. *Then only one of the relations*

$$m_P(A_1 \cap A_2) = 0, \;\; m_P(A_1 \Delta A_2) = 0$$

is possible.

The property of m_P in Proposition 3 is common to additive measures, but not to m_N, the restriction of m to N.

Example 3 Let $X = \mathbb{N}$ and $\Sigma = \mathcal{P}(X)$. Take a positive integer k, and define $m(E) = 1 \wedge [(|E| - k) \vee 0]$. Then $N = X$ (i.e. $m_N = m$), and any set A such that $k < |A| \leq 2k + 1$ is an atom.

Take $A_1 = \{1, 2, ..., 2k + 1\}$ and $A_2 = \{k + 1, k + 2, ..., 3k + 1\}$. Then

$$A_1 \cap A_2 = \{k + 1, ..., 2k + 1\} \quad \text{and}$$

$$A_1 \Delta A_2 = \{1, ..., k\} \cup \{2k + 2, ..., 3k + 1\},$$

and thus follows $m(A_1 \cap A_2) = 1 = m(A_1) = m(A_2)$ while $m(A_1 \Delta A_2) = 1$.

Now we consider the family of all the finite or countable infinite unions of atoms, and denote it by $\mathcal{A}(m)$.

Theorem 4 *If* m *is an* μ*-continuous positive monotone continuous set function, then there exists an element* $\mathbf{A} \in \mathcal{A}(m)$ *such that*

(i) $A \in \mathcal{A}(m)$ *implies* $\mu(A \backslash \mathbf{A}) = 0$ *(maximality),*

(ii) m *is non-atomic on* $X \backslash \mathbf{A}$, *and*

(iii) $P \cap \mathbf{A}$ *is represented as a disjoint union of atoms* $\{A_n : n \in \mathbb{N}\}$ *in the sense of* $m((P \cap \mathbf{A}) \backslash (\cup_n A_n)) = 0$.

Theorem 5 *Let m be an μ-continuous monotone set function defined on Σ. If m is null-additive, then there exist a map $r : \Sigma \cap \mathbf{A} \to \mathcal{P}(\mathbf{N})$ and monotone measure v on $(\mathbb{N}, \mathcal{P}(\mathbf{N}))$ such that*

(i) $r(X) = \mathbb{N}$, $r(X \backslash E) = \mathbb{N} \backslash r(E)$ and $r(\cup_j E_j) = \cup_j r(E_j)$,

(ii) $m(E) = v(r(E))$ for each $E \in \Sigma \cap \mathbf{A}$.

In other words the monotone measure space $(\mathbf{A}, \Sigma \cap \mathbf{A}, m)$ is represented by a simple fuzzy measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), v)$.

Theorem 6 *(Saks Decomposition) Let m be a null-additive set function, which is finite, exhaustive and order continuous or it is a finite fuzzy measure. Then for every $A \in \Sigma$ and every $\epsilon > 0$ there exists a finite number $A_0, A_1, ..., A_r$ of pairwise disjoint elements of Σ such that (i) $A = \cup_{i=0}^{r} A_i$; (ii) each A_i (i = 0,1,...,r) is either an atom of m and $m(A_i) > \epsilon$, or $m(A_i) \leq \epsilon$.*

Remark 2 It is easy to check that the preceding proof works also for finite concave exhaustive and order continuous set function, although it may not be null-additive.

4 The Sugeno integral and the representation of the comonotone-$\check{\vee}$-additive functional

Let m be a monotone positive set function defined on a σ-algebra Σ. The Choquet integral of a non-negative measurable function f on $A \in \Sigma$ is given by (see [5, 6, 17])

$$(C) \int_A f \, dm = \int_0^\infty m(A \cap \{x : f(x) \geq r\}) \, dr.$$

Sugeno has introduced another integral on a set $A \subset X$

$$(S) \int_A f(x) \, dm = \sup_{r \in [0, +\infty]} \min[r, m(A \cap \{x : f(x) \geq r\})].$$

CPT holds if there exist two fuzzy measures, μ^+ and μ^-, which ensure that the utility functional L, model for preference representation, can be represented by the difference of two Choquet integrals, i.e.,

$$L(f) = (C) \int f^+ \, d\mu^+ - (C) \int f^- \, d\mu^-,$$

where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. The functional L is defined on the class of functions (prospects, alternatives) $f : X \longrightarrow \mathbb{R}$, mapping a state space X in some subset of the real line. The f^+ is called the gain part of the prospect f, and $-f^-$ is called the loss part of f.

There is a close connection between general functionals with such properties and the Choquet integral. A real valued functional L on the class $\mathcal{M}$ of all measurable functions is monotone if for all $f, g \in \mathcal{M}$ with $f \leq g$ we have $L(f) \leq L(g)$. We say that L is comonotonic additive if for all $f, g \in \mathcal{M}$ which are comonotonic, i.e., $f(x) < f(x_1) \Rightarrow g(x) \leq g(x_1)$ $(x, x_1 \in X)$, we have $L(f + g) = L(f) + L(g)$. There was proved in [22] that a monotone and comonotonic additive functional on $\mathcal{M}$ can be represented by a Choquet integral with respect to a fuzzy measure. Narukawa proved in [16] that comonotone-additive and monotone functional can be represented as a difference of two Choquet integrals and gave the conditions for which it can be represented by one Choquet integral.

The Sugeno integral-based operator is one of the non-linear (w.r.t $\vee$) functionals on the class of measurable functions which is comonotone-maxitive, monotone and $\wedge$-homogeneous. The extension of the Sugeno integral to the bipolar scale $[-1, 1]$ in the spirit of the symmetric extension of Choquet integral, has been proposed by M. Grabisch in [12], and it is useful as a framework for Cumulative Prospect Theory in an ordinal context.

We assume that X is an universal set. Let $\mathcal{A}$ be a σ-algebra of subsets of X. Let $\mathcal{M}(X)$ be a class of all measurable functions from X to $[0, 1]$ and let $\mu : \mathcal{A} \longrightarrow [0, 1]$ be a normalized fuzzy measure.

We recall that the operation $\check{\vee}$ is *the symmetric maximum* originally introduced in [11] and which is defined by:

$$a \check{\vee} b := \begin{cases} -(|a| \vee |b|)\,, & b \neq -a \text{ and } |a| \vee |b| = -a \text{ or } \;= -b, \\ 0\,, & b = -a, \\ |a| \vee |b|\,, & \text{otherwise}, \end{cases}$$

The operation $\check{\wedge}$ is *symmetric minimum* given by:

$$a \check{\wedge} b := \begin{cases} -(|a| \wedge |b|)\,, & \operatorname{sign} a \neq \operatorname{sign} b, \\ |a| \wedge |b|\,, & \text{otherwise}. \end{cases}$$

We consider now the class $\mathcal{K}(X) = \{\, f \mid f : X \to [-1, 1]\,\}$. The class of all non-negative functions in $\mathcal{K}(X)$ is denoted by $\mathcal{K}(X)^+$.

Let X be a finite set, $X = \{x_1, x_2, \ldots, x_n\}$, and let $\mu : 2^X \longrightarrow [0, 1]$ be a normalized fuzzy measure. Recall that two measurable functions f and g on X are called *comonotone* [6] if they are measurable with respect to the same chain $\mathcal{C}$ in $\mathcal{A}$ ($\mathcal{A}$ is a σ-algebra of subsets of X). Equivalently, comonotonicity of the functions f and g can be expressed as follows: $f(x) < f(x_1) \;\; \Rightarrow \;\; g(x) \leq g(x_1)$ for all $x\,,\, x_1 \in X$.

Let X be a countable set, $X = \{x_1, x_2, \ldots,\}$ and let $\mathcal{K}_s(X)$ be a class of functions with finite support, $\mathcal{K}_s(X) = \{\, f \mid f : X \to [-1, 1]\,, |\operatorname{supp}(f)| < \infty\,\}$ where the support of f is given by $\operatorname{supp}(f) = \{\, x \mid f(x) \neq 0\}$.

For comonotone functions $f,\; g \in \mathcal{K}_s(X)$ we have that $f(x) > 0$ implies $g(x) \geq 0$.

Let L be a functional $L : \mathcal{K}(X) \longrightarrow [-1, 1]$. We introduce now the fuzzy rank and sign dependent functional and comonotone-$\check{\vee}$-additive functional.

Definition 6 *A functional $L : \mathcal{K}(X) \longrightarrow [-1,1]$ is a* fuzzy rank and sign dependent functional *on $\mathcal{K}(X)$ if there exist two fuzzy measures μ^+ and μ^- such that for all $f \in \mathcal{K}(X)$*

$$L(f) = \Big((S)\int f^+ \, d\mu^+\Big) \check{\vee} \Big(-(S)\int f^- \, d\mu^-\Big),$$

where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$.

Note that in the case when $\mu^+ = \mu^-$ and X is finite the fuzzy rank and sign dependent functional (f.r.s.d. functional for short) is exactly the Symmetric Sugeno integral. If the f.r.s.d. functional L is the Symmetric Sugeno integral then we have

$$L(-f) = -L(f).$$

Definition 7 *Let L be a functional on $\mathcal{K}(X)$, $L : \mathcal{K}(X) \longrightarrow [-1,1]$.*
(i) L is comonotone-$\check{\vee}$-additive *iff*

$$L(f \check{\vee} g) \;=\; L(f) \check{\vee} L(g) \tag{3}$$

for all comonotone functions f, $g \in \mathcal{K}(X)$.
(ii) L is monotone *iff*

$$f \le g \quad \Longrightarrow \quad L(f) \le L(g) \tag{4}$$

for all functions f, $g \in \mathcal{K}(X)$.
(iii) L is $\check{\wedge}$-homogeneous *iff*

$$L(a \check{\wedge} f) \;=\; a \check{\wedge} L(f) \tag{5}$$

for every $f \in \mathcal{K}(X)$ and $a \ge 0$.

For a monotone and idempotent functional L we have that $L(a \cdot \mathbf{1}_X) = a$ for all $a \in [-1,1]$.

Proposition 4 *A monotone and idempotent functional L on $\mathcal{K}(X)$ is $\check{\wedge}$-homogeneous.*

Let X be a finite set and L a functional $L : \mathcal{K}(X) \longrightarrow [-1,1]$. Let $C \subset X$. We define two set functions μ_L^+ and μ_L^- induced by the functional L:

$$\mu_L^+(C) := L(\mathbf{1}_C) \quad \text{and} \quad \mu_L^-(C) := -L(-\mathbf{1}_C)\,.$$

For $A \subset B$ we have $\mathbf{1}_A \le \mathbf{1}_B$ and if monotonicity of L was supposed than we obtain $\mu_L^+(A) \le \mu_L^+(B)$ and $\mu_L^-(A) \le \mu_L^-(B)$, i.e., μ_L^+ and μ_L^- are the fuzzy measures. Finally, from Proposition 2. we have the next result.

Theorem 7 *[19] If $L : \mathcal{K}(X) \longrightarrow [-1,1]$ is an idempotent comonotone-$\check{\vee}$-additive and monotone functional on $\mathcal{K}(X)$, then L is a f.r.s.d functional, i.e., there exist two fuzzy measures μ_L^+ and μ_L^- such that*

$$L(f) = \Big((S)\int f^+ \, d\mu_L^+\Big) \check{\vee} \Big(-(S)\int f^- \, d\mu_L^-\Big).$$

A f.r.s.d. functional on $\mathcal{K}(X)$ is not always comonotone-$\check{\vee}$-additive.

Example 4 Let $X = \{1, 2\}$. We take fuzzy measure μ defined by

$$\mu(A) = \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases}$$

We consider the f.r.s.d. functional L defined by

$$L(f) = \Big((S)\int f^+ \, d\mu\Big) \check{\vee} \Big(-(S)\int f^- \, d\mu\Big).$$

If we take comonotone functions $f,\, g \in \mathcal{K}(X)$ defined by $f(1) = 0.5, f(2) = 0$ and $g(1) = 0.5, g(2) = -0.5$, respectively, then $L(f) = 0.5$ and $L(g) = 0$, but $L(f \check{\vee} g) = 0$.

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