

# An extension principle for interactive fuzzy numbers

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## Abstract

In this paper we shall formulate an extension principle for interactive fuzzy numbers.

## 1 Introduction

A fuzzy number  $A$  is a fuzzy set of the real line  $\mathbb{R}$  with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by  $\mathcal{F}$ . A  $\gamma$ -level set of a fuzzy number  $A$  is defined by  $[A]^\gamma = \{t \in \mathbb{R} | A(t) \geq \gamma\}$  if  $\gamma > 0$  and  $[A]^\gamma = \text{cl}\{t \in \mathbb{R} | A(t) > 0\}$  (the closure of the support of  $A$ ) if  $\gamma = 0$ . An  $n$ -dimensional possibility distribution  $C$  is a fuzzy set in  $\mathbb{R}^n$  with a normal, continuous membership function of bounded support. The family of possibility distributions of dimension  $n$  will be denoted by  $\mathcal{F}_n$ .

Let us recall the concept and some basic properties of joint possibility distribution introduced in [4]. If  $A_1, \dots, A_n \in \mathcal{F}$  are fuzzy numbers, and  $C \in \mathcal{F}_n$  is a possibility distribution, then  $C$  is said to be their joint possibility distribution if

$$A_i(x_i) = \max_{x_j \in \mathbb{R}, j \neq i} C(x_1, \dots, x_n), \quad (1)$$

holds for all  $x_i \in \mathbb{R}, i = 1, \dots, n$ . Furthermore,  $A_i$  is called the  $i$ -th marginal possibility distribution of  $C$ . If  $A_1, \dots, A_n \in \mathcal{F}$  are fuzzy numbers, and  $C$  is their joint possibility distribution then,

$$C(x_1, \dots, x_n) \leq \min\{A_1(x_1), \dots, A_n(x_n)\},$$

or equivalently,

$$[C]^\gamma \subseteq [A_1]^\gamma \times \dots \times [A_n]^\gamma,$$

holds for all  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  and  $\gamma \in [0, 1]$ .

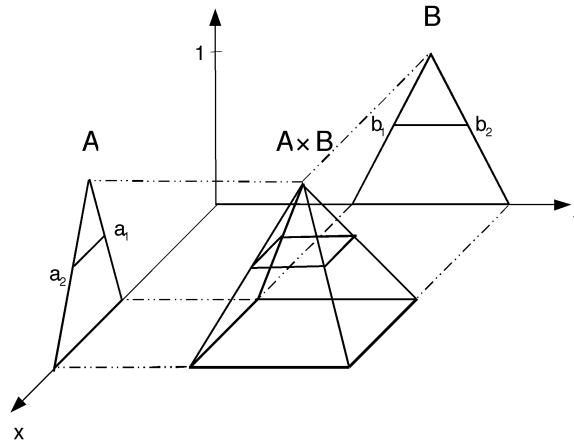


Figure 1: Non-interactive fuzzy numbers.

The biggest (in the sense of subethood of fuzzy sets) possibility distribution plays a special role; it defines the concept of non-interactivity of fuzzy numbers. Fuzzy numbers  $A_1, \dots, A_n$  are said to be non-interactive if their joint possibility distribution  $C$  satisfies the relationship

$$C(x_1, \dots, x_n) = \min\{A_1(x_1), \dots, A_n(x_n)\},$$

or equivalently,

$$[C]^\gamma = [A_1]^\gamma \times \dots \times [A_n]^\gamma,$$

for all  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  and  $\gamma \in [0, 1]$ .

## 2 The extension principle for interactive fuzzy numbers

Following [1] we will formulate the following extension principle for interactive fuzzy numbers.

**Definition 2.1.** [1] Let  $A_1, \dots, A_n \in \mathcal{F}$  and let  $C$  be their joint possibility distribution, and let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function. Then

$$f_C(A_1, \dots, A_n)$$

is defined by

$$f_C(A_1, \dots, A_n)(y) = \sup_{y=f(x_1, \dots, x_n)} C(x_1, \dots, x_n). \quad (2)$$

Since the joint possibility distribution uniquely defines its marginal distributions - by the rule of *falling shadows* (1) - we shall use the notation

$$f(C) = f_C(A_1, \dots, A_n).$$

**Remark 2.1.** If  $A_1, \dots, A_n$  are non-interactive, that is, their joint possibility distribution is defined by

$$C(x_1, \dots, x_n) = \min\{A_1(x_1), \dots, A_n(x_n)\},$$

then (2) turns into the extension principle introduced by Zadeh in 1965 [6]. Furthermore, if

$$C(x_1, \dots, x_n) = T(A_1(x_1), \dots, A_n(x_n)),$$

where  $T$  is a  $t$ -norm then we get the  $t$ -norm-based extension principle.

Let  $C$  be the joint distribution of  $A_1, \dots, A_n \in \mathcal{F}$ , and let  $\pi_i(x_1, \dots, x_n) = x_i$  be the projection function onto the  $i$ -th axis. Then, using Definition 2.1 we find

$$\pi_i(C)(y) = \sup_{y=\pi_i(x_1, \dots, x_n)} C(x_1, \dots, x_n) = \sup_{y=x_i} C(x_1, \dots, x_n) = A_i(y),$$

for any  $y \in \mathbb{R}$ , that is,

$$\pi_i(A_1, \dots, A_n) = A_i,$$

for  $i = 1, \dots, n$ . Furthermore, for any  $\gamma \in [0, 1]$  we find

$$[f(C)]^\gamma = f([C]^\gamma).$$

Hence, we have the following lemma, which can be interpreted as a generalization of Nguyen's theorem [5].

**Lemma 1.** [1] Let  $A_1, \dots, A_n \in \mathcal{F}$  be fuzzy numbers, let  $C$  be their joint possibility distribution, and let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function. Then,

$$[f_C(A_1, \dots, A_n)]^\gamma = [f(C)]^\gamma = f([C]^\gamma)$$

for all  $\gamma \in [0, 1]$ .

In particular,  $[f(C)]^\gamma \subset \mathbb{R}$  is a compact interval for any  $\gamma \in [0, 1]$  since continuous functions map compact and connected sets into compact and connected sets. Furthermore, from the relationships  $[C]^\mu \subseteq [C]^\tau$  and  $[f(C)]^\mu \subseteq [f(C)]^\tau$  for any  $0 \leq \tau \leq \mu \leq 1$  we get that  $f(C)$  is always a fuzzy number.

### 3 Example

We illustrate Lemma 1 by a simple example.

Let  $A = (a, \alpha)$  and  $B = (b, \alpha)$  be symmetrical triangular fuzzy numbers with

$$[A]^\gamma = [a - (1 - \gamma)\alpha, a + (1 - \gamma)\alpha], [B]^\gamma = [b - (1 - \gamma)\alpha, b + (1 - \gamma)\alpha],$$

and let their joint possibility distribution  $C$  be defined by the product  $t$ -norm, i.e.

$$C(x, y) = A(x) \cdot B(y),$$

and let  $f(x_1, x_2) = x_1 + x_2$  be the addition operator on  $\mathbb{R}^2$ . Then,

$$\begin{aligned} [C]^\gamma &= \text{cl}\{(x, y) \in \mathbb{R}^2 \mid C(x, y) > \gamma\} \\ &= \{(x, y) \in \mathbb{R}^2 \mid (\alpha - |x - a|)(\alpha - |y - b|) \\ &\geq \alpha^2\gamma, |x - a| \leq \alpha, |y - b| \leq \alpha\}, \end{aligned}$$

and according to Lemma 1 a  $\gamma$ -level set of  $A + B$  is computed by

$$\begin{aligned} [A + B]^\gamma &= \{x + y \in \mathbb{R} \mid (x, y) \in [C]^\gamma\} \\ &= \{x + y \in \mathbb{R} \mid (\alpha - |x - a|)(\alpha - |y - b|) \\ &\geq \alpha^2\gamma, |x - a| \leq \alpha, |y - b| \leq \alpha\}. \end{aligned}$$

Since

$$a + b - |x - a| - |y - b| \leq x + y \leq a + b + |x - a| + |y - b|,$$

and by using the inequality between the arithmetic and geometric means we get

$$\alpha\sqrt{\gamma} \leq \sqrt{(\alpha - |x - a|)(\alpha - |y - b|)} \leq \alpha - \frac{|x - a| + |y - b|}{2},$$

that is  $|x - a| + |y - b| \leq 2\alpha(1 - \sqrt{\gamma})$  holds for all  $\gamma \in [0, 1]$  and  $x, y \in \mathbb{R}$  for which  $|x - a| \leq \alpha$  and  $|y - b| \leq \alpha$ , therefore

$$x + y \leq a + b + |x - a| + |y - b| \leq a + b + 2\alpha(1 - \sqrt{\gamma}) \quad (3)$$

and

$$x + y \geq a + b - |x - a| - |y - b| \geq a + b - 2\alpha(1 - \sqrt{\gamma}). \quad (4)$$

Furthermore, inequalities (3) and (4) are strong, which can be seen by setting

$$x = a + \alpha(1 - \sqrt{\gamma}), y = b + \alpha(1 - \sqrt{\gamma})$$

and

$$x = a - \alpha(1 - \sqrt{\gamma}), y = b - \alpha(1 - \sqrt{\gamma}),$$

respectively. Finally, we get

$$[A + B]^\gamma = [a + b - 2\alpha(1 - \sqrt{\gamma}), a + b + 2\alpha(1 - \sqrt{\gamma})]$$

for all  $\gamma \in [0, 1]$ , and the membership function of the sum has the following form

$$(A + B)(z) = \left(1 - \frac{|a + b - z|}{2\alpha}\right)^2 \cdot \chi_{[a+b-2\alpha, a+b+2\alpha]}(z),$$

where  $\chi_{[a+b-2\alpha, a+b+2\alpha]}$  denotes the characteristic function of interval

$$[a + b - 2\alpha, a + b + 2\alpha].$$

## 4 Summary

We have generalized Zadeh's extension principle for interactive fuzzy numbers via joint possibility distributions. We have shown that Nguyen's theorem remains valid in this environment. Other results on interactive fuzzy numbers can be found in [2, 3, 4].

## References

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