Opinion Dynamics on Multiple Interdependent Topics: Modeling and Analysis

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GIST DCASL

- Members: 1 Professor/ 9 PhD students/ 5 MS students/ 1 Post-Doc.
- Alumni: 13 PhD, 25 MS, 10+ Interns, etc..
- Collaborations: with ANU, SNU, Technion, Kyoto U., Tokyo Tech., Colorado School of Mines, etc.
- Research: Formation control, Autonomous systems (Group of drones), Autonomous vehicles, Distributed coordination, Complex networks

Social networks

The ability to collect and analyze such social network data provides unique opportunities to understand the underlying principles of social networks, their formation, evolution and characteristics.

- **Algorithms:** **Design of novel algorithms, algorithms for analyzing social networks, as well to improve the performance of information sharing in social networks**.
- **Systems:** Development of new systems to harvest, collect and analyze data from online social networks, as well building novel social networking applications.
- **User Behavior:** Understanding the user behavior in social networks, in particular understanding incentives for users to form and participate in social networks, as well as **understand the importance of communities, influence and reputation in social networks**.

http://web.cs.toronto.edu/research/areas/sn.htm

Opinion dynamics in social networks

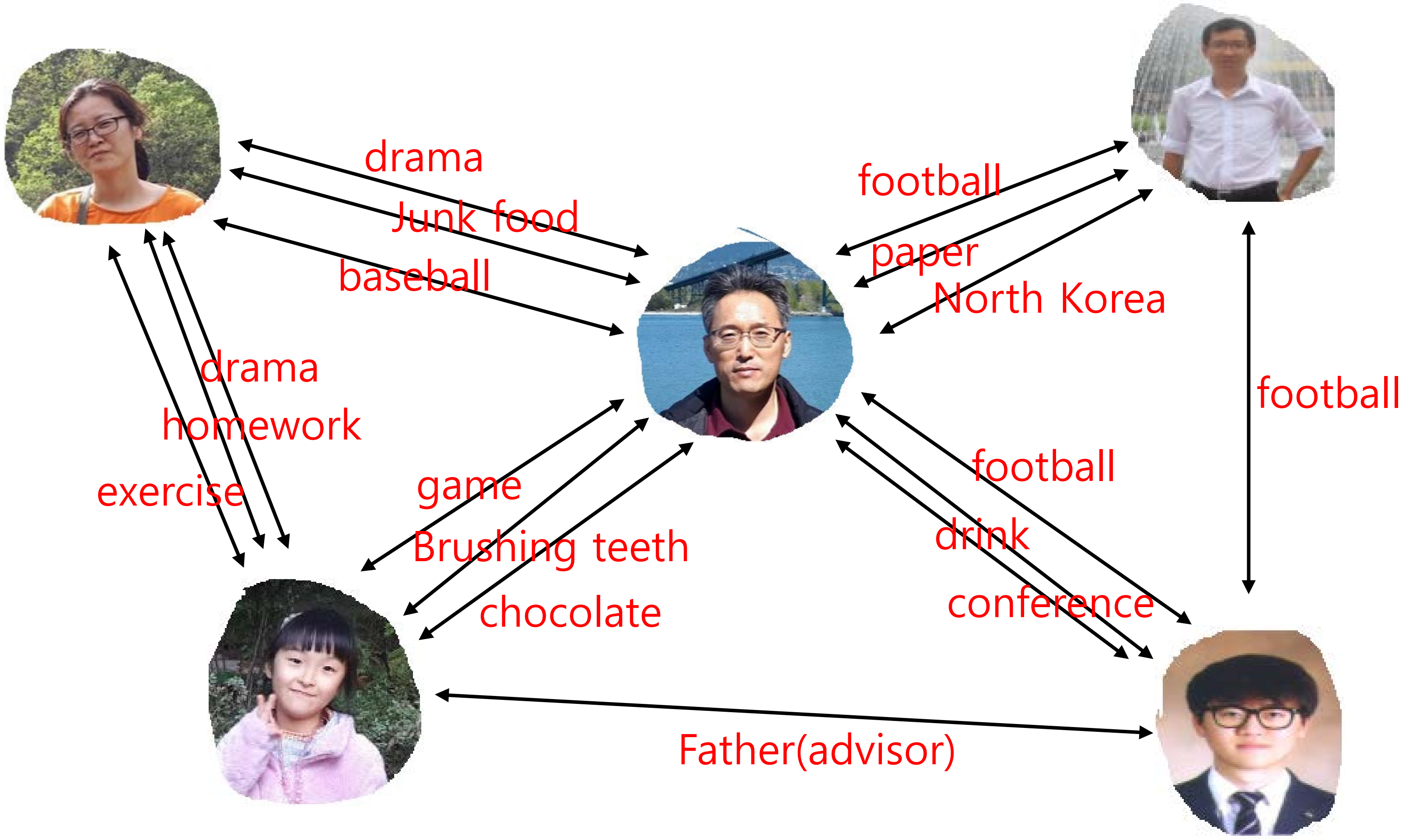
Knowing more people gives one greater access, enhances the sharing of information, and makes it easier to influence others for the simple reason that influencing people you know is easier than influencing strangers.

https://www.livetradingnews.com/share-network-powerful-becomes-7578.html#.WwumA0iFO70

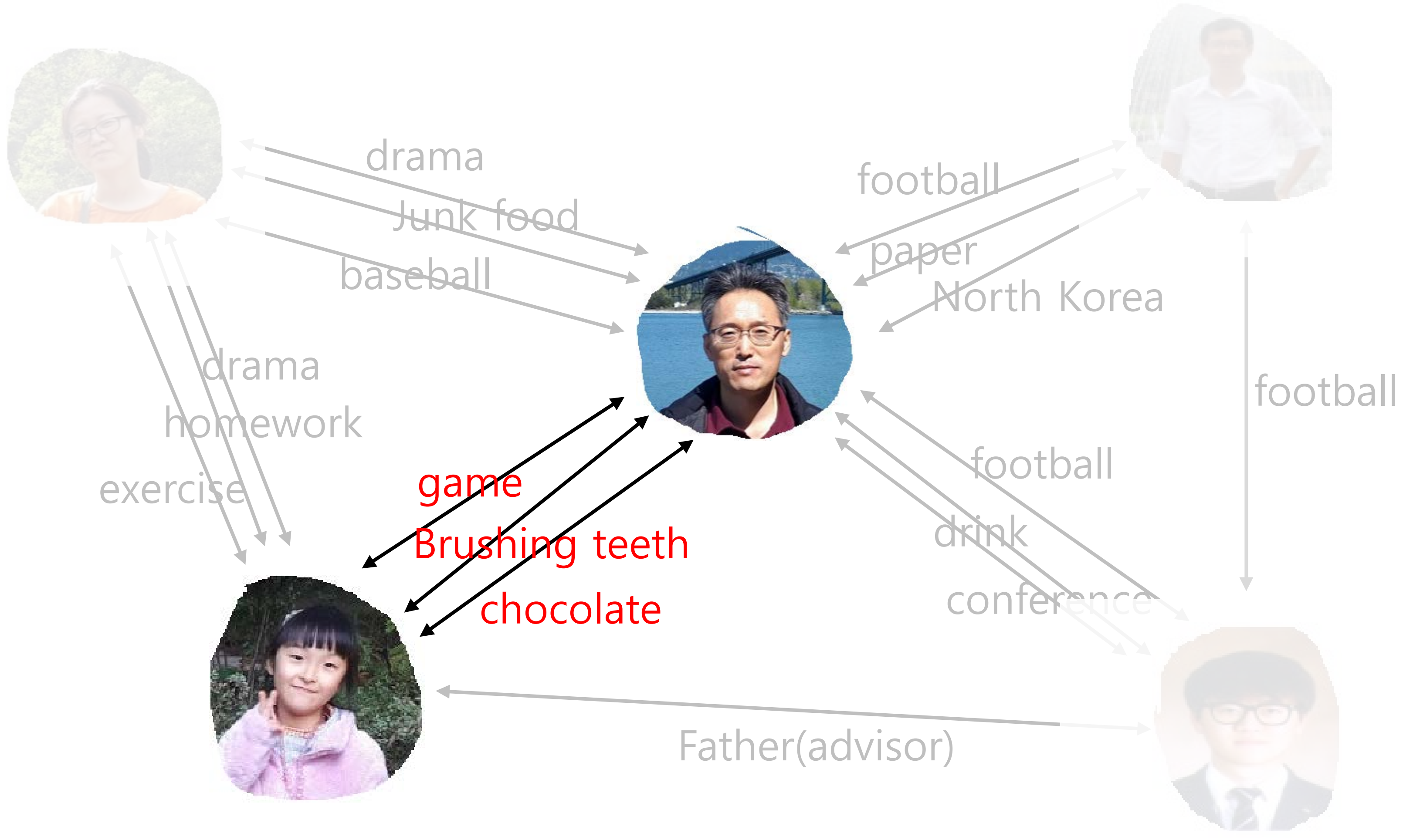
Part-1: Modeling

Multiple Interdependent Topics

Multiple Interdependent Topics



Multiple Interdependent Topics

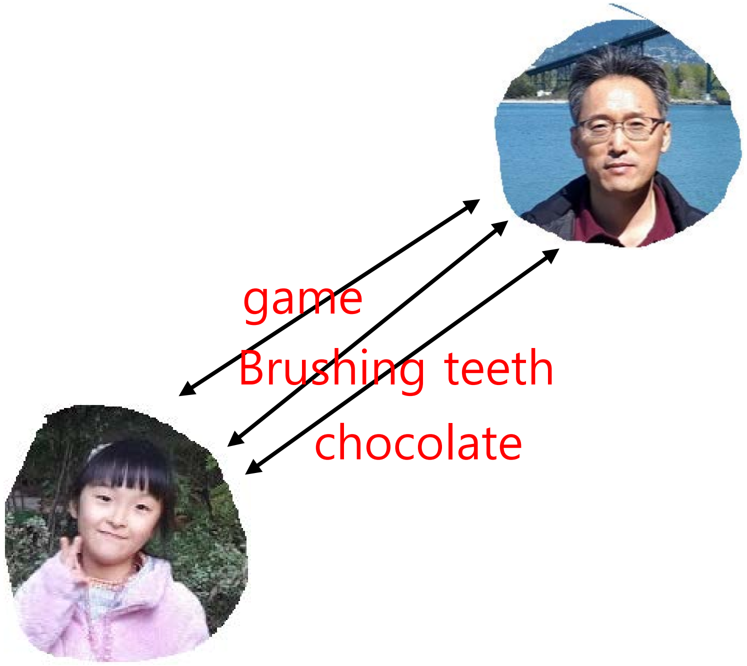


Multiple Interdependent Topics

Game: S_g = 0.7 vs. H_g = 0.3

Brushing: S_b = 0.1 vs. H_b = 1.0

chocolate: S_c = 0.9 vs. H_c = 0.01



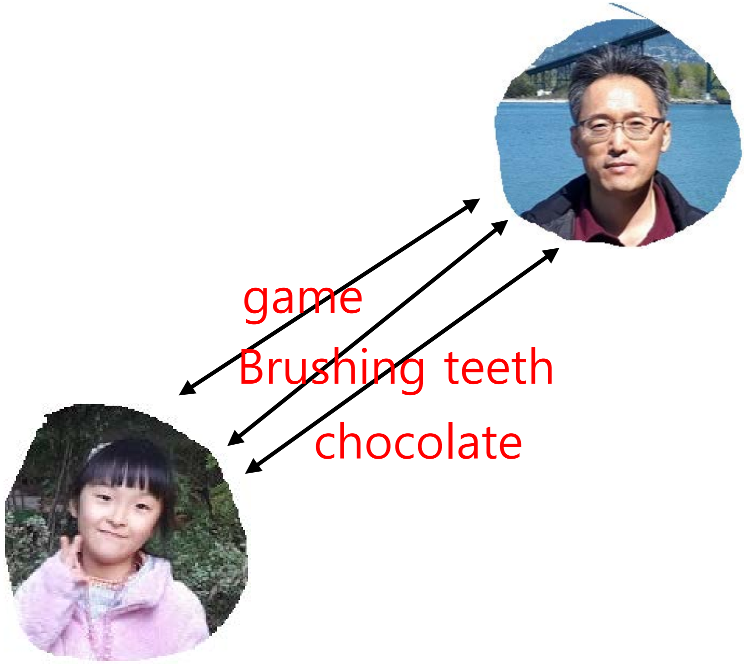
Multiple Interdependent Topics

Preference or disfavor

Game: S_g = 0.7 vs. H_g = 0.3

Brushing: S_b = 0.1 vs. H_b = 1.0

chocolate: S_c = 0.9 vs. H_c = 0.01



Multiple Interdependent Topics

Game: $S_g = 0.7$ vs. $H_g = 0.3$

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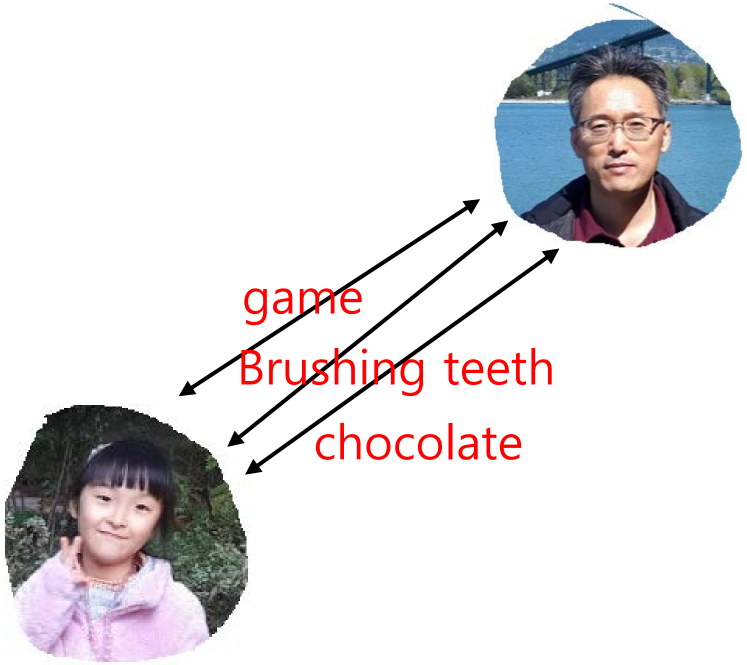
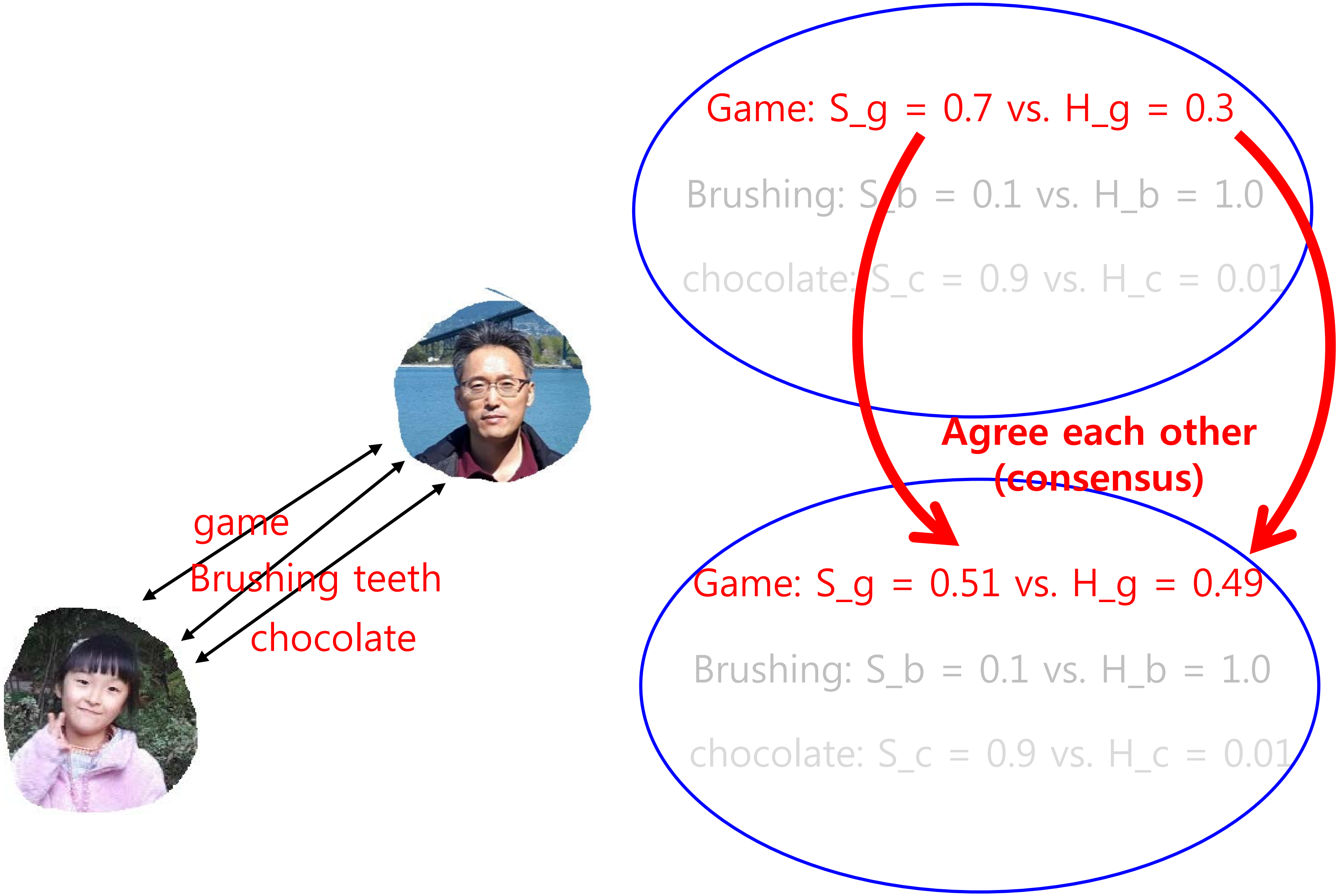
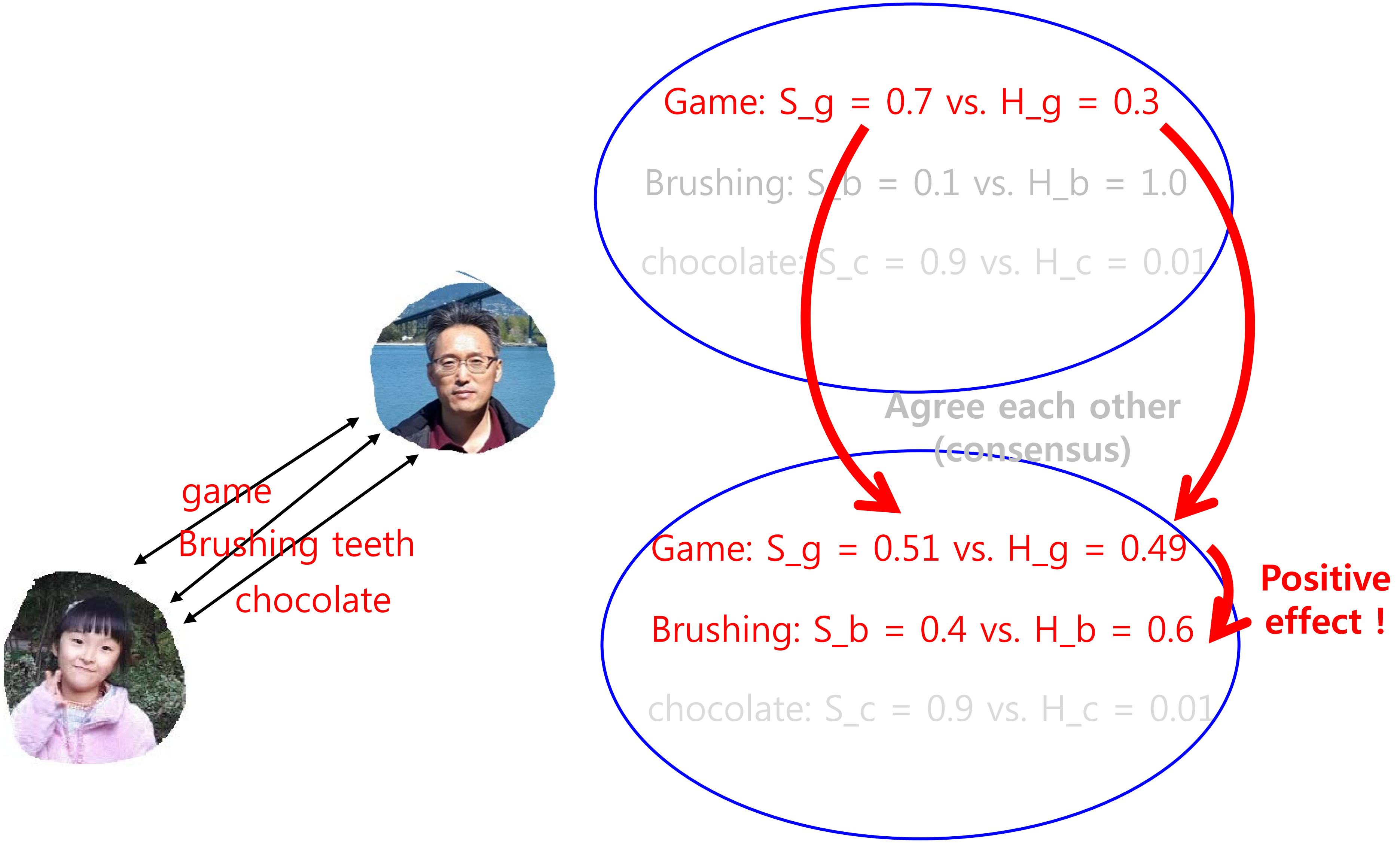
chocolate: $S_c = 0.9$ vs. $H_c = 0.01$



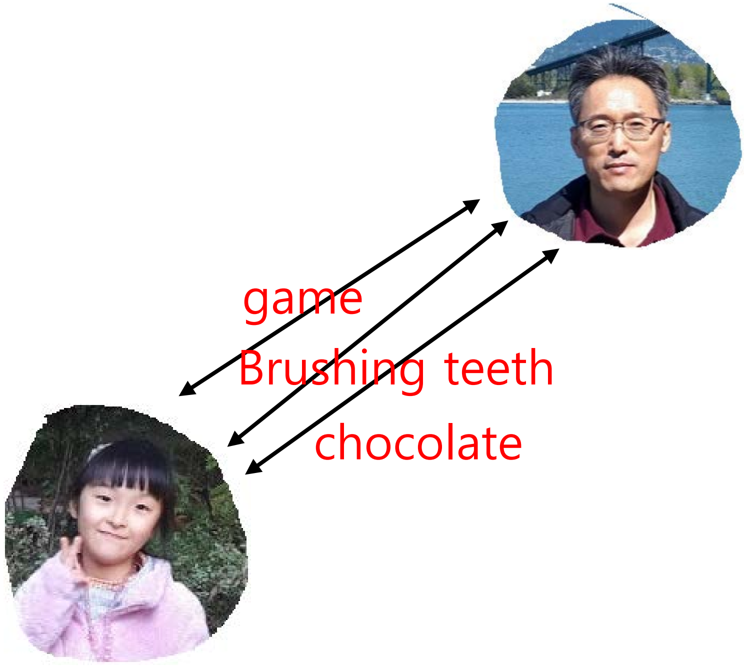
Multiple Interdependent Topics



Multiple Interdependent Topics



Multiple Interdependent Topics



Game: S_g = 0.7 vs. H_g = 0.3

Brushing: S_b = 0.1 vs. H_b = 1.0

chocolate: S_c = 0.9 vs. H_c = 0.01

Agree each other (consensus)

Game: S_g = 0.51 vs. H_g = 0.49

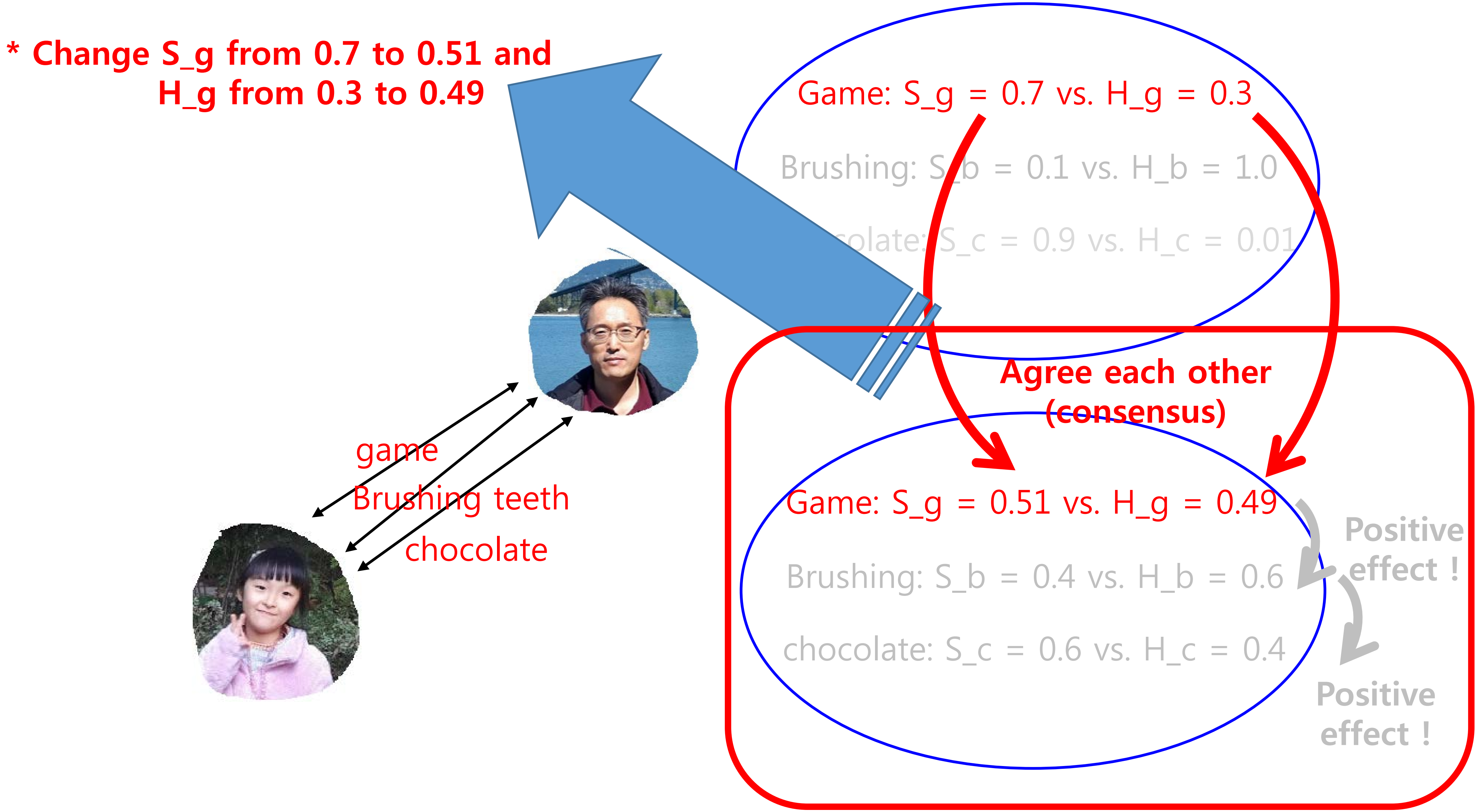
Brushing: S_b = 0.4 vs. H_b = 0.6

chocolate: S_c = 0.6 vs. H_c = 0.4

Positive effect !

Positive effect !

Multiple Interdependent Topics



Multiple Interdependent Topics

*** Change S_g from 0.7 to 0.51 and H_g from 0.3 to 0.49**

Then, it will give you positive feedbacks…

Game: S_g = 0.7 vs. H_g = 0.3

Brushing: S_b = 0.1 vs. H_b = 1.0

olate: S_c = 0.9 vs. H_c = 0.01

game

Brushing teeth

chocolate

Agree each other (consensus)

Game: S_g = 0.51 vs. H_g = 0.49

Brushing: S_b = 0.4 vs. H_b = 0.6

chocolate: S_c = 0.6 vs. H_c = 0.4

Positive effect !

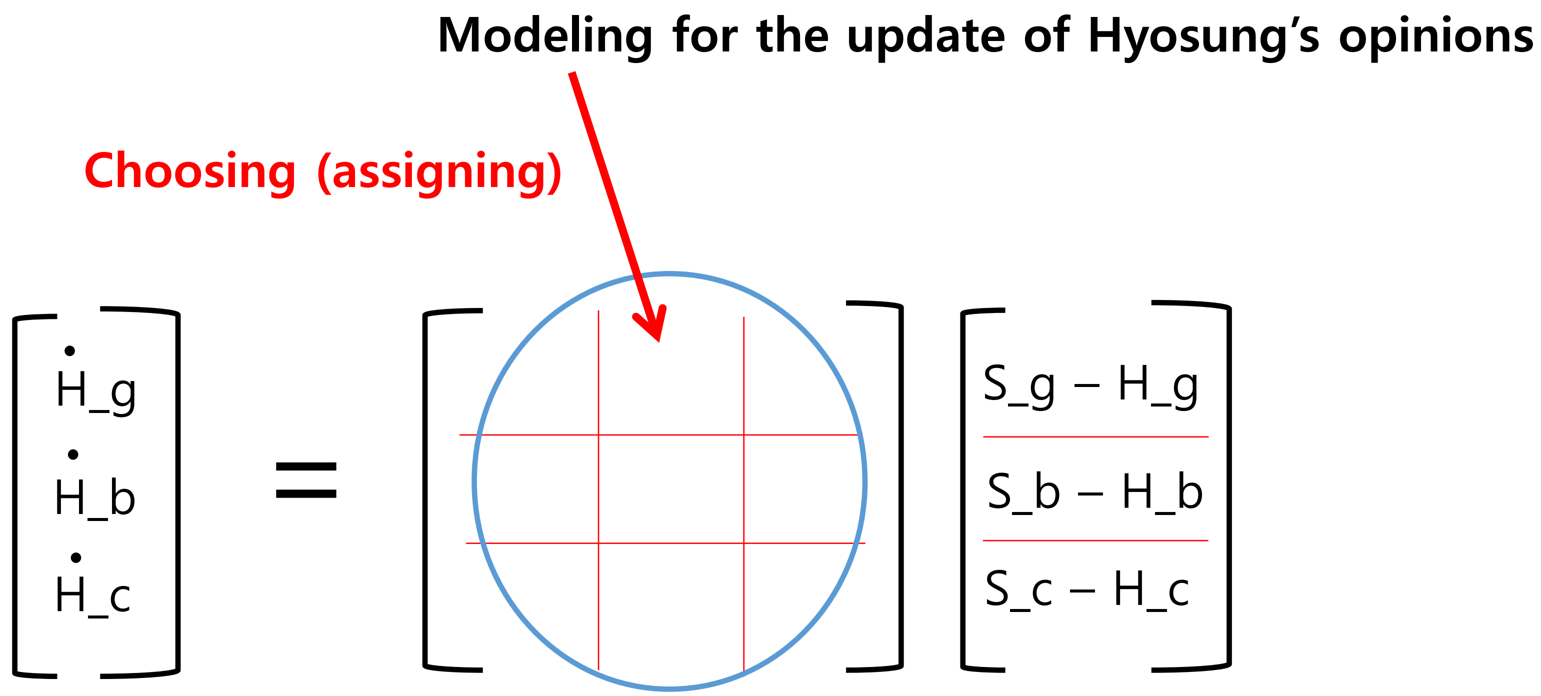
Positive effect !

Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

$$\begin{bmatrix}\dot{H}_g\\ \dot{H}_b\\ \dot{H}_c\end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix}S_g - H_g\\ S_b - H_b\\ S_c - H_c\end{bmatrix}$$

Multiple Interdependent Topics



Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

Change to the bigger,

$$\begin{bmatrix} >0 & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} S\_g - H\_g \\ S\_b - H\_b \\ S\_c - H\_c \end{bmatrix}$$

Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

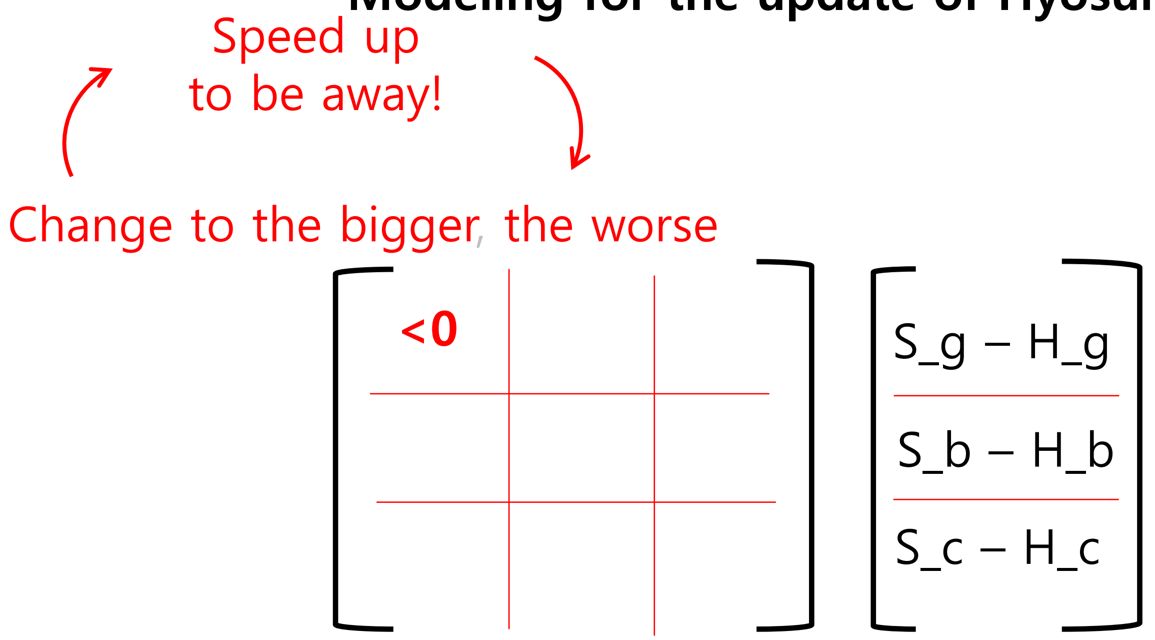
Speed up
to agree!

Change to the bigger, the better

$$\begin{bmatrix} >0 & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} S_g - H_g \\ S_b - H_b \\ S_c - H_c \end{bmatrix}$$

Multiple Interdependent Topics

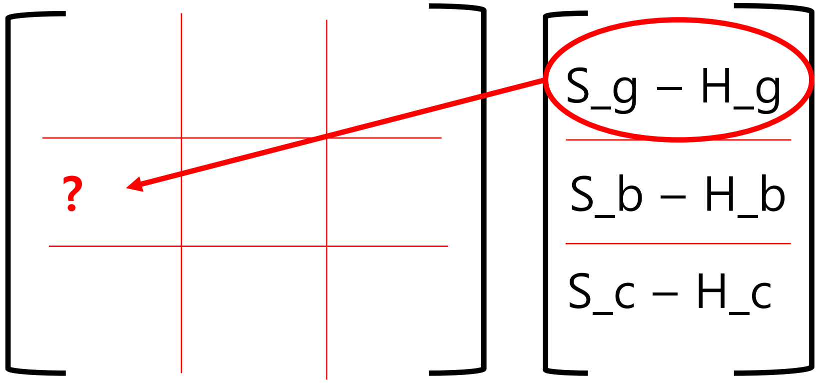
Modeling for the update of Hyosung's opinions



Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

Positive effect (coupling) vs. negative coupling



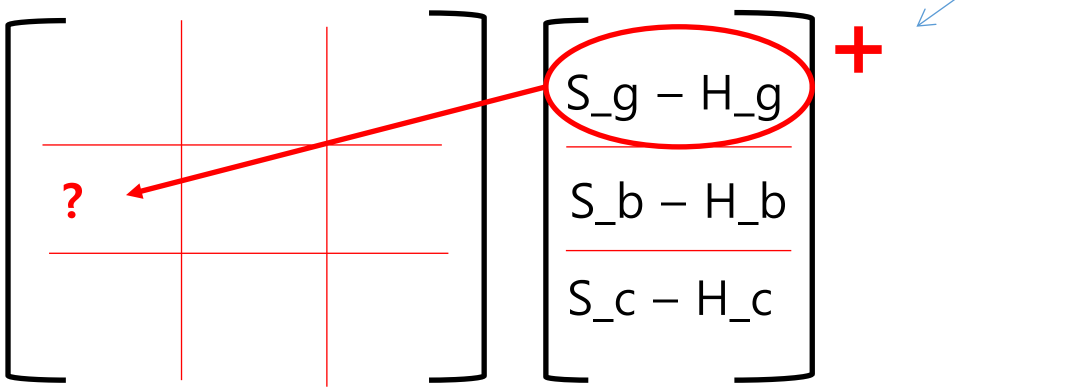
Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

Game: S_g = 0.7 vs. H_g = 0.3

0.7-0.3

Positive effect (coupling) vs. negative coupling



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Game: S_g = 0.7 vs. H_g = 0.3

0.7-0.3

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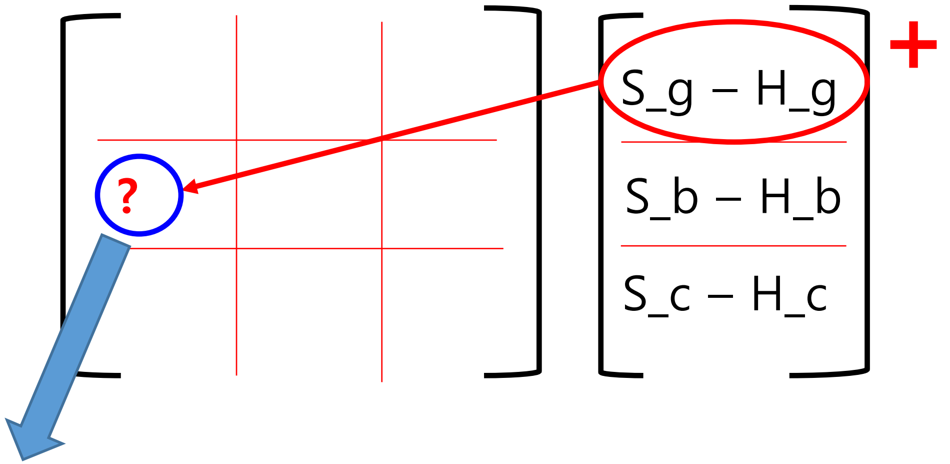


Positive effect: sign(S_b- H_b)

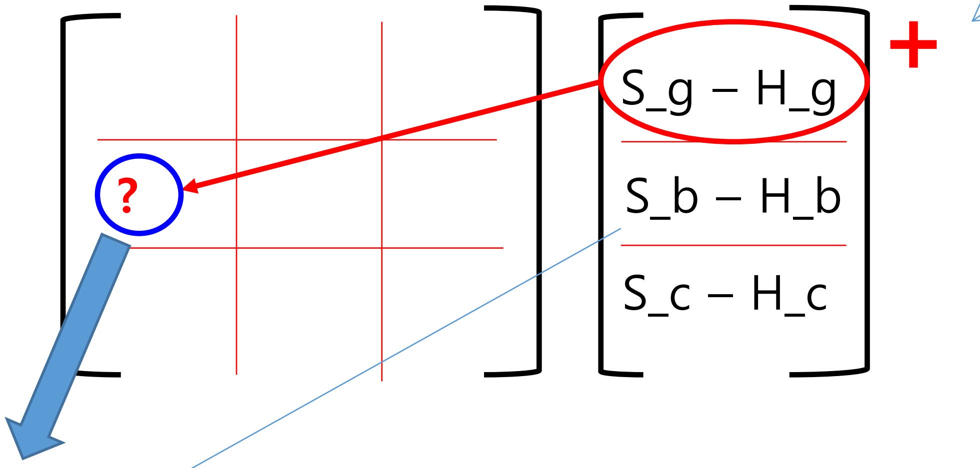
Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

Game: S_g = 0.7 vs. H_g = 0.3

0.7-0.3

Positive effect (coupling) vs. negative coupling



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Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

Game: S_g = 0.7 vs. H_g = 0.3

0.7-0.3

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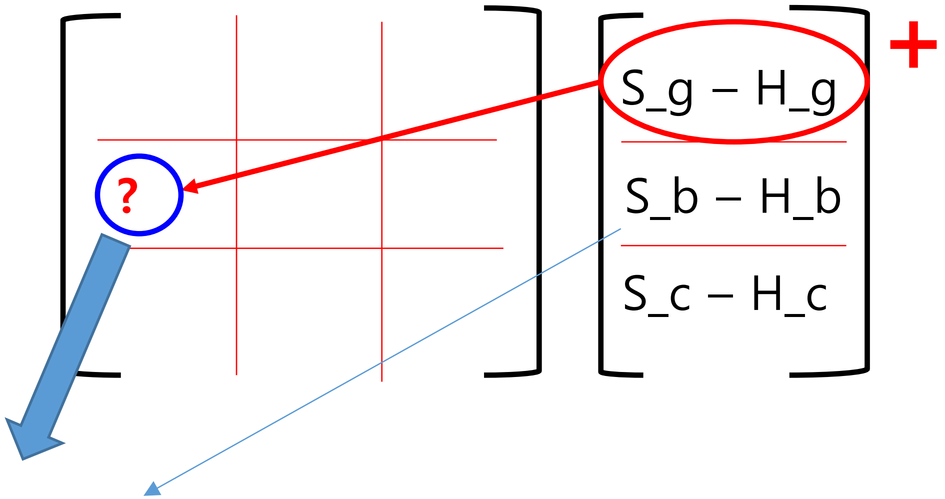


Positive effect: sign(S_b- H_b)   Why?

sign(S_b- H_b) → positive → H_b needs to be increased

sign(S_b- H_b) → negative → H_b needs to be decreased

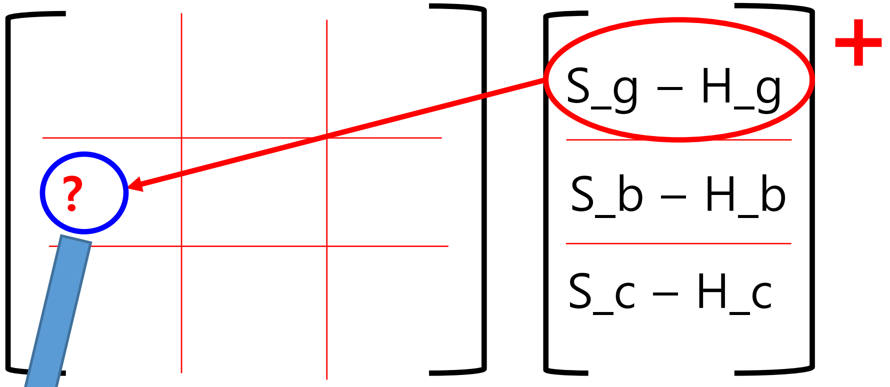
Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

Game: S_g = 0.7 vs. H_g = 0.3

0.7-0.3

Positive effect (coupling) vs. negative coupling



Positive effect: sign(S_b- H_b)

Negative effect: -sign(S_b- H_b)

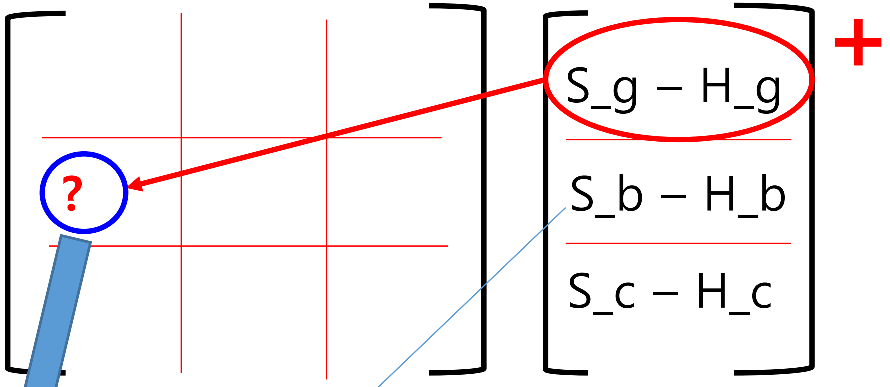
Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

Game: $S_g = 0.7$ vs. $H_g = 0.3$

0.7-0.3

Positive effect (coupling) vs. negative coupling



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Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

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0.7-0.3

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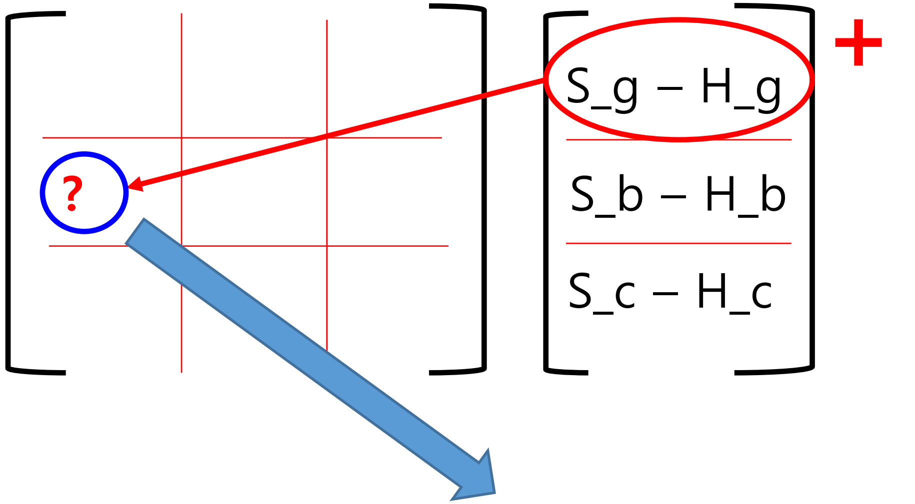


Positive effect: sign(S_b- H_b)

Magnitude: inverse relationship of abs(S_g – H_g)

Negative effect: -sign(S_b- H_b)

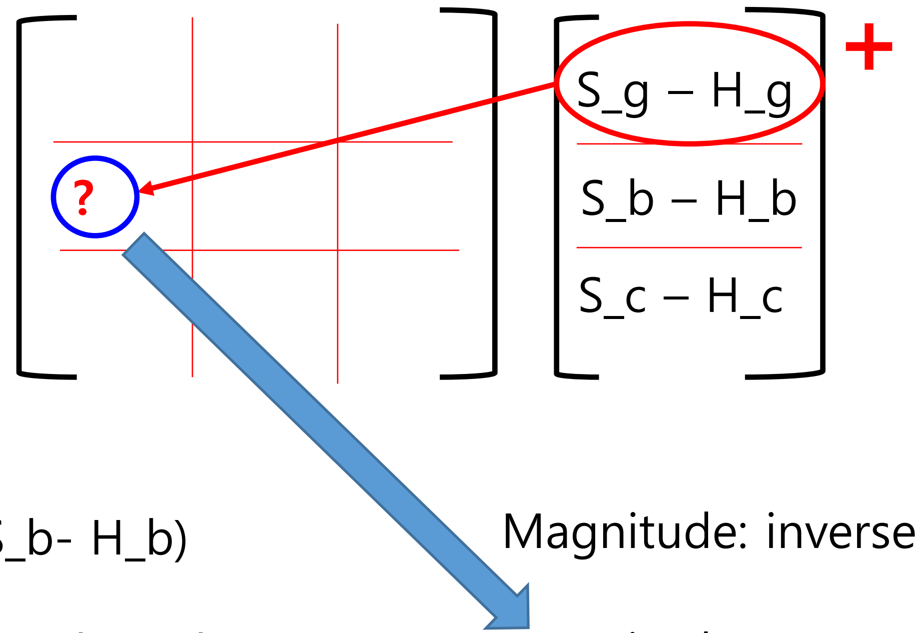
Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

Game: S_g = 0.7 vs. H_g = 0.3

0.7-0.3

Positive effect (coupling) vs. negative coupling



Positive effect: sign(S_b- H_b)

Negative effect: -sign(S_b- H_b)

Magnitude: inverse relationship of abs(S_g – H_g)

Magnitude: or, proportional to abs(S_g – H_g)

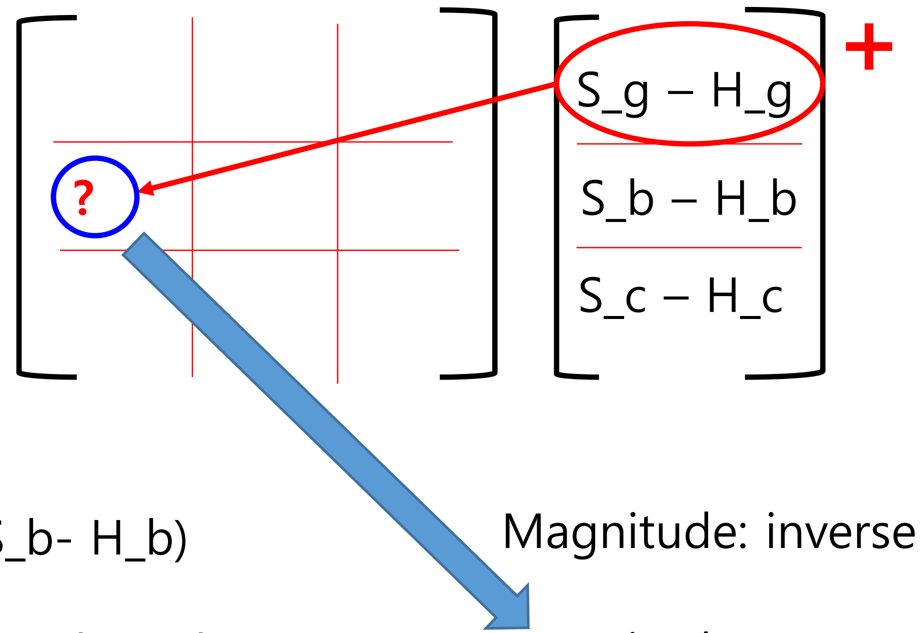
Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

Game: S_g = 0.7 vs. H_g = 0.3

0.7-0.3

Positive effect (coupling) vs. negative coupling



Positive effect: sign(S_b- H_b)

Negative effect: -sign(S_b- H_b)

Magnitude: inverse relationship of abs(S_g – H_g)

Magnitude: or, proportional to abs(S_g – H_g)

(S_g = 0.7, H_g = 0.3) vs. (S_g = 0.49, H_g = 0.51)

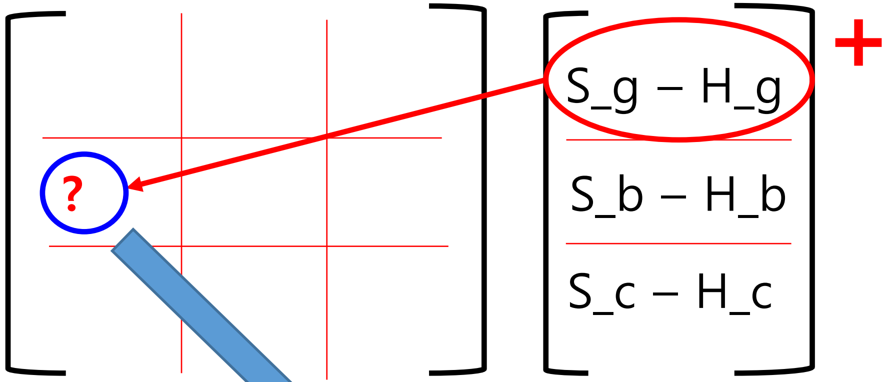
Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

Game: S_g = 0.7 vs. H_g = 0.3

0.7-0.3

Positive effect (coupling) vs. negative coupling



Positive effect: sign(S_b- H_b)

Negative effect: -sign(S_b- H_b)

Magnitude: inverse relationship of abs(S_g – H_g)

Magnitude: or, proportional to abs(S_g – H_g)

(S_g = 0.7, H_g = 0.3) vs. (S_g = 0.49, H_g = 0.51)

(Less close) Almost agreement (close)

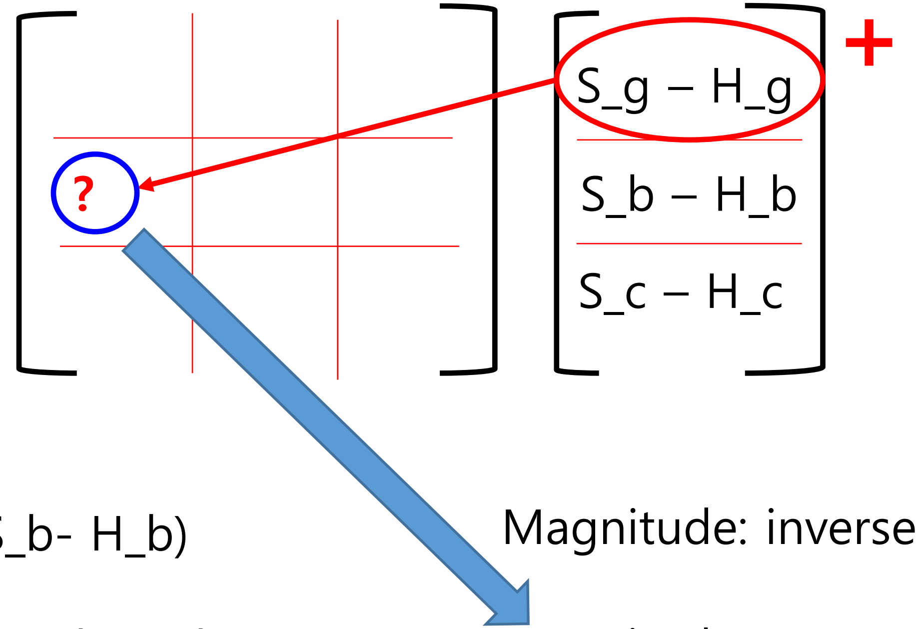
Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

Game: S_g = 0.7 vs. H_g = 0.3

0.7-0.3

Positive effect (coupling) vs. negative coupling



Positive effect: sign(S_b- H_b)

Negative effect: -sign(S_b- H_b)

Magnitude: inverse relationship of abs(S_g – H_g)

Magnitude: or, proportional to abs(S_g – H_g)

(S_g = 0.7, H_g = 0.3) vs. (S_g = 0.49, H_g = 0.51)

(Less close) Almost agreement (close)

Positive coupling

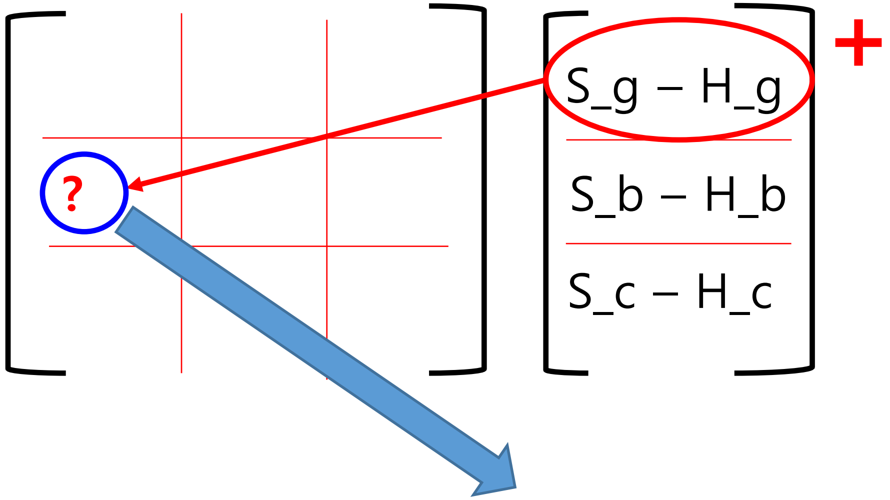
Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

Game: S_g = 0.7 vs. H_g = 0.3

0.7-0.3

Positive effect (coupling) vs. negative coupling



Positive effect: sign(S_b- H_b)

Negative effect: -sign(S_b- H_b)

Magnitude: inverse relationship of abs(S_g – H_g)

Magnitude: or, proportional to abs(– H_g)

Positive coupling

(S_g = 0.7, H_g = 0.3) vs. (S_g = 0.49, H_g = 0.51)

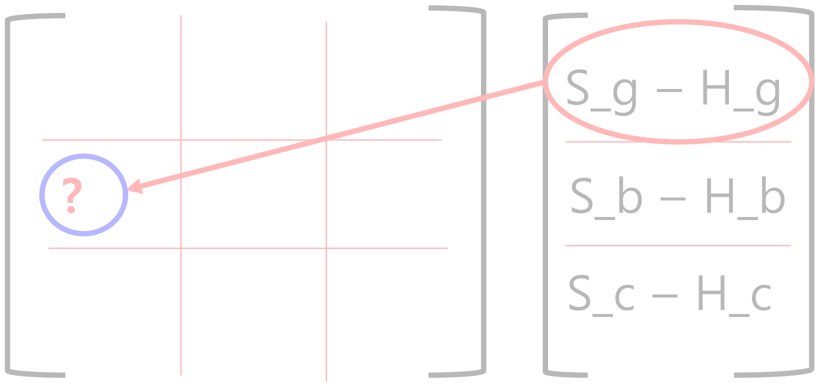
(Less close) Almost agreement (close)

Multiple Interdependent Topics

Modeling for the update of Hyosung's opinions

It may be complicated!

Positive effect (coupling) vs. negative coupling



Positive effect: sign(S_b- H_b)

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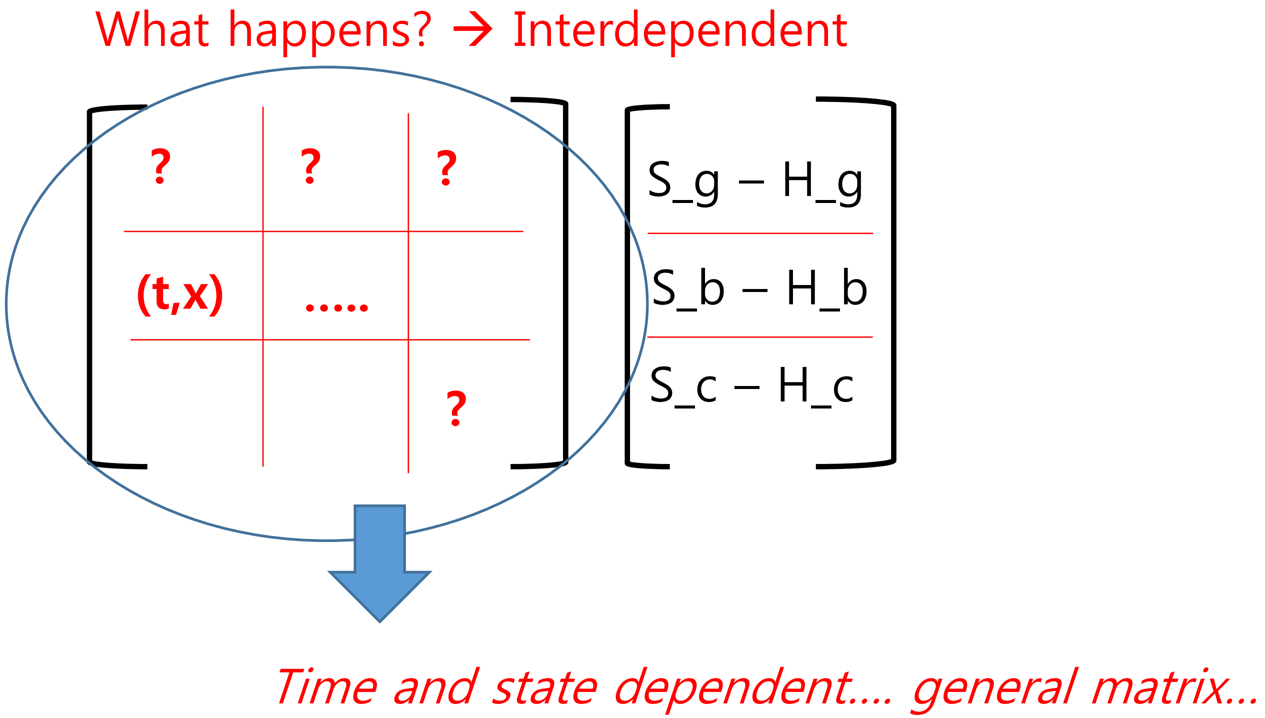
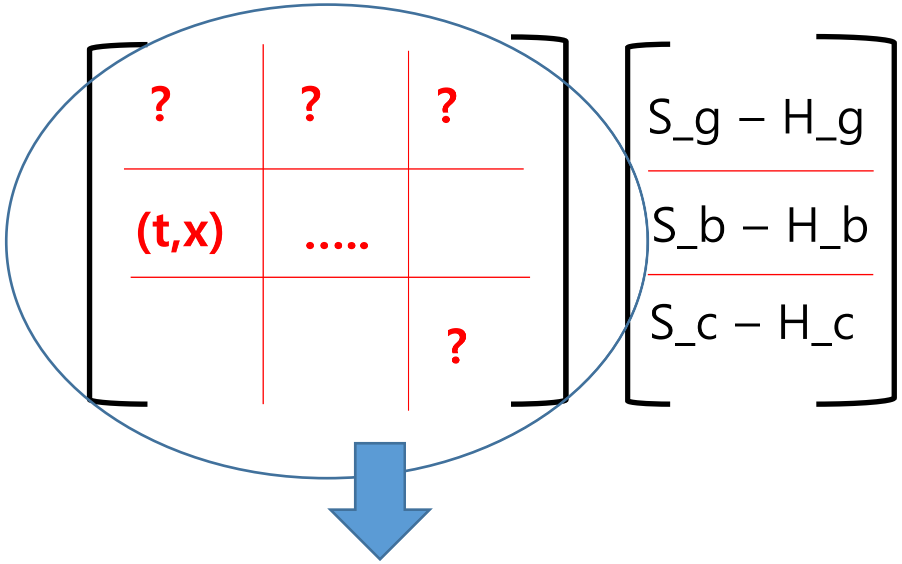
Magnitude: or, proportional to abs(S_g – H_g)

Multiple Interdependent Topics



Multiple Interdependent Topics

What happens? → Interdependent



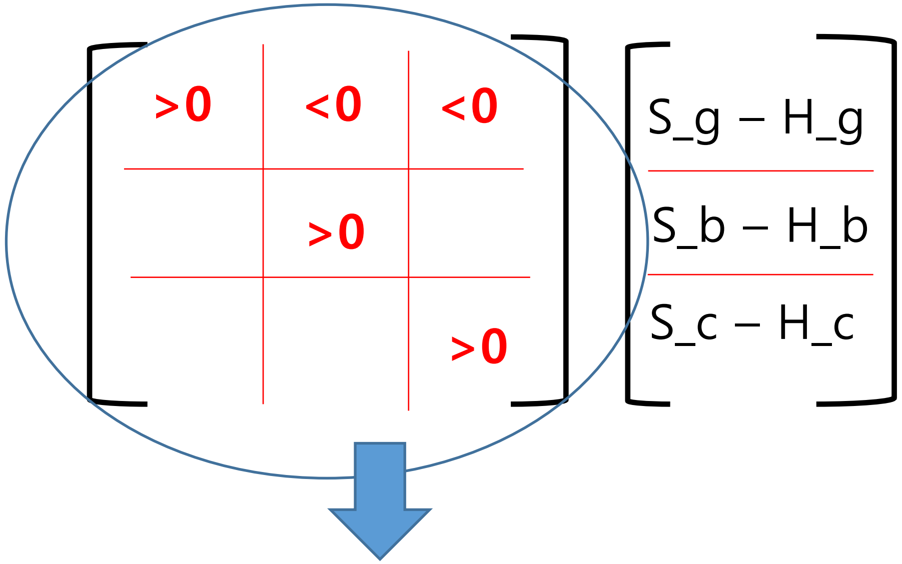
Time and state dependent…. general matrix…

Nominal model or linearization or some specific forms…

Multiple Interdependent Topics

Deterministic model – Static case!

What happens? → Interdependent



Fixed matrix elements → *Linearized interdependent model around a nominal (temporal-instant) social opinion network!*

Multiple Interdependent Topics

<u>Deterministic model – Static case!</u>

What happens? → Interdependent

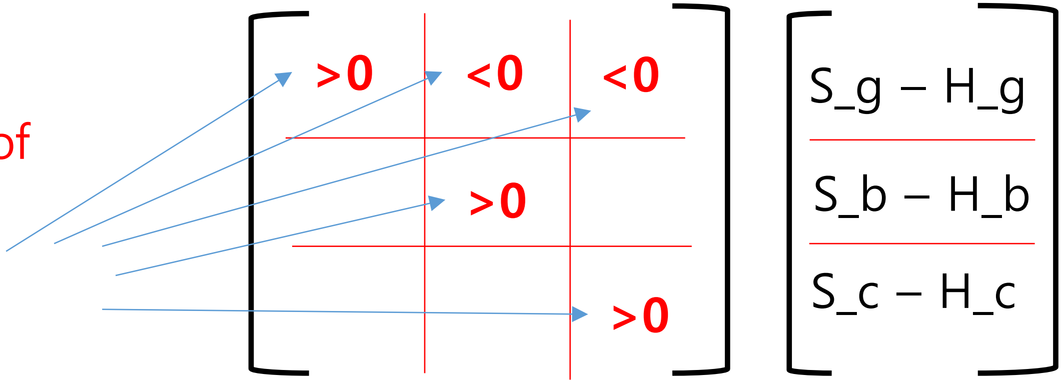
$$
\begin{bmatrix} >0 & <0 & <0 \\ & >0 & \\ & & >0 \end{bmatrix}
\begin{bmatrix} S\_g - H\_g \\ S\_b - H\_b \\ S\_c - H\_c \end{bmatrix}
$$

Multiple Interdependent Topics

Deterministic model – Static case!

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?



Multiple Interdependent Topics

<u>Deterministic model – Static case!</u>

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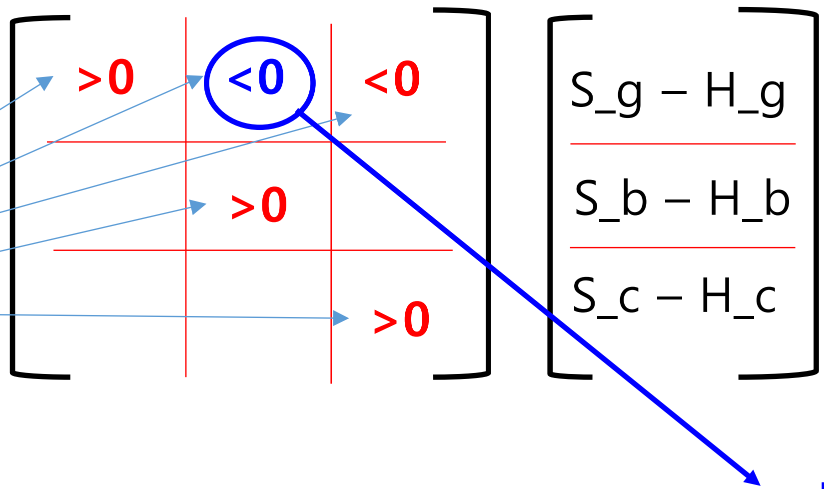


Ex. -1 ➔ what is the physical meaning?

Multiple Interdependent Topics

<u>Deterministic model – Static case!</u>

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?

>0	<0	<0	$S\_g - H\_g$
	>0		$S\_b - H\_b$
		>0	$S\_c - H\_c$

Ex. -1 ➔ what is the physical meaning?

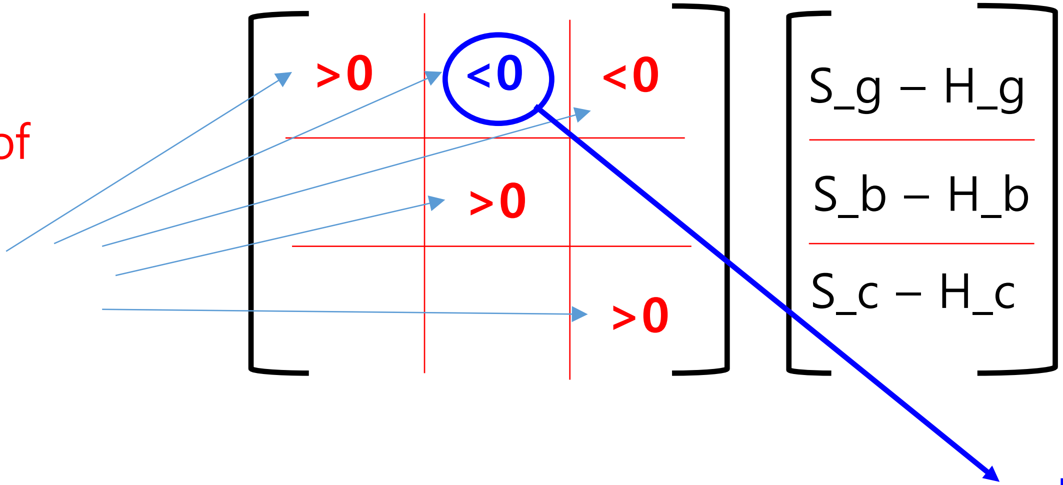
Case 1: $S\_g - H\_g < 0$, $S\_b - H\_b < 0$

Multiple Interdependent Topics

<u>Deterministic model – Static case!</u>

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?



Ex. -1 ➔ what is the physical meaning?

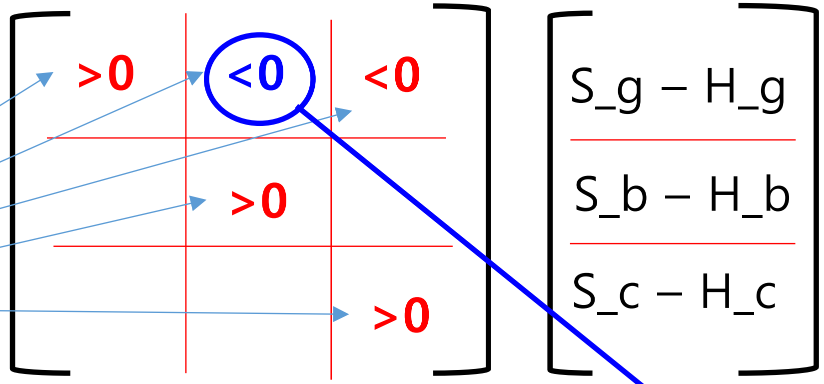
Case 1: S_g – H_g <0, S_b- H_b <0 ⇨ **Anti (or non)-cooperative**

Multiple Interdependent Topics

<u>Deterministic model – Static case!</u>

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?



Ex. -1 ➔ what is the physical meaning?

Case 1: $S_g - H_g < 0$, $S_b - H_b < 0$ ⇨ **Anti (or non)-cooperative**

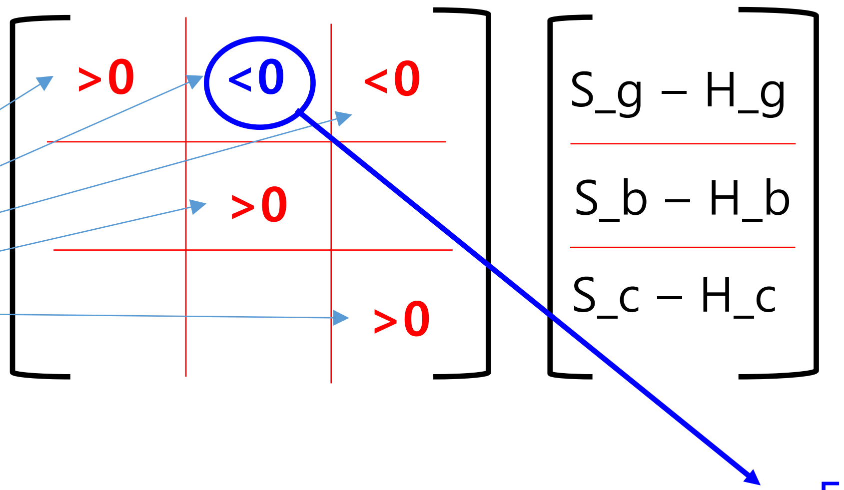
Case 2: $S_g - H_g < 0$, $S_b - H_b > 0$

Multiple Interdependent Topics

Deterministic model – Static case!

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?



Ex. -1 ➔ what is the physical meaning?

Case 1: $S_g - H_g < 0$, $S_b - H_b < 0$ ⇨ **Anti (or non)-cooperative**

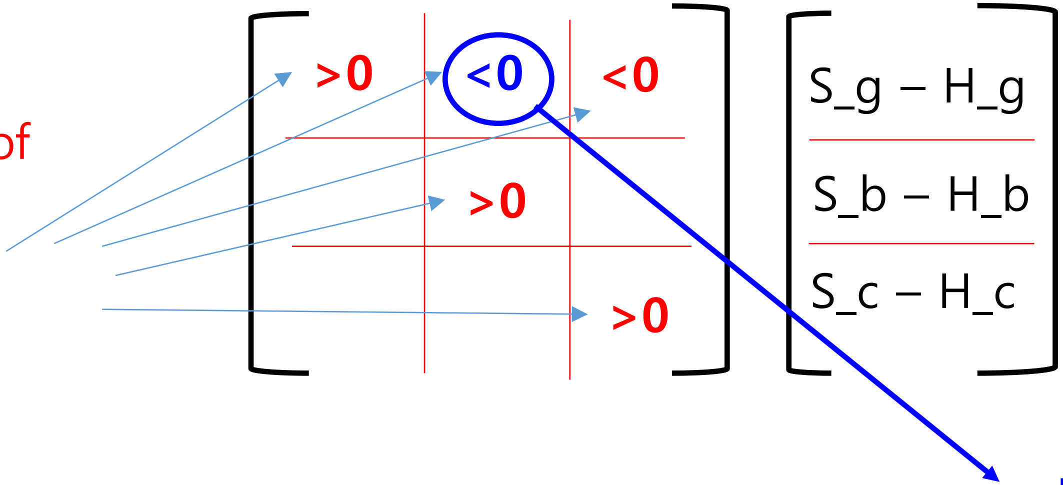
Case 2: $S_g - H_g < 0$, $S_b - H_b > 0$ ⇨ **cooperative**

Multiple Interdependent Topics

<u>Deterministic model – Static case!</u>

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?



Ex. -1 ➔ what is the physical meaning?

Case 1: $S_g - H_g < 0$, $S_b - H_b < 0$ ⇨ **Anti (or non)-cooperative**

Case 2: $S_g - H_g < 0$, $S_b - H_b > 0$ ⇨ **cooperative**

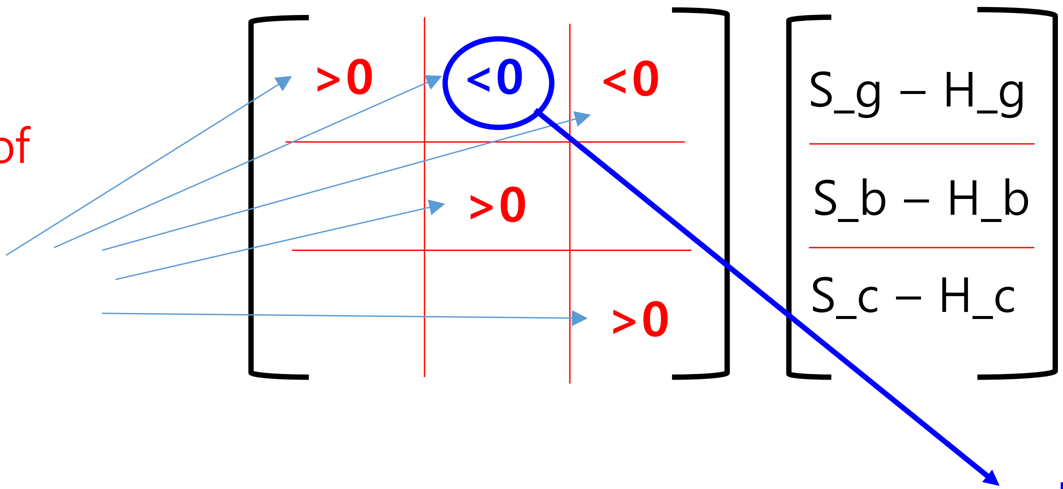
Case 3: $S_g - H_g > 0$, $S_b - H_b < 0$ ⇨ **cooperative**

Multiple Interdependent Topics

Deterministic model – Static case!

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?



Ex. -1 ➔ what is the physical meaning?

Case 1: S_g – H_g <0, S_b- H_b <0 ⇒ **Anti (or non)-cooperative**

Case 2: S_g – H_g <0, S_b- H_b >0 ⇒ **cooperative**

Case 3: S_g – H_g >0, S_b- H_b <0 ⇒ **cooperative**

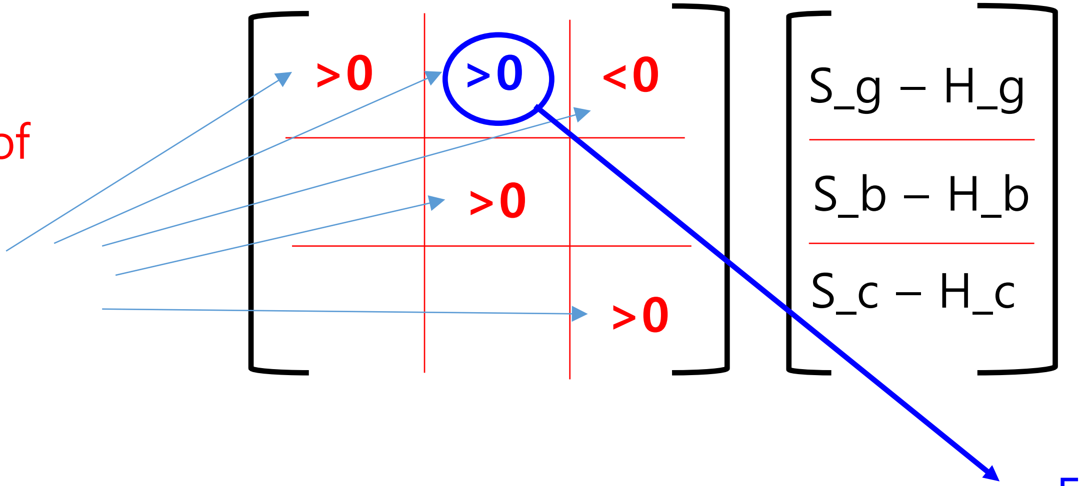
Case 4: S_g – H_g >0, S_b- H_b >0 ⇒ **Anti (or non)-cooperative**

Multiple Interdependent Topics

<u>Deterministic model – Static case!</u>

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?



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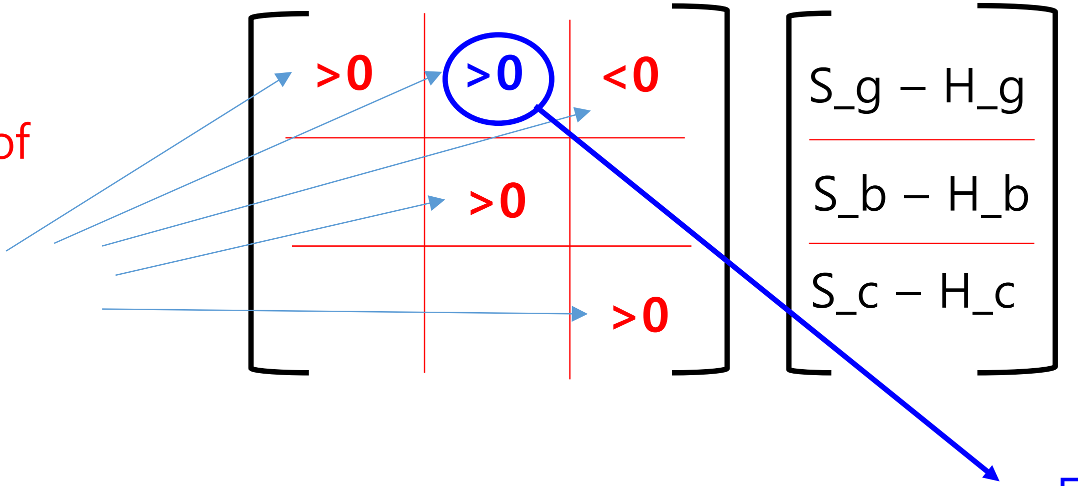

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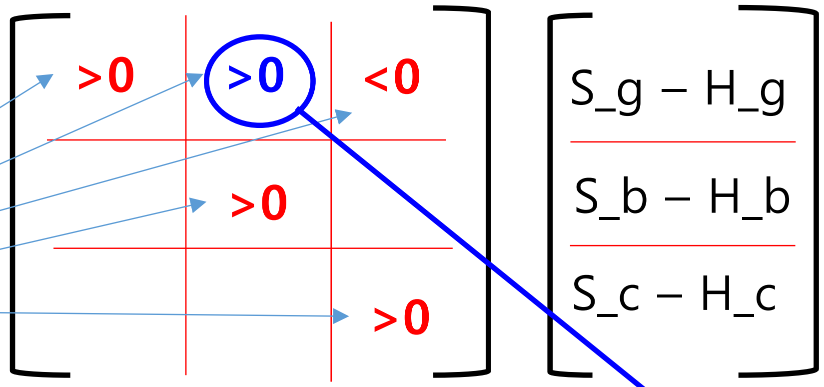
Case 1: $S_g – H_g < 0$, $S_b - H_b < 0$

Multiple Interdependent Topics

<u>Deterministic model – Static case!</u>

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?



Ex. -1 ➔ what is the physical meaning?

Case 1: S_g – H_g <0, S_b- H_b <0 ⇨ **Cooperative**

Multiple Interdependent Topics

Deterministic model – Static case!

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?

>0	>0	<0
	>0	
		>0

S_g – H_g
S_b – H_b
S_c – H_c

Ex. -1 ➔ what is the physical meaning?

Case 1: S_g – H_g <0, S_b- H_b <0 ⇒ **Cooperative**

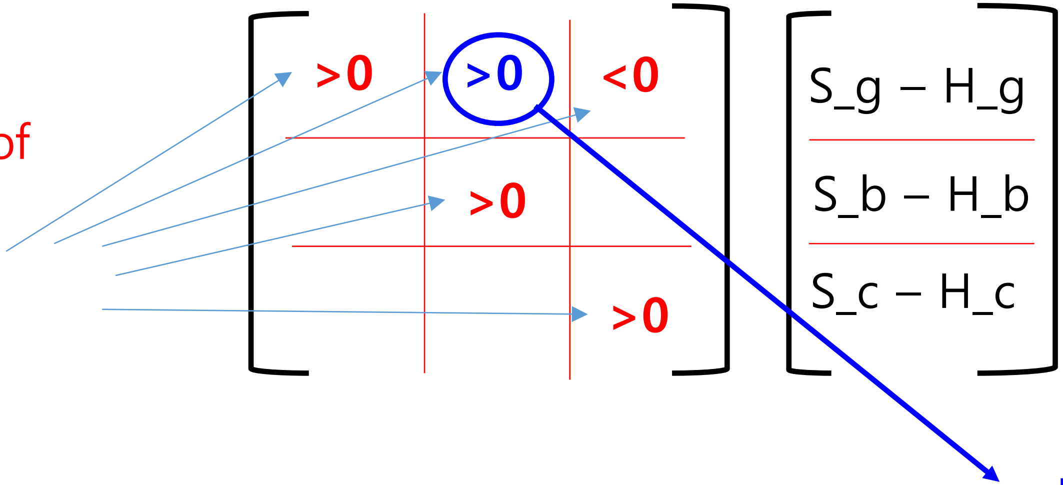
Case 2: S_g – H_g <0, S_b- H_b >0

Multiple Interdependent Topics

Deterministic model – Static case!

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?



Ex. -1 ➔ what is the physical meaning?

Case 1: $S_g - H_g < 0$, $S_b - H_b < 0$ ⇒ **Cooperative**

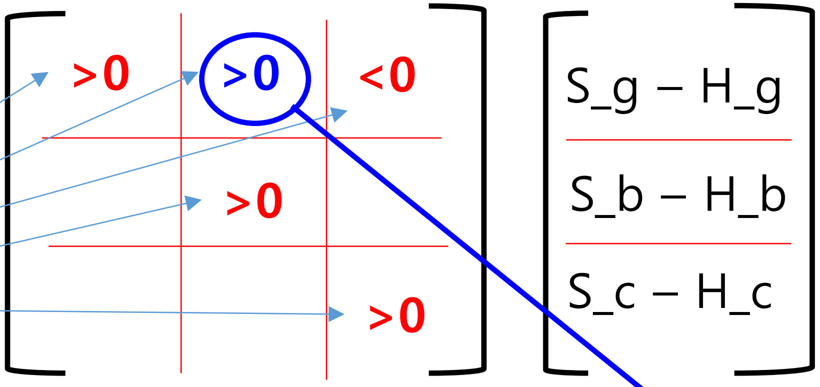
Case 2: $S_g - H_g < 0$, $S_b - H_b > 0$ ⇒ **Anti (or non)-cooperative**

Multiple Interdependent Topics

Deterministic model – Static case!

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?



Ex. -1 ➔ what is the physical meaning?

Case 1: $S_g - H_g < 0$, $S_b - H_b < 0$ ⇨ **Cooperative**

Case 2: $S_g - H_g < 0$, $S_b - H_b > 0$ ⇨ **Anti (or non)-cooperative**

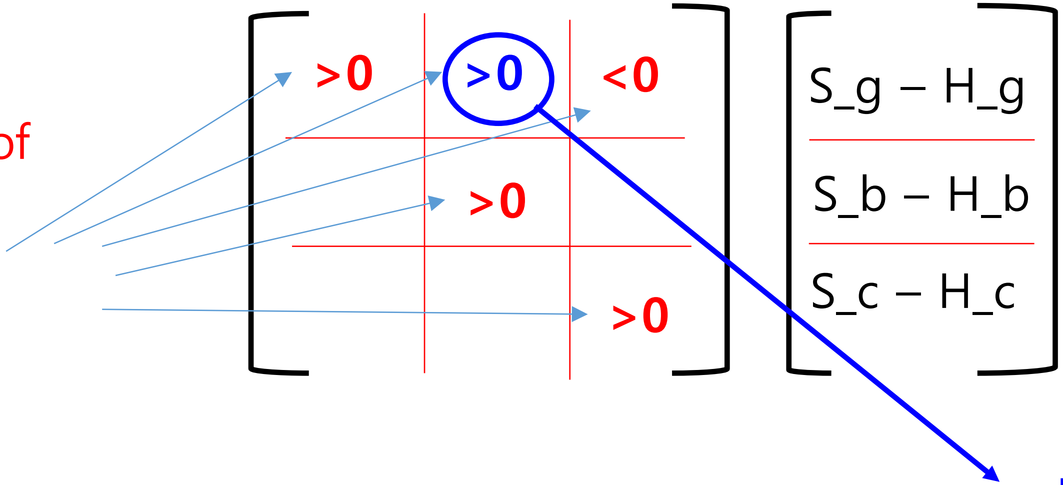
Case 3: $S_g - H_g > 0$, $S_b - H_b < 0$ ⇨ **Anti (or non)-cooperative**

Multiple Interdependent Topics

Deterministic model – Static case!

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?



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Case 3: $S_g - H_g > 0$, $S_b - H_b < 0$ ⇨ **Anti (or non)-cooperative**

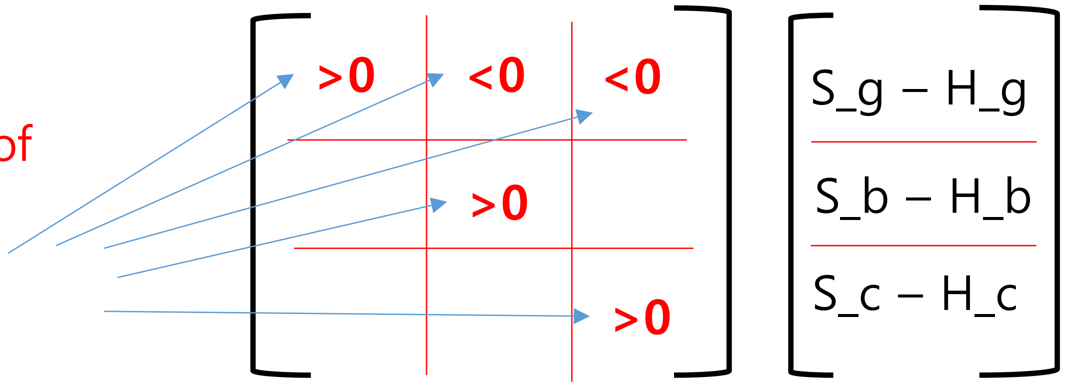
Case 4: $S_g - H_g > 0$, $S_b - H_b > 0$ ⇨ **Cooperative**

Multiple Interdependent Topics

Deterministic model – Static case!

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?



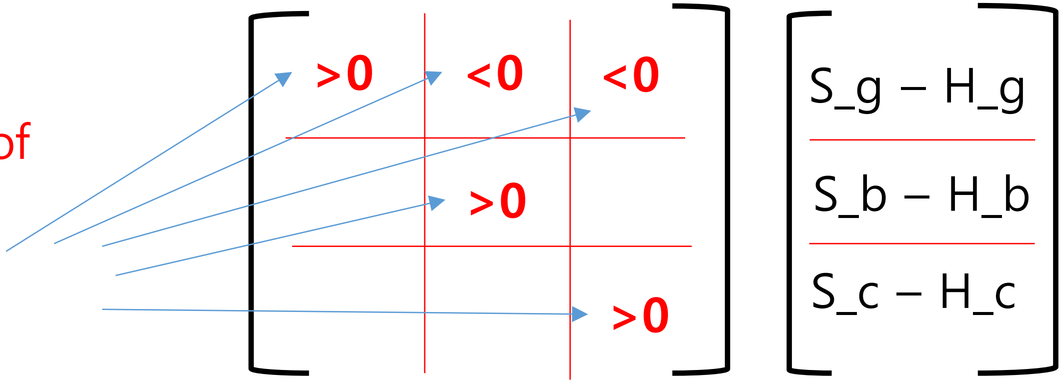
What is the optimal way (ex. change minimum number of elements) for a consensus (or cluster consensus)?

Multiple Interdependent Topics

<u>Deterministic model – Static case!</u>

What happens? → Interdependent

How (what values) to design the elements of matrix weights for a perfect consensus (or cluster consensus)?



What is the optimal way (ex. change minimum number of elements) for a consensus (or cluster consensus)?

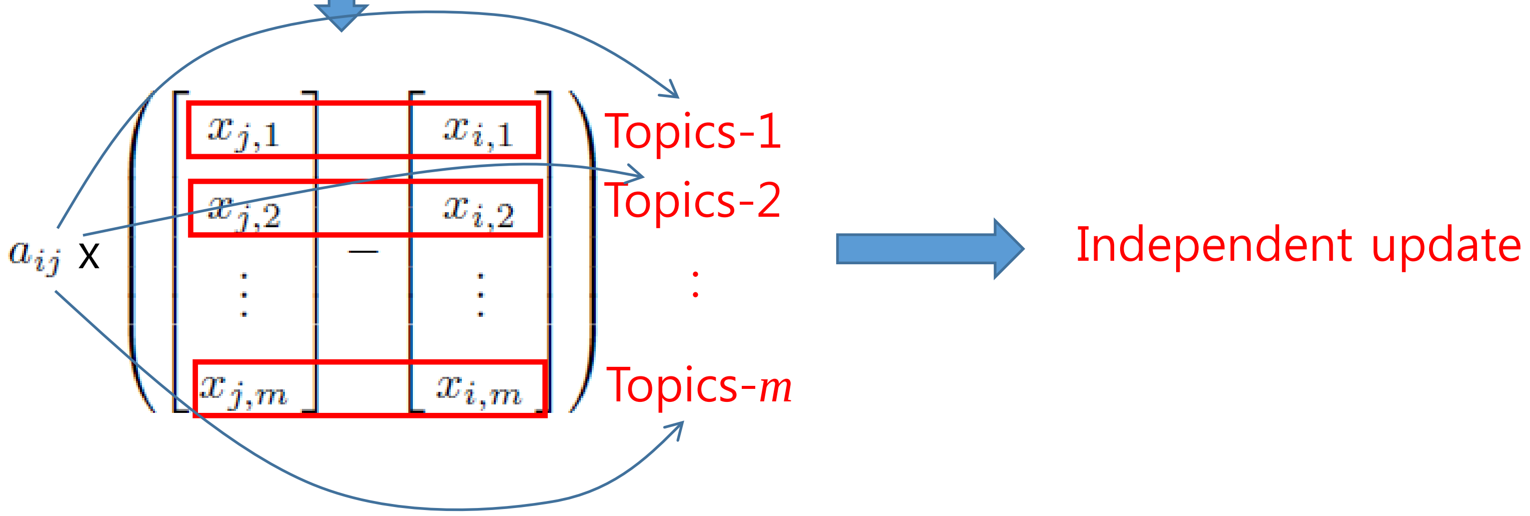
<u>For example, *change your mind for the game for a complete consensus..^^*</u>

Independent Update

$$\dot{\mathbf{x}}_i = -\sum_{j=1}^{n} a_{ij}(\mathbf{x}_j - \mathbf{x}_i), \forall i = 1, \ldots, n,$$

$a_{ij} > 0, \ \forall (i,j) \in \mathcal{E}$

$\exists\, a_{ij} < 0, \text{ for some } (i,j) \in \mathcal{E}$



Multiple Interdependent Topics

$$\dot{\mathbf{x}}_i(t) = -\sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),$$

$$\mathbf{A}_{ij} \text{ x } \left(\begin{bmatrix} x_{j,1} \\ x_{j,2} \\ \vdots \\ x_{j,m} \end{bmatrix} - \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,m} \end{bmatrix} \right)$$

Multiple Interdependent Topics

$$\dot{\mathbf{x}}_i(t) = -\sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),$$

$$\begin{bmatrix} a^{ij}_{1,1} & a^{ij}_{1,2} & \cdots & a^{ij}_{1,m} \\ a^{ij}_{2,1} & \cdots & \cdots & a^{ij}_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a^{ij}_{m,1} & a^{ij}_{m,2} & \cdots & a^{ij}_{m,m} \end{bmatrix} \text{x} \left(\begin{bmatrix} x_{j,1} \\ x_{j,2} \\ \vdots \\ x_{j,m} \end{bmatrix} - \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,m} \end{bmatrix} \right)$$

Multiple Interdependent Topics

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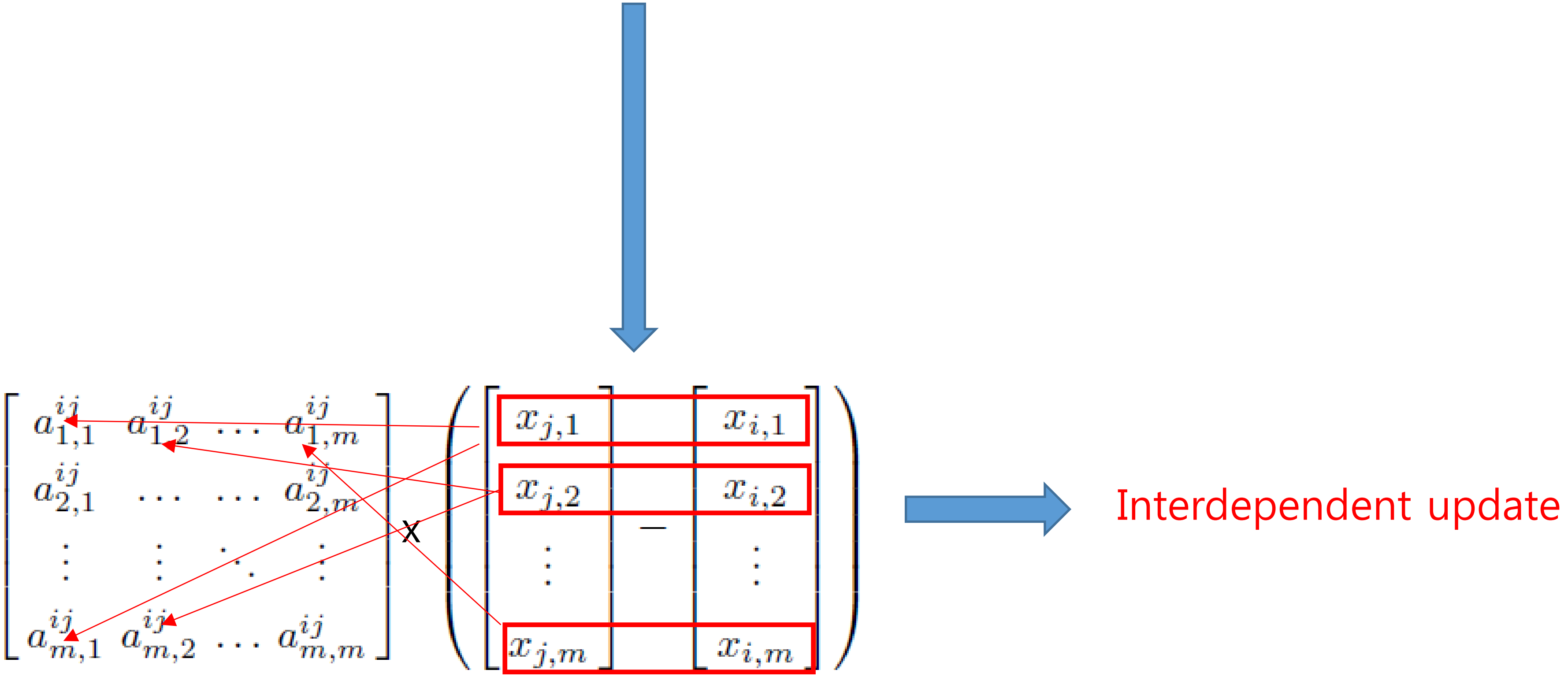

Scalar vs. Matrix

$$\dot{\mathbf{x}}_i = -\sum_{j=1}^{n} a_{ij}(\mathbf{x}_j - \mathbf{x}_i), \forall i = 1, \ldots, n,$$

$$\dot{\mathbf{x}}_i(t) = -\sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),$$

$$a_{ij}$$

vs.

$$\begin{bmatrix} a^{ij}_{1,1} & a^{ij}_{1,2} & \ldots & a^{ij}_{1,m} \\ a^{ij}_{2,1} & \ldots & \ldots & a^{ij}_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a^{ij}_{m,1} & a^{ij}_{m,2} & \ldots & a^{ij}_{m,m} \end{bmatrix}$$

$$-, 0, +$$

$0 \prec A_{ij}, 0 \preceq A_{ij}, 0 \succ A_{ij}, 0 \succeq A_{ij}$, or indefinite

Scalar vs. Matrix

$$\dot{\mathbf{x}}_i = -\sum_{j=1}^{n} a_{ij}(\mathbf{x}_j - \mathbf{x}_i), \forall i = 1, \ldots, n,$$

$$\dot{\mathbf{x}}_i(t) = -\sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),$$

$$a_{ij}$$

vs.

$$\begin{bmatrix} a_{1,1}^{ij} & a_{1,2}^{ij} & \cdots & a_{1,m}^{ij} \\ a_{2,1}^{ij} & \cdots & \cdots & a_{2,m}^{ij} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}^{ij} & a_{m,2}^{ij} & \cdots & a_{m,m}^{ij} \end{bmatrix}$$

$$-, 0, +$$

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Connected?
- Positive connected
- Semi-positive connected

Scalar vs. Matrix

$$\dot{\mathbf{x}}_i = -\sum_{j=1}^{n} a_{ij}(\mathbf{x}_j - \mathbf{x}_i), \forall i = 1, \ldots, n,$$

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a_{ij}

vs.

$$\begin{bmatrix} a_{1,1}^{ij} & a_{1,2}^{ij} & \ldots & a_{1,m}^{ij} \\ a_{2,1}^{ij} & \ldots & \ldots & a_{2,m}^{ij} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}^{ij} & a_{m,2}^{ij} & \ldots & a_{m,m}^{ij} \end{bmatrix}$$

More general, realistic, but complicated different phenomenon

$-, 0, +$

$0 \prec A_{ij}, 0 \preceq A_{ij}, 0 \succ A_{ij}, 0 \succeq A_{ij}$, or indefinite

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- Positive connected
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$$\begin{bmatrix} a_{1,1}^{ij} & a_{1,2}^{ij} & \cdots & a_{1,m}^{ij} \\ a_{2,1}^{ij} & \cdots & \cdots & a_{2,m}^{ij} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}^{ij} & a_{m,2}^{ij} & \cdots & a_{m,m}^{ij} \end{bmatrix}$$

More general, realistic, but complicated different phenomenon

$-, 0, +$

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- Positive connected
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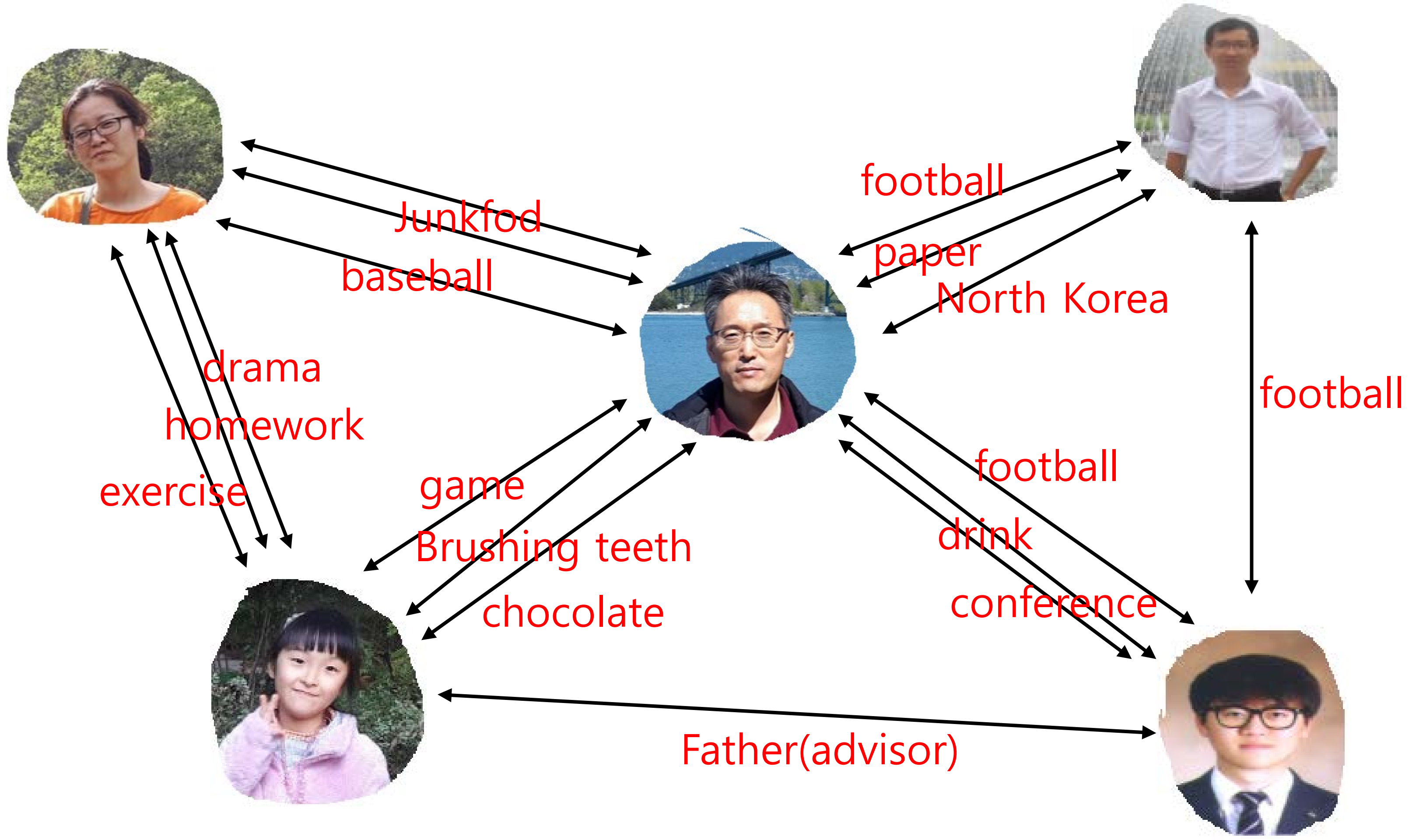
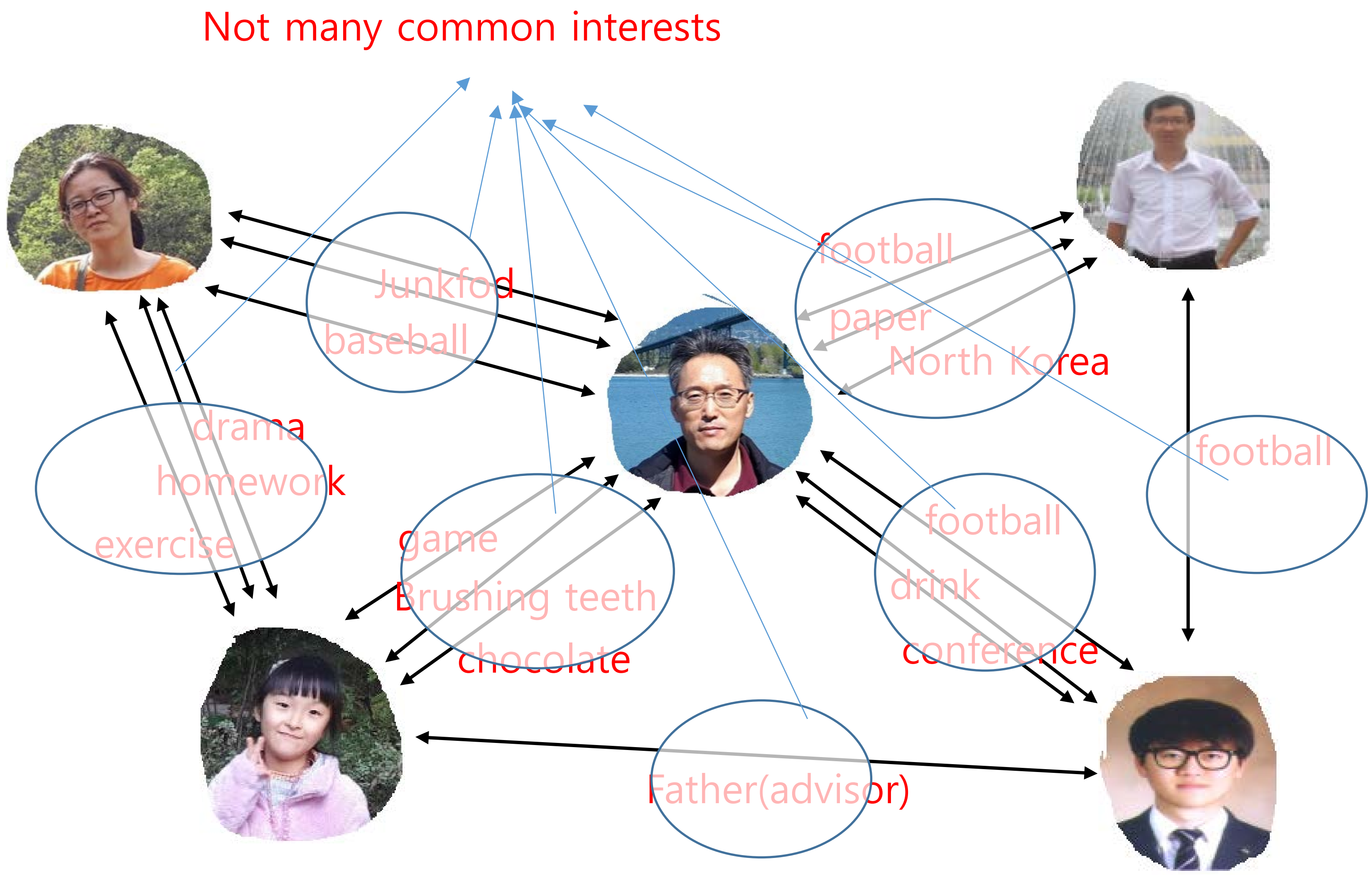
Clusters

Physical Meaning of P.D and P.S.D

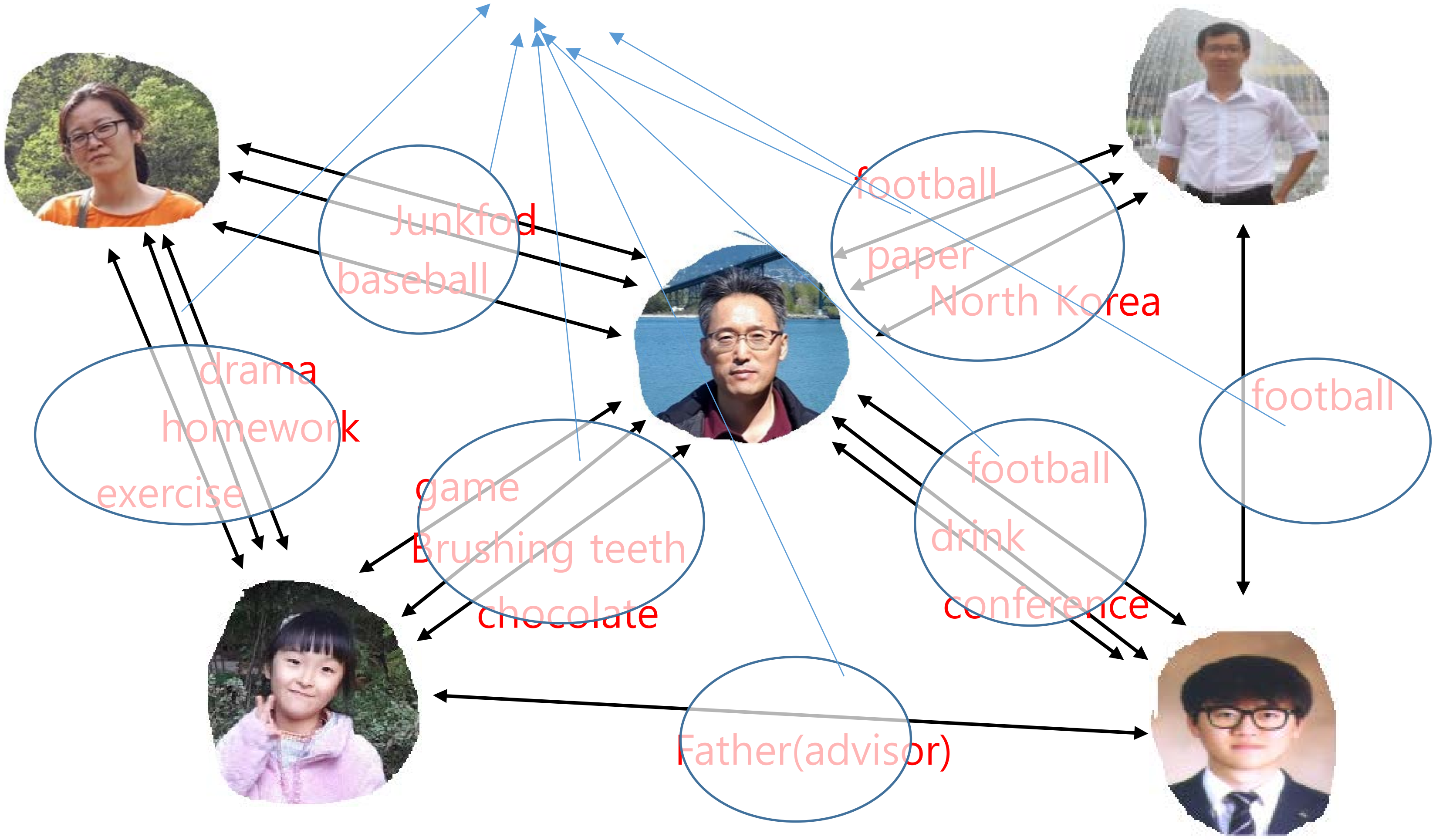


Physical Meaning of P.D and P.S.D

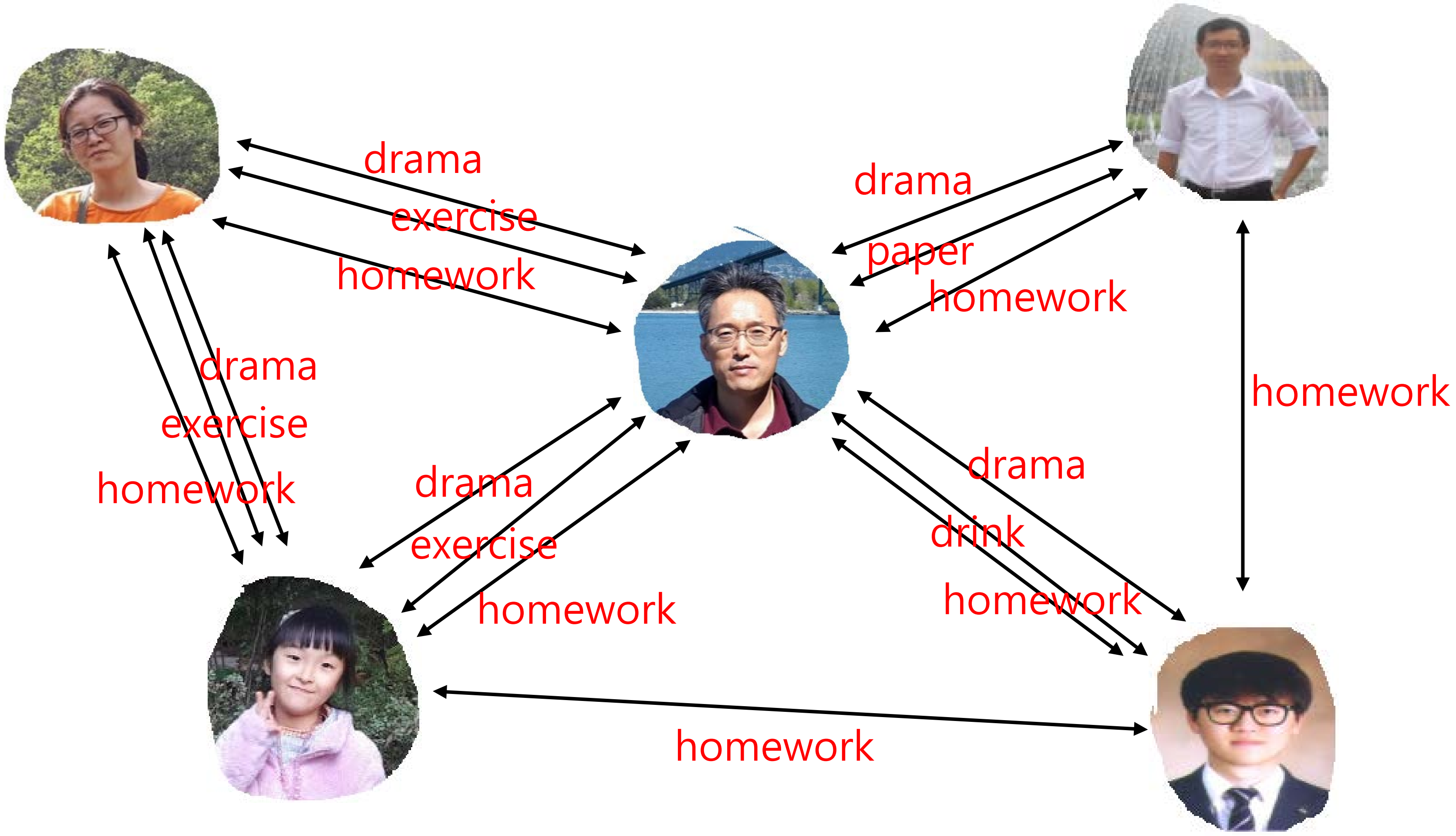


Physical Meaning of P.D and P.S.D

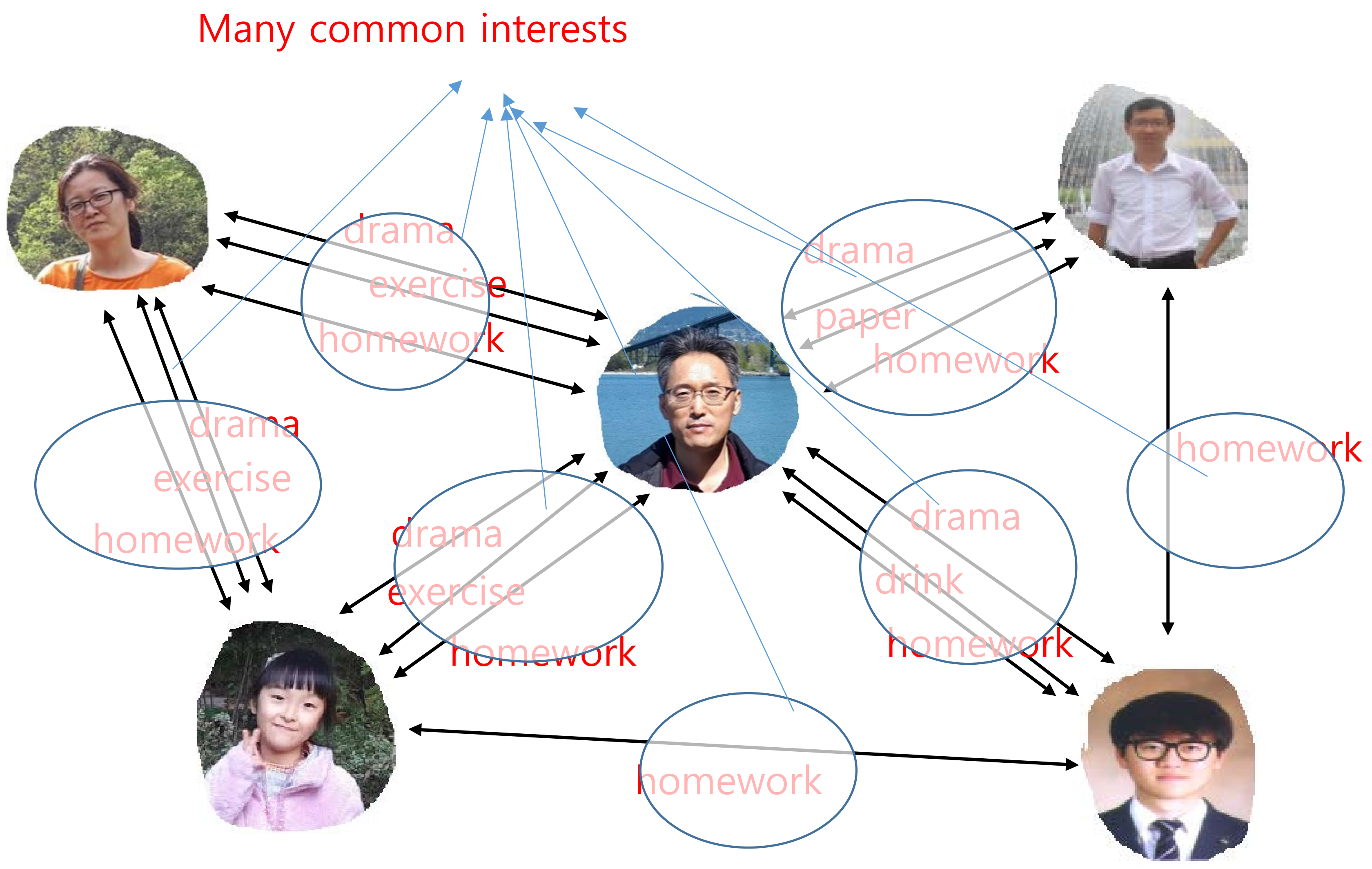
Not many common interests ➔ *Positive semi-definite !*



Physical Meaning of P.D and P.S.D

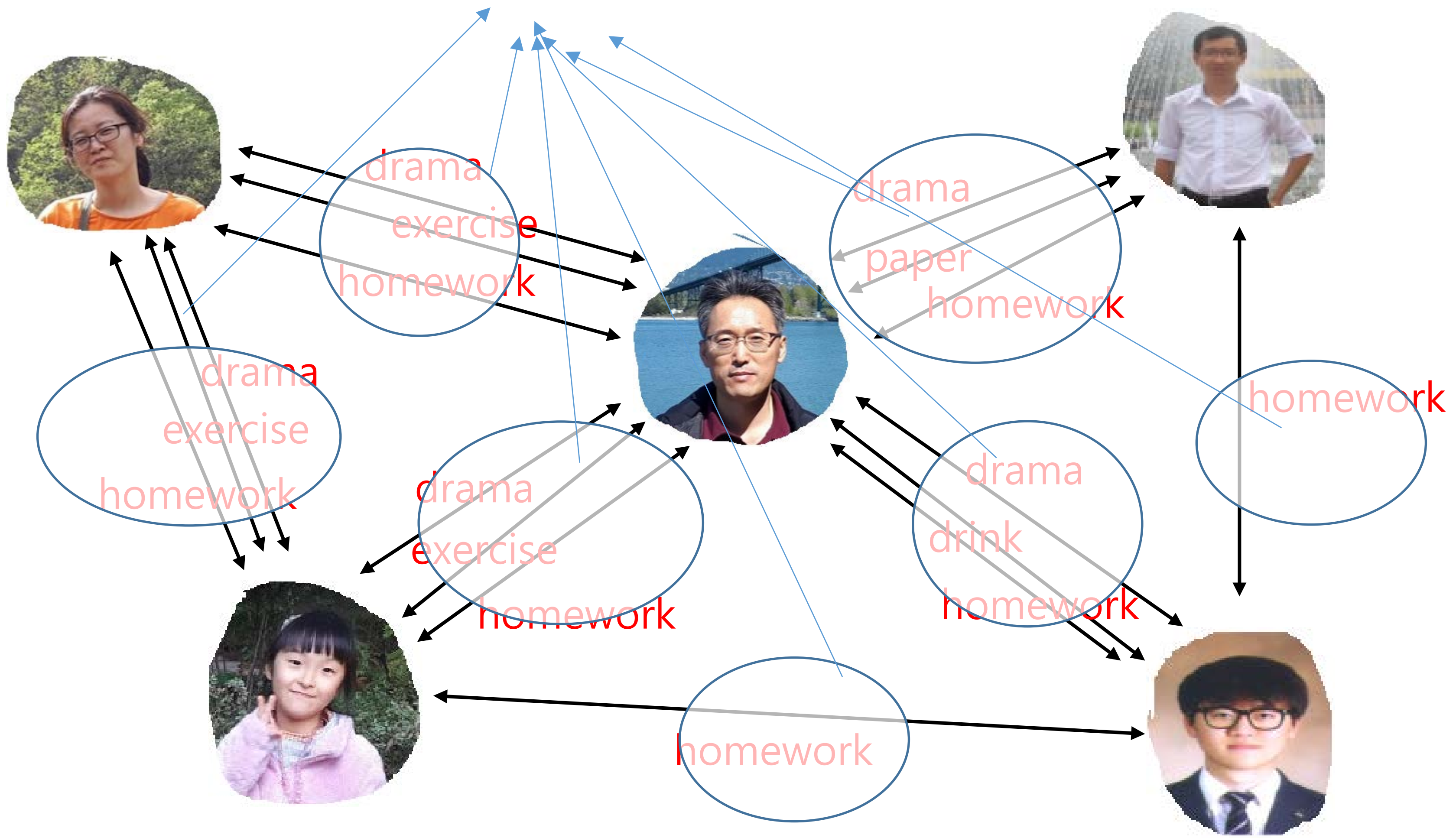


Physical Meaning of P.D and P.S.D



Physical Meaning of P.D and P.S.D

Many common interests ➔ *Positive definite !*



Part-2: Analysis

Problem 1-Fixed Matrices

(Typical consensus-based ideas- Linearized/ Nominal

or, positive & negative mixed)

Model 1- static case

$$\dot{\mathbf{x}}_i(t) = -\sum_{j\in\mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),$$

Model 1- static case

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$$\mathbf{A}_{ij} = \mathbf{A}_{ij}^T \geq 0$$

Model 1- static case

$$\dot{\mathbf{x}}_i(t) = -\sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),$$

$$\mathbf{A}_{ij} = \mathbf{A}_{ij}^T \geq 0$$

$$\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x}.$$

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Def.: Clusters & cluster consensus (clustered opinions)

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Def.: Clusters & cluster consensus (clustered opinions)

A partition of $\mathcal{V}(\mathcal{G})$ is given by $\mathcal{C}_1, \ldots, \mathcal{C}_l$ $(1 \leq l \leq n)$ satisfying two properties: (i) $\mathcal{C}_i \bigcap \mathcal{C}_j = \emptyset$, for $i \neq j$, and (ii) $\bigcup_{k=1}^{l} \mathcal{C}_k = \mathcal{V}(\mathcal{G})$. We have the following definition.

The n-agent system under the consensus protocol achieves a cluster consensus if there exists a partition $\mathcal{C}_1, \ldots, \mathcal{C}_l$, such that all agents belonging to the same partition achieve consensus, while for any two agents i and j belonging to two different partitions, $\mathbf{x}_i \neq \mathbf{x}_j$. Each $\mathcal{C}_i$, $i = 1, \ldots, l$, is referred to as a cluster.

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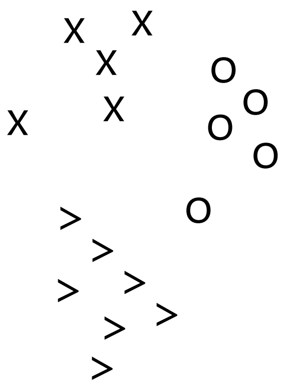

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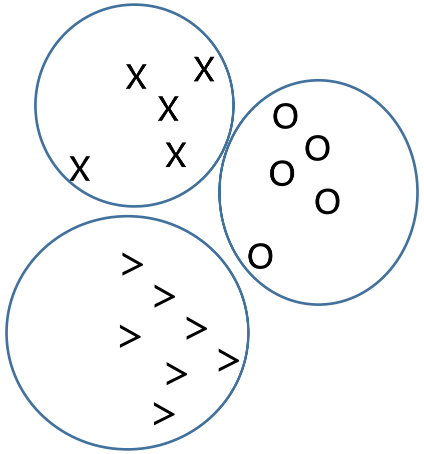

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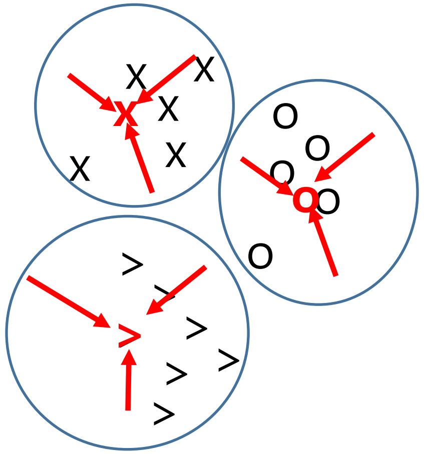

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Property: Null space of Laplacian

Model 1- static case

$$\dot{\mathbf{x}}_i(t) = -\sum_{j\in\mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),$$

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Property: Null space of Laplacian

$$\mathcal{N}(\mathbf{L}) = \mathrm{span}\{\mathrm{range}\{\mathbf{1}_n \otimes \mathbf{I}_{d\times d}\}, \{\mathbf{v} = [\mathbf{v}_1^T, \ldots, \mathbf{v}_n^T]^T \in \mathbb{R}^{dn} | (\mathbf{v}_j - \mathbf{v}_i) \in \mathcal{N}(\mathbf{A}_{ij}), \forall (i,j) \in \mathcal{E}\}\}$$

Model 1- static case

$$\dot{\mathbf{x}}_i(t) = -\sum_{j\in\mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),$$

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Def.: Clusters & cluster consensus (clustered opinions)

A partition of $\mathcal{V}(\mathcal{G})$ is given by $\mathcal{C}_1, \ldots, \mathcal{C}_l$ $(1 \leq l \leq n)$ satisfying two properties: (i) $\mathcal{C}_i \bigcap \mathcal{C}_j = \emptyset$, for $i \neq j$, and (ii) $\bigcup_{k=1}^{l} \mathcal{C}_k = \mathcal{V}(\mathcal{G})$. We have the following definition.

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Property: Null space of Laplacian

$\mathcal{N}(\mathbf{L}) = \mathrm{span}\{\mathrm{range}\{\mathbf{1}_n \otimes \mathbf{I}_{d\times d}\}, \{\mathbf{v} = [\mathbf{v}_1^T, \ldots, \mathbf{v}_n^T]^T \in \mathbb{R}^{dn} | (\mathbf{v}_j - \mathbf{v}_i) \in \mathcal{N}(\mathbf{A}_{ij}), \forall (i,j) \in \mathcal{E}\}\}$

A sole null space of scalar consensus

Model 1- static case

$$\dot{\mathbf{x}}_i(t) = -\sum_{j\in\mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),$$

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$$\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x}.$$

Def.: Clusters & cluster consensus (clustered opinions)

A partition of $\mathcal{V}(\mathcal{G})$ is given by $\mathcal{C}_1, \ldots, \mathcal{C}_l$ $(1 \leq l \leq n)$ satisfying two properties: (i) $\mathcal{C}_i \bigcap \mathcal{C}_j = \emptyset$, for $i \neq j$, and (ii) $\bigcup_{k=1}^{l} \mathcal{C}_k = \mathcal{V}(\mathcal{G})$. We have the following definition.

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A sole null space of scalar consensus

Property: Null space of Laplacian

$\mathcal{N}(\mathbf{L}) = \text{span}\{\text{range}\{\mathbf{1}_n \otimes \mathbf{I}_{d\times d}\}, \{\mathbf{v} = [\mathbf{v}_1^T, \ldots, \mathbf{v}_n^T]^T \in \mathbb{R}^{dn} | (\mathbf{v}_j - \mathbf{v}_i) \in \mathcal{N}(\mathbf{A}_{ij}), \forall (i, j) \in \mathcal{E}\}\}$

Additional null space !

Model 1- static case

$$\dot{\mathbf{x}}_i(t) = -\sum_{j\in\mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),$$

$$\mathbf{A}_{ij} = \mathbf{A}_{ij}^T \geq 0$$

$$\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x}.$$

Def.: Positive (semi-) connected

Def.: Clusters & cluster consensus (clustered opinions)

A partition of $\mathcal{V}(\mathcal{G})$ is given by $\mathcal{C}_1, \ldots, \mathcal{C}_l$ $(1 \leq l \leq n)$ satisfying two properties: (i) $\mathcal{C}_i \bigcap \mathcal{C}_j = \emptyset$, for $i \neq j$, and (ii) $\bigcup_{k=1}^{l} \mathcal{C}_k = \mathcal{V}(\mathcal{G})$. We have the following definition.

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Property: Null space of Laplacian

$$\mathcal{N}(\mathbf{L}) = \mathrm{span}\{\mathrm{range}\{\mathbf{1}_{\mathrm{n}} \otimes \mathbf{I}_{\mathrm{d}\times\mathrm{d}}\}, \{\mathbf{v} = [\mathbf{v}_1^{\mathrm{T}}, \ldots, \mathbf{v}_{\mathrm{n}}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{\mathrm{dn}} | (\mathbf{v}_{\mathrm{j}} - \mathbf{v}_{\mathrm{i}}) \in \mathcal{N}(\mathbf{A}_{\mathrm{ij}}), \forall (\mathrm{i},\mathrm{j}) \in \mathcal{E}\}\}$$

Model 1- static case

$$\dot{\mathbf{x}}_i(t) = -\sum_{j\in\mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),$$

$$\mathbf{A}_{ij} = \mathbf{A}_{ij}^T \geq 0$$

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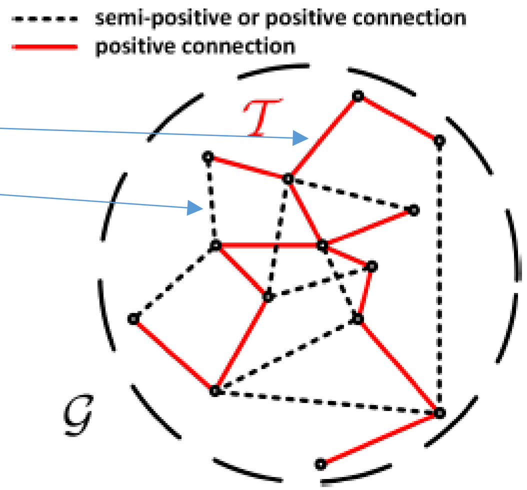
Def.: Positive (semi-) connected



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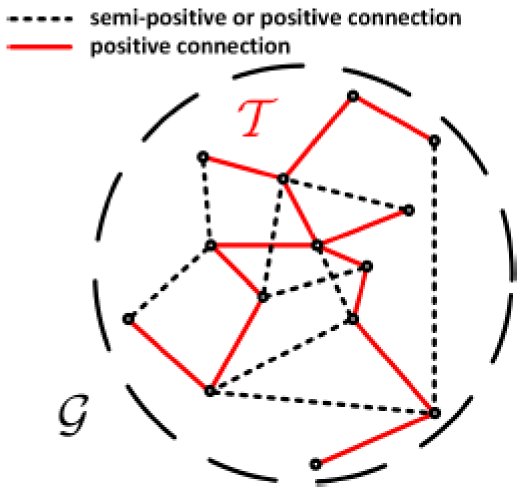
Def.: Positive (semi-) connected



$\mathcal{T}$ is a positive spanning tree of $\mathcal{G}$. The edges in $\mathcal{E}(\mathcal{T})$ are in red color.

Thm.: Exact condition for a consensus

The consensus system globally exponentially converges to $\mathbf{x}^* = \mathbf{1}_n \otimes \bar{\mathbf{x}}$ if and only if $\mathcal{N}(\mathbf{L}) = \mathrm{span}\{\mathbf{1}_n \otimes \mathbf{I}_{d\times d}\}$.

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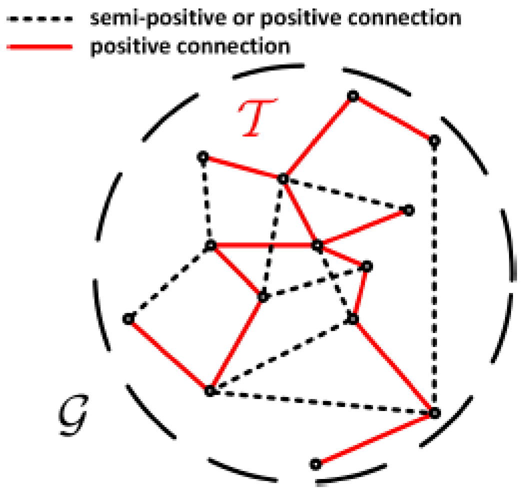

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No other null space!

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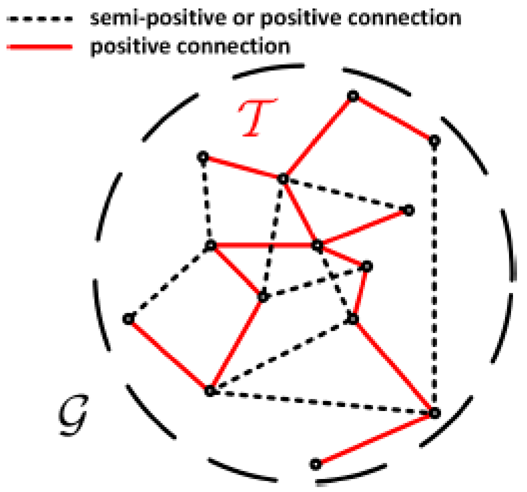

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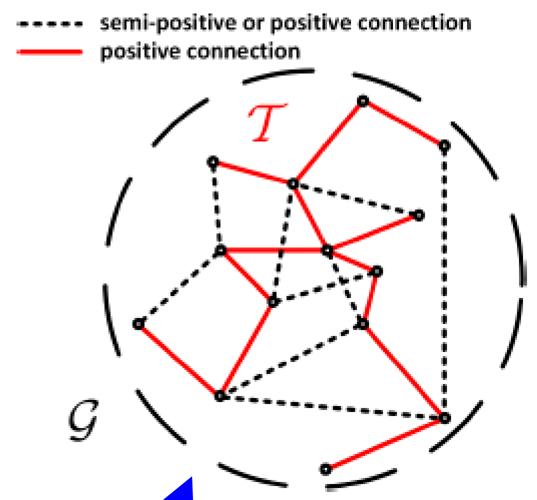

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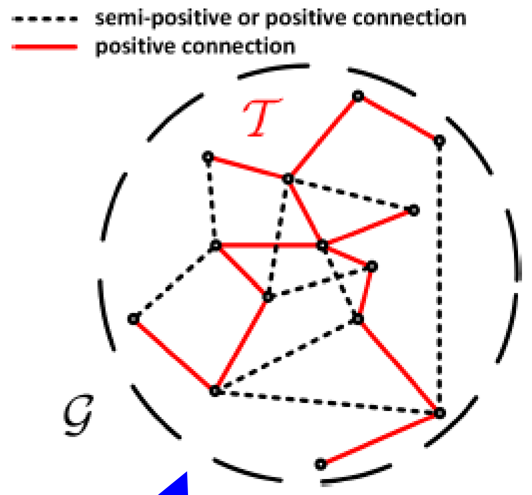

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Connected !

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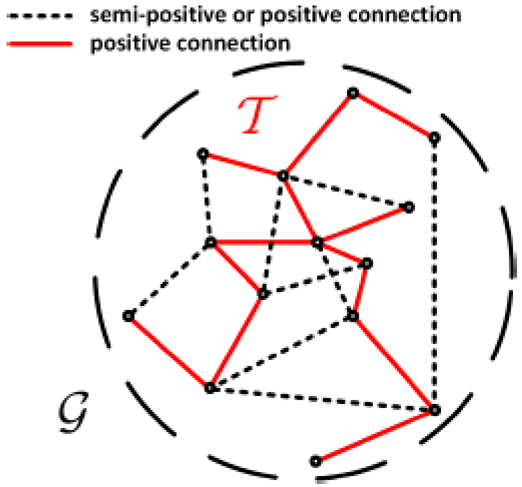

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If there exists a positive spanning tree in $\mathcal{G}$, then consensus is globally exponentially achieved.

Suppose there exists a positive tree $\mathcal{T} \subset \mathcal{G}$ of l vertices. Under the consensus protocol, $\mathbf{x}_i(t) \to \mathbf{x}_j(t)$, $\forall i, j \in \mathcal{T}$, as $t \to \infty$.

Model 1- static case

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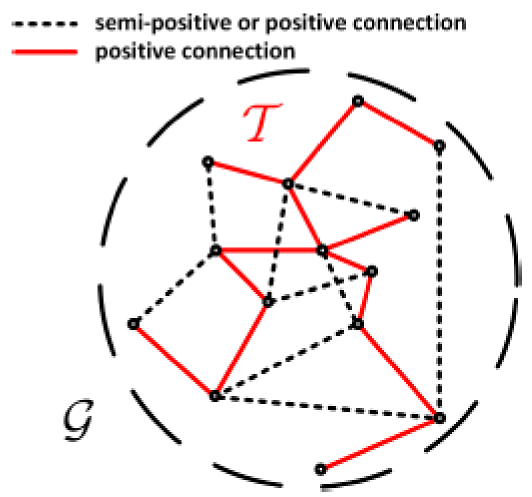

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If there exists a positive spanning tree in $\mathcal{G}$, then consensus is globally exponentially achieved.

Clusters!

Suppose there exists a positive tree $\mathcal{T} \subset \mathcal{G}$ of l vertices. Under the consensus protocol, $\mathbf{x}_i(t) \to \mathbf{x}_j(t)$, $\forall i, j \in \mathcal{T}$, as $t \to \infty$.

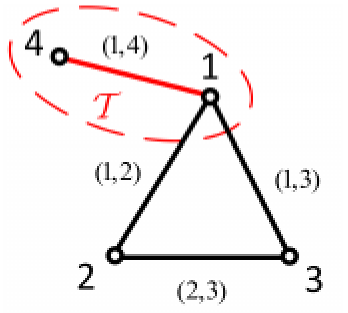
Model 1- static case

Algorithm 1 Finding all clusters of $\mathcal{G}$ under the matrix-weighted consensus protocol (8).

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$

$i = 0$;
Find the set of positive trees $\{\mathcal{T}_1, \ldots, \mathcal{T}_p\}$ in $\mathcal{G}$;
$\mathcal{C}_{\mathcal{G}}(0) = \{\mathcal{C}_m = \{\mathcal{V}(\mathcal{T}_m)\}, m = 1, \ldots, p\}$;
repeat
  $\mathcal{C}_{\mathcal{G}}(i+1) = \mathcal{C}_{\mathcal{G}}(i)$;
  check $\leftarrow$ false;
  for all $\mathcal{C}_m \in \mathcal{C}_{\mathcal{G}}(i)$ **do**
    for all $\mathcal{C}_l \in \mathcal{C}_{\mathcal{G}}(i), l \neq m$ **do**
      if $\exists i \in \mathcal{C}_l$ satisfies Proposition 1(ii) **then**
        $\mathcal{V}_{temp} = \mathcal{V}(\mathcal{T}_m) \cup \mathcal{V}(\mathcal{T}_l)$;
        $\mathcal{E}_{temp} = \mathcal{E}(\mathcal{T}_m) \cup \mathcal{E}(\mathcal{T}_l) \cup \mathcal{S}$;
        $\mathcal{C}_{temp} = \mathcal{C}_m \cup \mathcal{C}_l$;
        $\mathcal{C}_{\mathcal{G}}(i+1) = \mathcal{C}_{\mathcal{G}}(i+1) \setminus \{\mathcal{C}_m, \mathcal{C}_l\} \cup \{\mathcal{C}_{temp}\}$;
        check $\leftarrow$ true;
        break;
      end if
    end for
    if check == true **then**
      break;
    end if
  end for
until $\mathcal{C}_{\mathcal{G}}(i) = \mathcal{C}_{\mathcal{G}}(i-1)$

EX-1:



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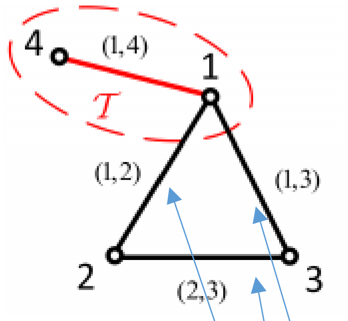

Positive semidefinite

Model 1- static case

Algorithm 1 Finding all clusters of $\mathcal{G}$ under the matrix-weighted consensus protocol (8).

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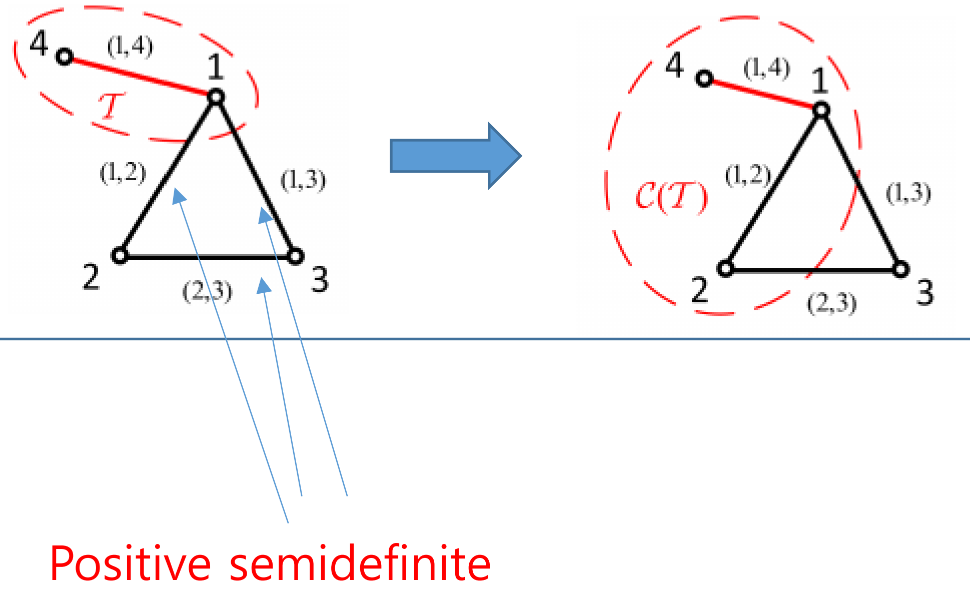

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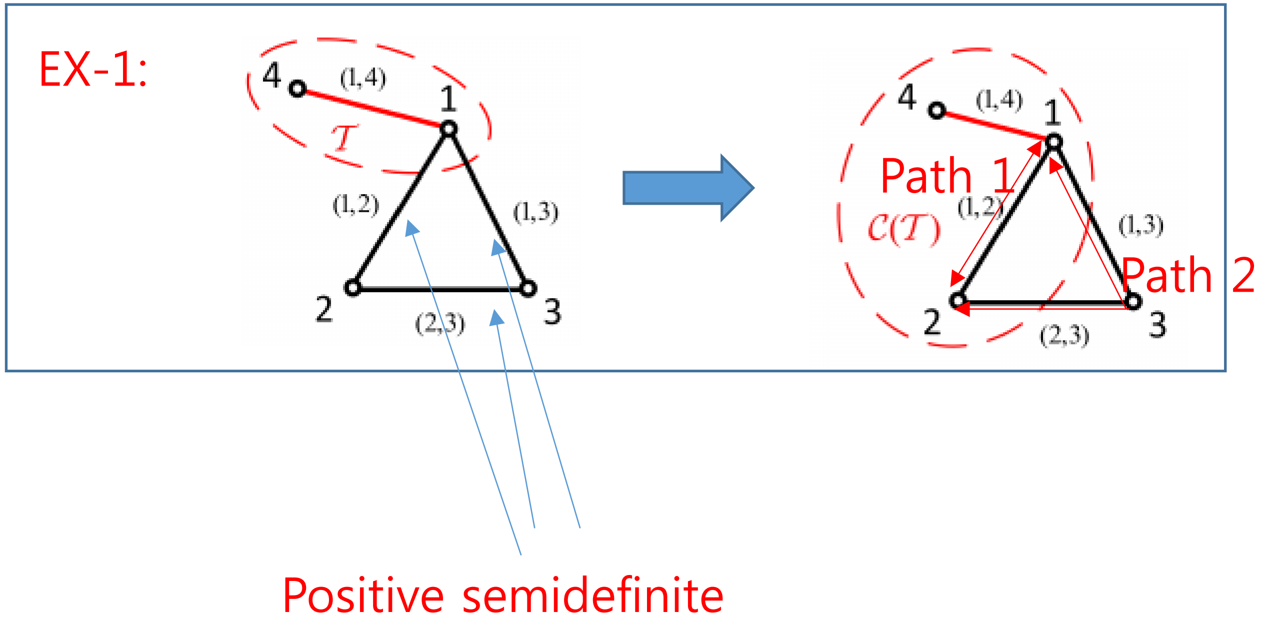
    for all $\mathcal{C}_l \in \mathcal{C}_\mathcal{G}(i), l \neq m$ **do**
      if $\exists i \in \mathcal{C}_l$ satisfies Proposition 1(ii) **then**
        $\mathcal{V}_{temp} = \mathcal{V}(\mathcal{T}_m) \cup \mathcal{V}(\mathcal{T}_l)$;
        $\mathcal{E}_{temp} = \mathcal{E}(\mathcal{T}_m) \cup \mathcal{E}(\mathcal{T}_l) \cup \mathcal{S}$;
        $\mathcal{C}_{temp} = \mathcal{C}_m \cup \mathcal{C}_l$;
        $\mathcal{C}_\mathcal{G}(i+1) = \mathcal{C}_\mathcal{G}(i+1) \setminus \{\mathcal{C}_m, \mathcal{C}_l\} \cup \{\mathcal{C}_{temp}\}$;
        check $\leftarrow$ true;
        break;
      end if
    end for
    if check == true **then**
      break;
    end if
  end for
until $\mathcal{C}_\mathcal{G}(i) = \mathcal{C}_\mathcal{G}(i-1)$

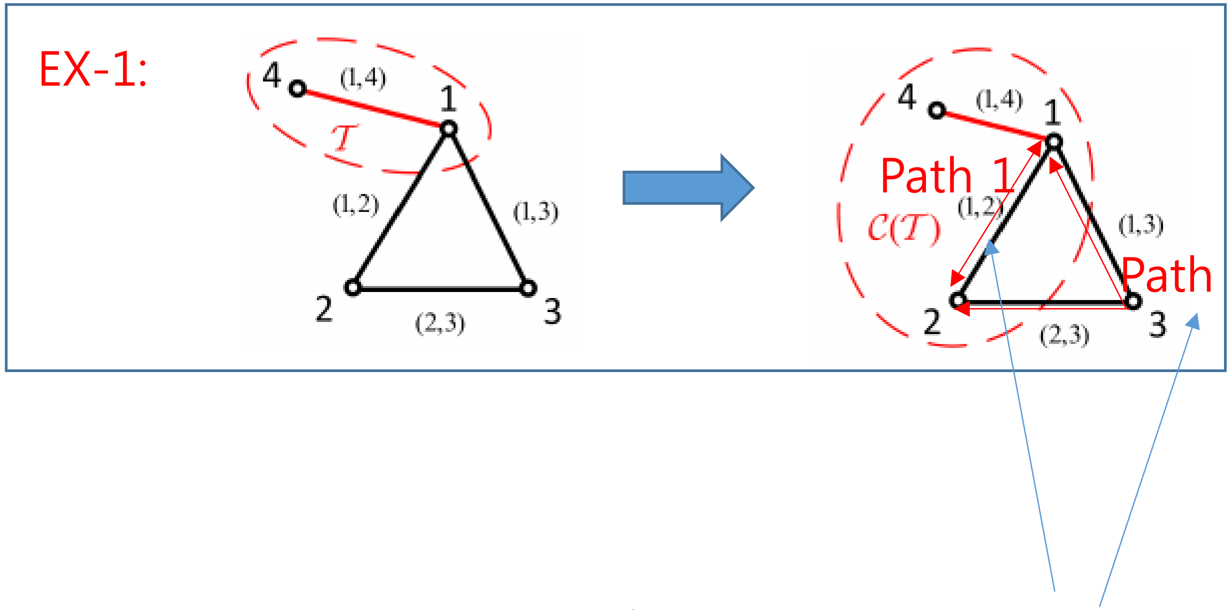


Model 1- static case

Algorithm 1 Finding all clusters of $\mathcal{G}$ under the matrix-weighted consensus protocol (8).

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$

$i = 0$;
Find the set of positive trees $\{\mathcal{T}_1, \ldots, \mathcal{T}_p\}$ in $\mathcal{G}$;
$\mathcal{C}_{\mathcal{G}}(0) = \{\mathcal{C}_m = \{\mathcal{V}(\mathcal{T}_m)\}, m = 1, \ldots, p\}$;
repeat
  $\mathcal{C}_{\mathcal{G}}(i+1) = \mathcal{C}_{\mathcal{G}}(i)$;
  check $\leftarrow$ false;
  for all $\mathcal{C}_m \in \mathcal{C}_{\mathcal{G}}(i)$ **do**
    for all $\mathcal{C}_l \in \mathcal{C}_{\mathcal{G}}(i), l \neq m$ **do**
      if $\exists i \in \mathcal{C}_l$ satisfies Proposition 1(ii) **then**
        $\mathcal{V}_{temp} = \mathcal{V}(\mathcal{T}_m) \cup \mathcal{V}(\mathcal{T}_l)$;
        $\mathcal{E}_{temp} = \mathcal{E}(\mathcal{T}_m) \cup \mathcal{E}(\mathcal{T}_l) \cup \mathcal{S}$;
        $\mathcal{C}_{temp} = \mathcal{C}_m \cup \mathcal{C}_l$;
        $\mathcal{C}_{\mathcal{G}}(i+1) = \mathcal{C}_{\mathcal{G}}(i+1) \setminus \{\mathcal{C}_m, \mathcal{C}_l\} \cup \{\mathcal{C}_{temp}\}$;
        check $\leftarrow$ true;
        break;
      end if
    end for
    if check == true **then**
      break;
    end if
  end for
until $\mathcal{C}_{\mathcal{G}}(i) = \mathcal{C}_{\mathcal{G}}(i-1)$



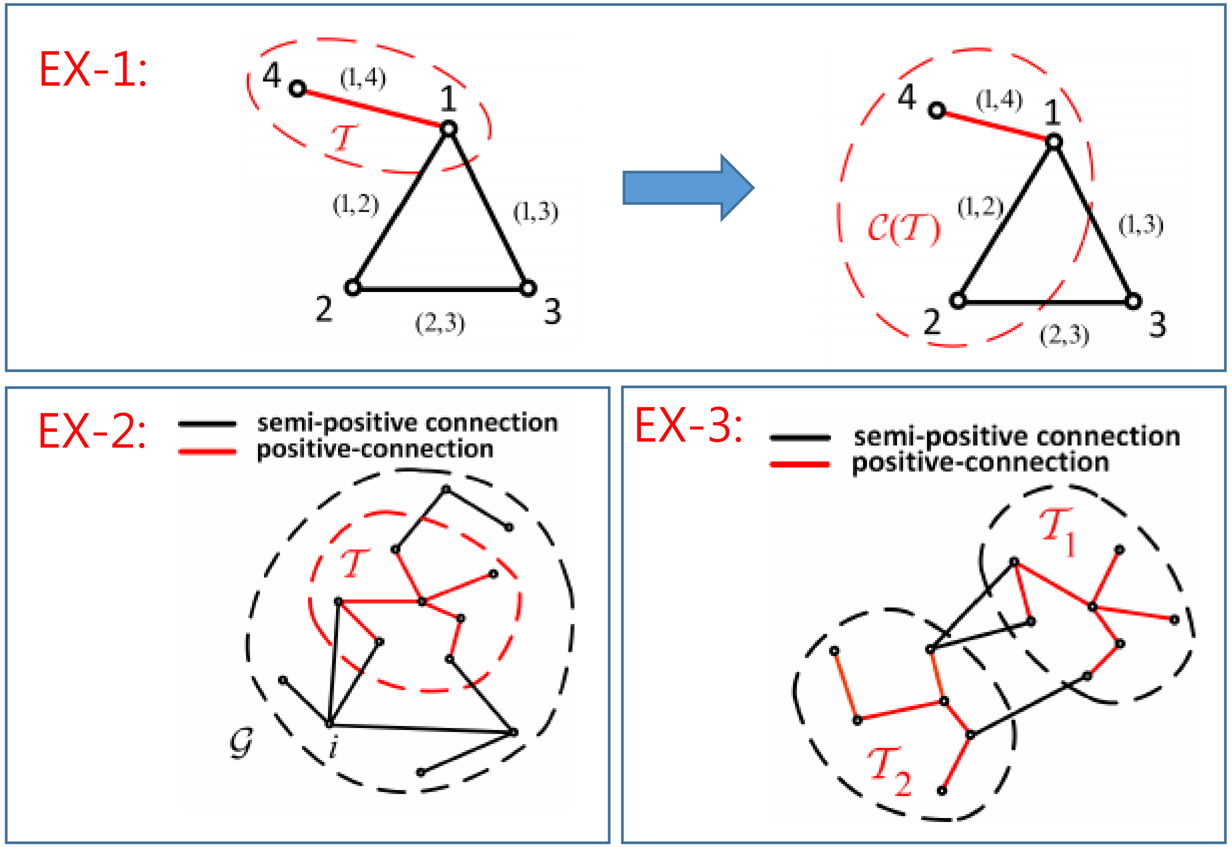
Path 1 + path 2 = positive path

Model 1- static case

Algorithm 1 Finding all clusters of $\mathcal{G}$ under the matrix-weighted consensus protocol (8).

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$

```
i = 0;
Find the set of positive trees {T_1, ..., T_p} in G;
C_G(0) = {C_m = {V(T_m)}, m = 1, ..., p};
repeat
    C_G(i+1) = C_G(i);
    check ← false;
    for all C_m ∈ C_G(i) do
        for all C_l ∈ C_G(i), l ≠ m do
            if ∃i ∈ C_l satisfies Proposition 1(ii) then
                V_temp = V(T_m) ∪ V(T_l);
                E_temp = E(T_m) ∪ E(T_l) ∪ S;
                C_temp = C_m ∪ C_l;
                C_G(i+1) = C_G(i+1) \ {C_m, C_l} ∪ {C_temp};
                check ← true;
                break;
            end if
        end for
        if check == true then
            break;
        end if
    end for
until C_G(i) = C_G(i − 1)
```

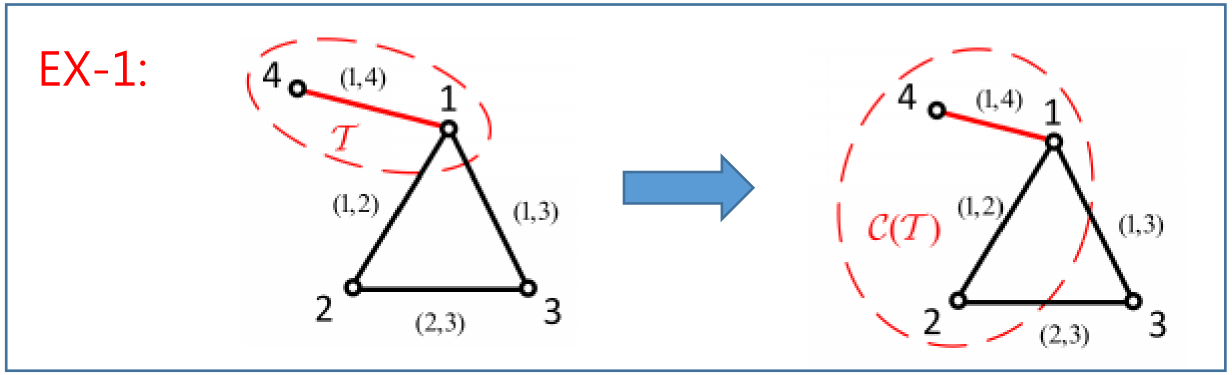


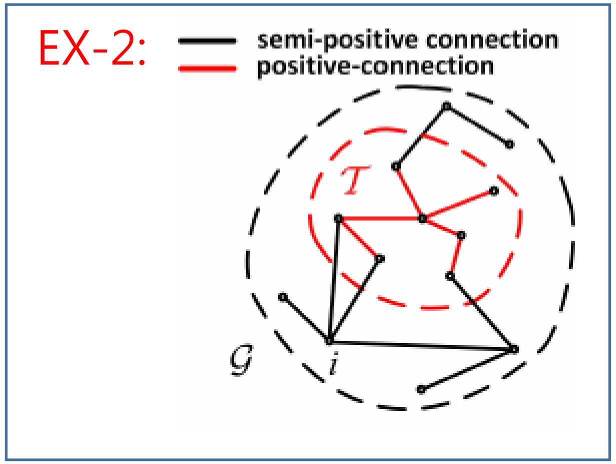
Model 1- static case

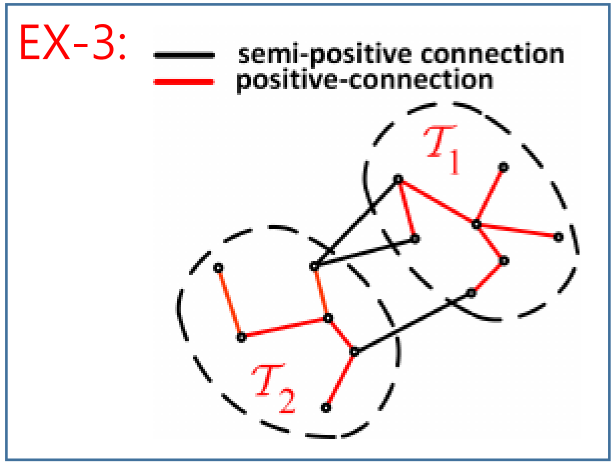
Algorithm 1 Finding all clusters of $\mathcal{G}$ under the matrix-weighted consensus protocol (8).

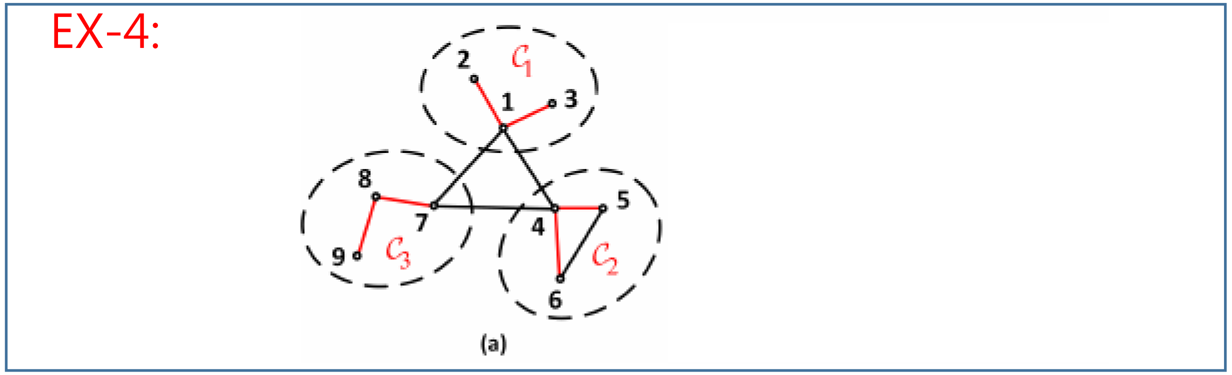
Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$

$i = 0$;
Find the set of positive trees $\{\mathcal{T}_1, \ldots, \mathcal{T}_p\}$ in $\mathcal{G}$;
$\mathcal{C}_{\mathcal{G}}(0) = \{\mathcal{C}_m = \{\mathcal{V}(\mathcal{T}_m)\}, m = 1, \ldots, p\}$;
repeat
  $\mathcal{C}_{\mathcal{G}}(i+1) = \mathcal{C}_{\mathcal{G}}(i)$;
  check $\leftarrow$ false;
  for all $\mathcal{C}_m \in \mathcal{C}_{\mathcal{G}}(i)$ **do**
    for all $\mathcal{C}_l \in \mathcal{C}_{\mathcal{G}}(i), l \neq m$ **do**
      if $\exists i \in \mathcal{C}_l$ satisfies Proposition 1(ii) **then**
        $\mathcal{V}_{temp} = \mathcal{V}(\mathcal{T}_m) \cup \mathcal{V}(\mathcal{T}_l)$;
        $\mathcal{E}_{temp} = \mathcal{E}(\mathcal{T}_m) \cup \mathcal{E}(\mathcal{T}_l) \cup \mathcal{S}$;
        $\mathcal{C}_{temp} = \mathcal{C}_m \cup \mathcal{C}_l$;
        $\mathcal{C}_{\mathcal{G}}(i+1) = \mathcal{C}_{\mathcal{G}}(i+1) \setminus \{\mathcal{C}_m, \mathcal{C}_l\} \cup \{\mathcal{C}_{temp}\}$;
        check $\leftarrow$ true;
        break;
      end if
    end for
    if check == true **then**
      break;
    end if
  end for
until $\mathcal{C}_{\mathcal{G}}(i) = \mathcal{C}_{\mathcal{G}}(i-1)$









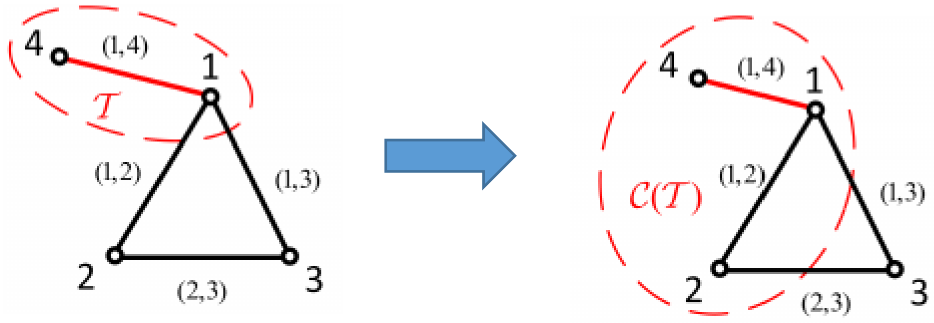
Model 1- static case

Algorithm 1 Finding all clusters of $\mathcal{G}$ under the matrix-weighted consensus protocol (8).

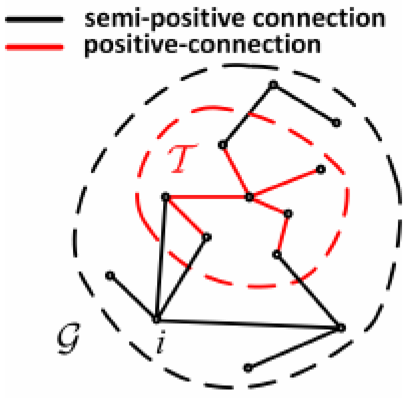
Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$

$i = 0$;
Find the set of positive trees $\{\mathcal{T}_1, \ldots, \mathcal{T}_p\}$ in $\mathcal{G}$;
$\mathcal{C}_{\mathcal{G}}(0) = \{\mathcal{C}_m = \{\mathcal{V}(\mathcal{T}_m)\}, m = 1, \ldots, p\}$;
repeat
    $\mathcal{C}_{\mathcal{G}}(i+1) = \mathcal{C}_{\mathcal{G}}(i)$;
    check $\leftarrow$ false;
    for all $\mathcal{C}_m \in \mathcal{C}_{\mathcal{G}}(i)$ **do**
        for all $\mathcal{C}_l \in \mathcal{C}_{\mathcal{G}}(i), l \neq m$ **do**
            if $\exists i \in \mathcal{C}_l$ satisfies Proposition 1(ii) **then**
                $\mathcal{V}_{temp} = \mathcal{V}(\mathcal{T}_m) \cup \mathcal{V}(\mathcal{T}_l)$;
                $\mathcal{E}_{temp} = \mathcal{E}(\mathcal{T}_m) \cup \mathcal{E}(\mathcal{T}_l) \cup \mathcal{S}$;
                $\mathcal{C}_{temp} = \mathcal{C}_m \cup \mathcal{C}_l$;
                $\mathcal{C}_{\mathcal{G}}(i+1) = \mathcal{C}_{\mathcal{G}}(i+1) \setminus \{\mathcal{C}_m, \mathcal{C}_l\} \cup \{\mathcal{C}_{temp}\}$;
                check $\leftarrow$ true;
                break;
            end if
        end for
        if check == true **then**
            break;
        end if
    end for
until $\mathcal{C}_{\mathcal{G}}(i) = \mathcal{C}_{\mathcal{G}}(i-1)$

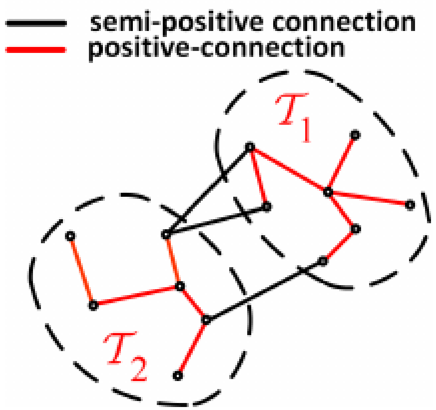
EX-1:



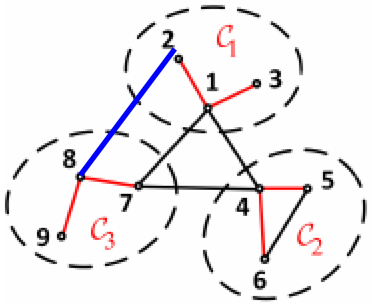
EX-2:



EX-3:



EX-4:



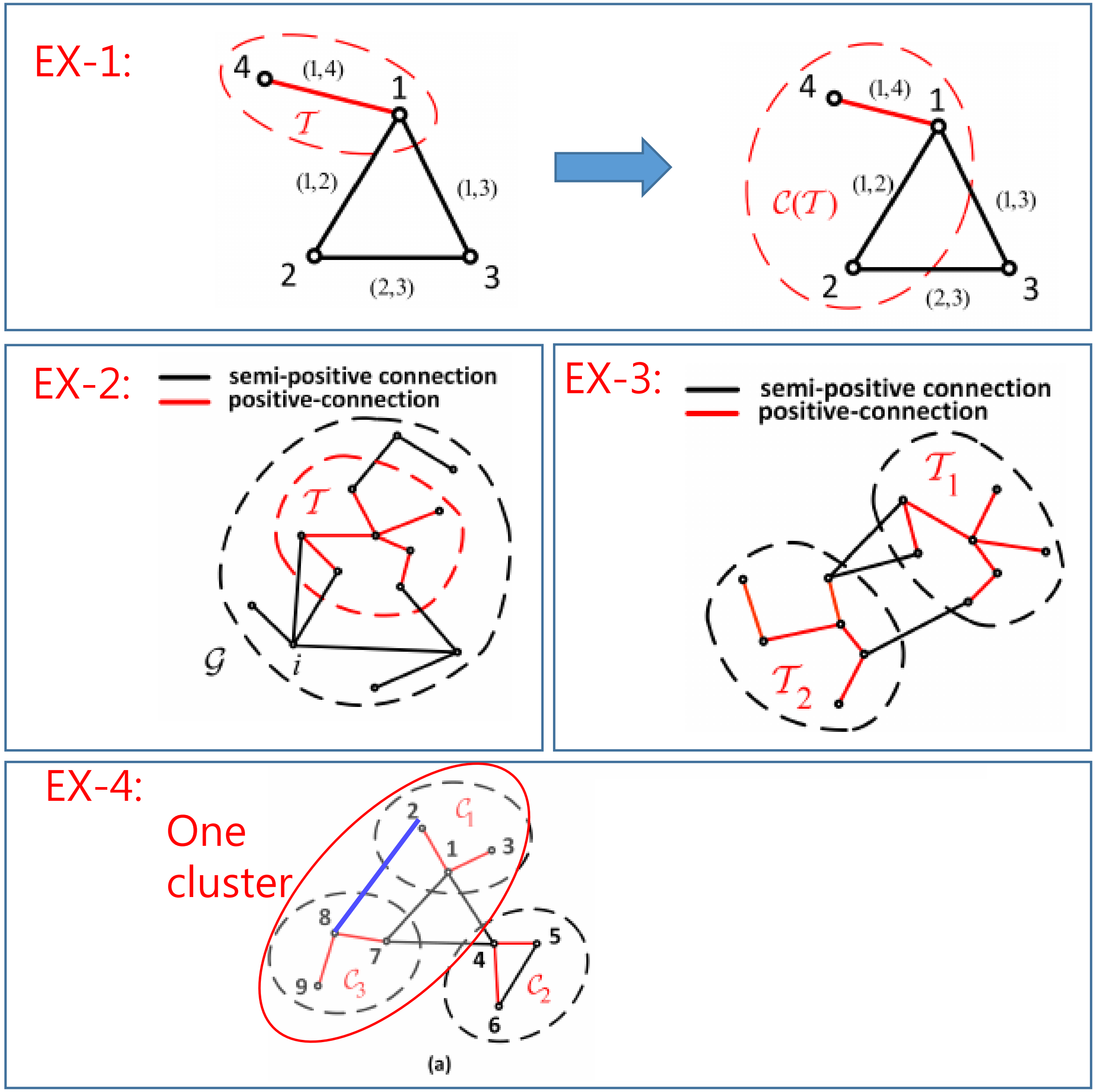
(a)

Model 1- static case

Algorithm 1 Finding all clusters of $\mathcal{G}$ under the matrix-weighted consensus protocol (8).

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$

$i = 0$;
Find the set of positive trees $\{\mathcal{T}_1, \ldots, \mathcal{T}_p\}$ in $\mathcal{G}$;
$\mathcal{C}_{\mathcal{G}}(0) = \{\mathcal{C}_m = \{\mathcal{V}(\mathcal{T}_m)\}, m = 1, \ldots, p\}$;
repeat
  $\mathcal{C}_{\mathcal{G}}(i+1) = \mathcal{C}_{\mathcal{G}}(i)$;
  check $\leftarrow$ false;
  for all $\mathcal{C}_m \in \mathcal{C}_{\mathcal{G}}(i)$ **do**
    for all $\mathcal{C}_l \in \mathcal{C}_{\mathcal{G}}(i), l \neq m$ **do**
      if $\exists i \in \mathcal{C}_l$ satisfies Proposition 1(ii) **then**
        $\mathcal{V}_{temp} = \mathcal{V}(\mathcal{T}_m) \cup \mathcal{V}(\mathcal{T}_l)$;
        $\mathcal{E}_{temp} = \mathcal{E}(\mathcal{T}_m) \cup \mathcal{E}(\mathcal{T}_l) \cup \mathcal{S}$;
        $\mathcal{C}_{temp} = \mathcal{C}_m \cup \mathcal{C}_l$;
        $\mathcal{C}_{\mathcal{G}}(i+1) = \mathcal{C}_{\mathcal{G}}(i+1) \setminus \{\mathcal{C}_m, \mathcal{C}_l\} \cup \{\mathcal{C}_{temp}\}$;
        check $\leftarrow$ true;
        break;
      end if
    end for
    if check == true **then**
      break;
    end if
  end for
until $\mathcal{C}_{\mathcal{G}}(i) = \mathcal{C}_{\mathcal{G}}(i-1)$



Model 1- static case

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Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$

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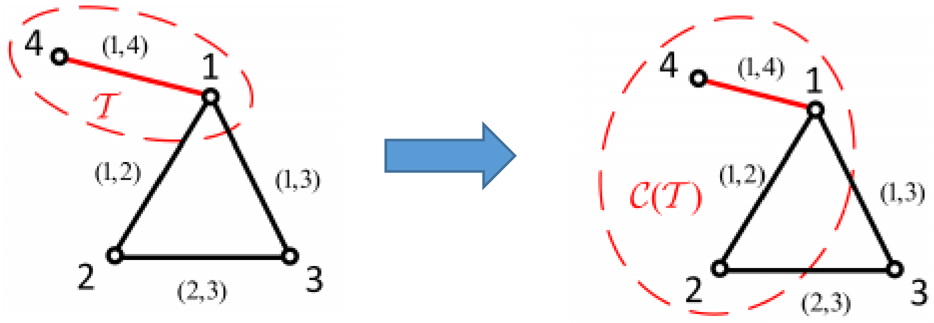
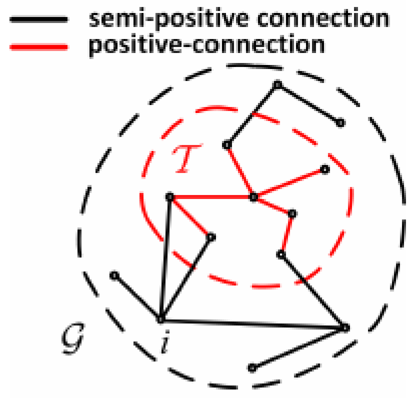
        break;
      end if
    end for
    if check == true **then**
      break;
    end if
  end for
until $\mathcal{C}_{\mathcal{G}}(i) = \mathcal{C}_{\mathcal{G}}(i-1)$

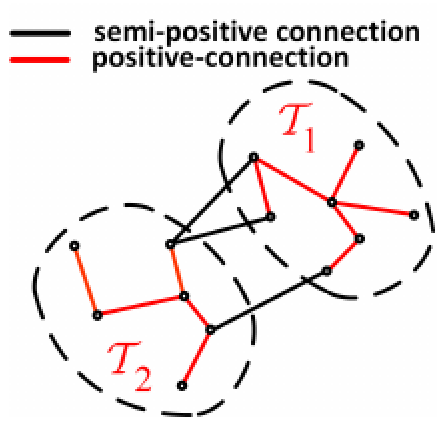
EX-1:



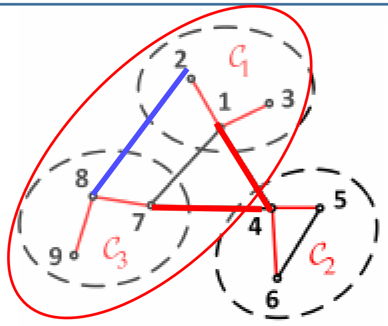
EX-2:



EX-3:



EX-4:



(a)

Model 1- static case

Algorithm 1 Finding all clusters of $\mathcal{G}$ under the matrix-weighted consensus protocol (8).

Require: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$

$i = 0$;
Find the set of positive trees $\{\mathcal{T}_1, \ldots, \mathcal{T}_p\}$ in $\mathcal{G}$;
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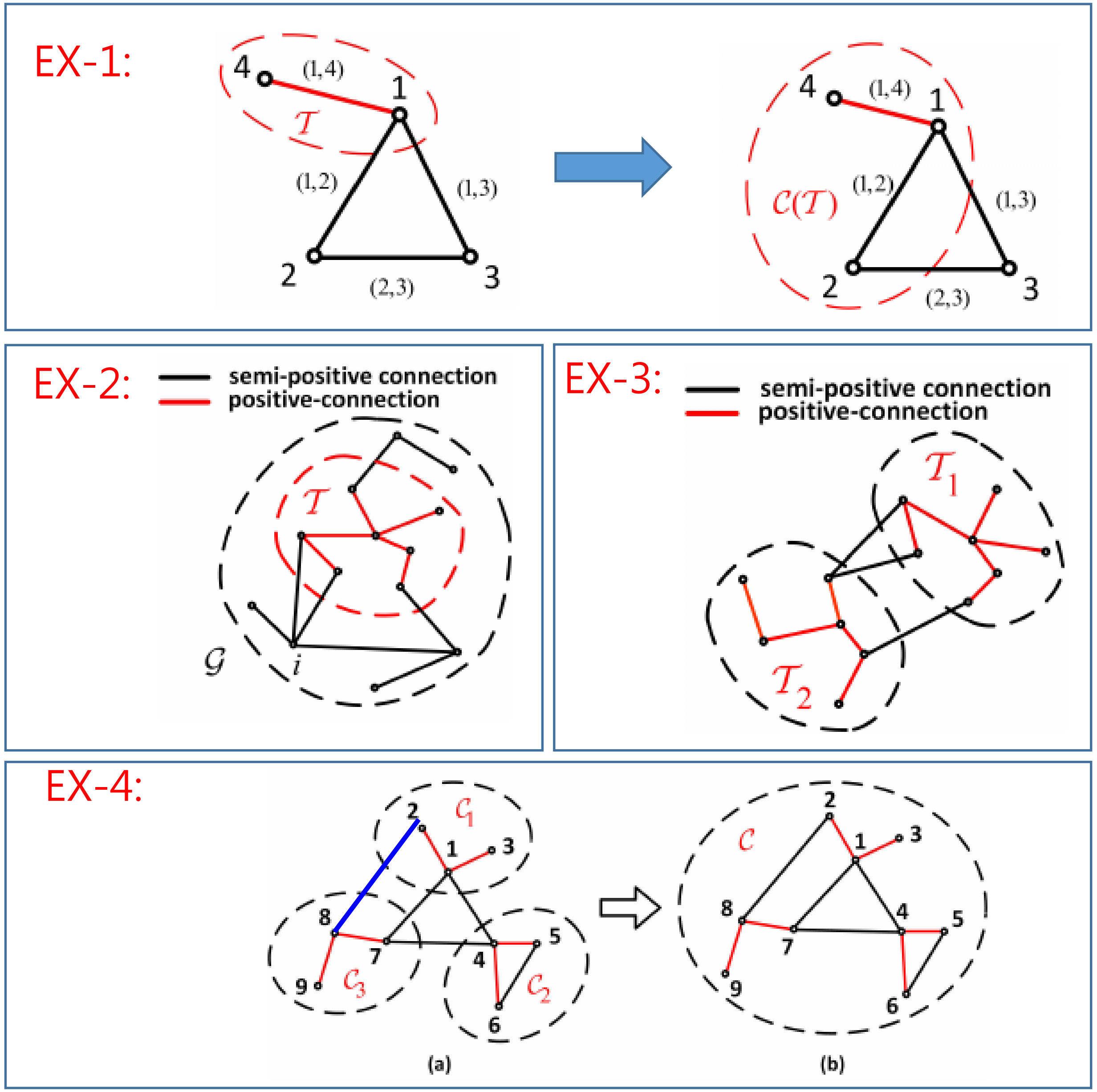
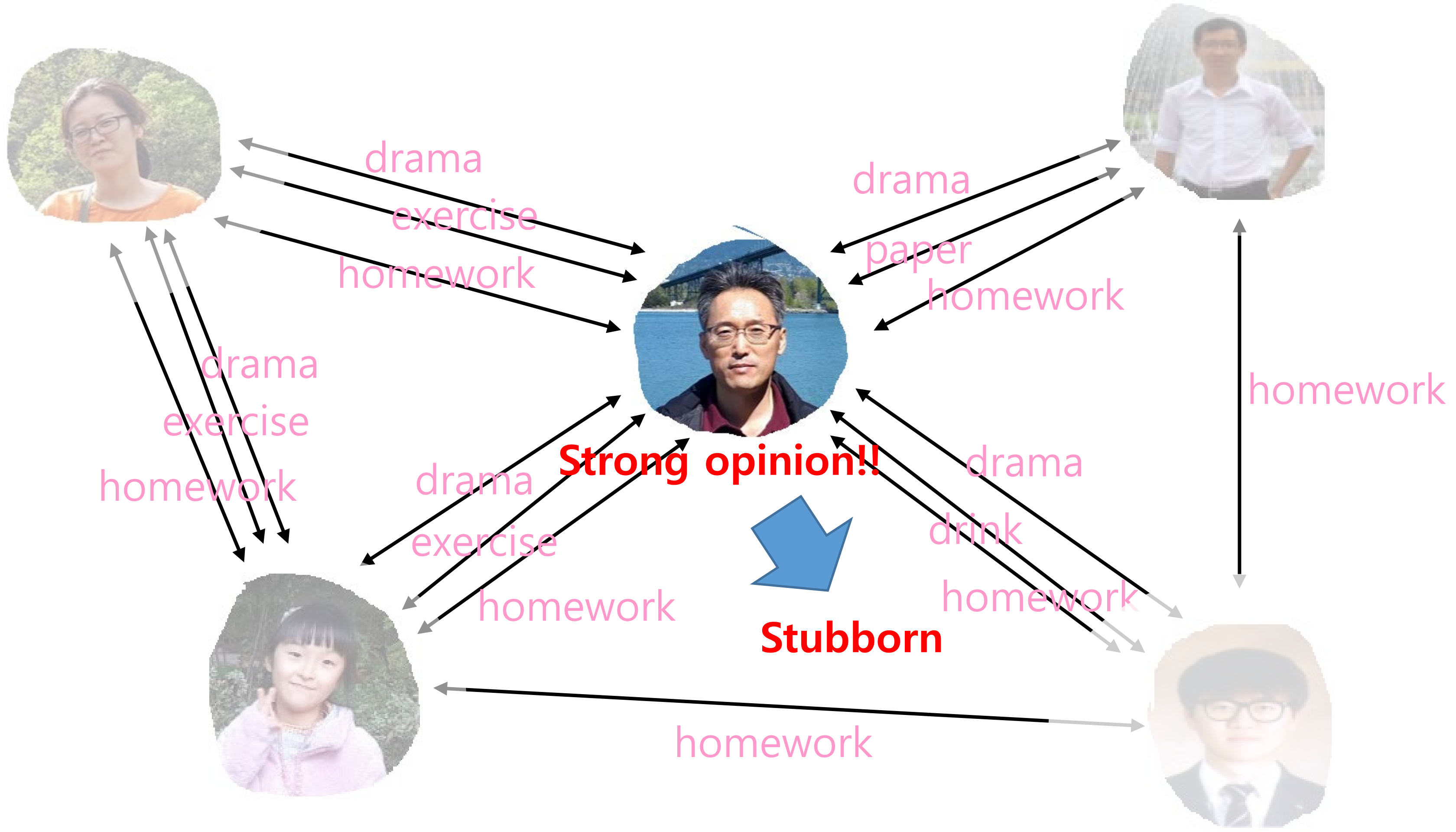
repeat
    $\mathcal{C}_{\mathcal{G}}(i+1) = \mathcal{C}_{\mathcal{G}}(i)$;
    check $\leftarrow$ false;
    for all $\mathcal{C}_m \in \mathcal{C}_{\mathcal{G}}(i)$ **do**
        for all $\mathcal{C}_l \in \mathcal{C}_{\mathcal{G}}(i), l \neq m$ **do**
            if $\exists i \in \mathcal{C}_l$ satisfies Proposition 1(ii) **then**
                $\mathcal{V}_{temp} = \mathcal{V}(\mathcal{T}_m) \cup \mathcal{V}(\mathcal{T}_l)$;
                $\mathcal{E}_{temp} = \mathcal{E}(\mathcal{T}_m) \cup \mathcal{E}(\mathcal{T}_l) \cup \mathcal{S}$;
                $\mathcal{C}_{temp} = \mathcal{C}_m \cup \mathcal{C}_l$;
                $\mathcal{C}_{\mathcal{G}}(i+1) = \mathcal{C}_{\mathcal{G}}(i+1) \setminus \{\mathcal{C}_m, \mathcal{C}_l\} \cup \{\mathcal{C}_{temp}\}$;
                check $\leftarrow$ true;
                break;
            end if
        end for
        if check == true **then**
            break;
        end if
    end for
until $\mathcal{C}_{\mathcal{G}}(i) = \mathcal{C}_{\mathcal{G}}(i-1)$



Part-2: Analysis

Problem 2-With Stubborn Nodes

Model 2 - Opinion Dynamics with Stubborn Agents



Model 2 - Opinion Dynamics with Stubborn Agents

$$x_i(k+1) = \lambda_i \sum_{j=1}^{n} w_{ij} C x_j(k) + (1-\lambda_i) x_i(0)$$

Friedkin-Johnsen algorithm

Model 2 - Opinion Dynamics with Stubborn Agents

$$x_i(k+1) = \lambda_i \sum_{j=1}^{n} w_{ij} C x_j(k) + (1-\lambda_i) x_i(0)$$

Friedkin-Johnsen algorithm

Degree of stubborn

Model 2 - Opinion Dynamics with Stubborn Agents

Friedkin-Johnsen algorithm

$$x_i(k+1) = \lambda_i \sum_{j=1}^{n} w_{ij} C x_j(k) + (1-\lambda_i) x_i(0)$$

Model:

Cont. time

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} C \left(x_j(t) - x_i(t)\right) + (C - I_n) x_i(t) + b_i(x_i(0) - x_i(t))$$

Model 2 - Opinion Dynamics with Stubborn Agents

$$x_i(k+1) = \lambda_i \sum_{j=1}^{n} w_{ij} C x_j(k) + (1-\lambda_i) x_i(0)$$

Friedkin-Johnsen algorithm

Cont. time

Model:

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} C (x_j(t) - x_i(t)) + (C - I_n) x_i(t) + b_i(x_i(0) - x_i(t))$$

Matrix weighted

Stubbornness

Model 2 - Opinion Dynamics with Stubborn Agents

Friedkin-Johnsen algorithm

$$x_i(k+1) = \lambda_i \sum_{j=1}^{n} w_{ij} C x_j(k) + (1-\lambda_i) x_i(0)$$

Model:

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} C\,(x_j(t) - x_i(t)) + (C - I_n) x_i(t) + b_i(x_i(0) - x_i(t))$$

Stubbornness

Assumption:

Assumption 1 (The matrix C) *The matrix $C \in \mathbb{R}^{d \times d}$ is such that $\lambda_1(C) = 1$ is a simple eigenvalue with an associated right (respectively left) eigenvector ζ (respectively $\xi^\top$), such that the eigenvectors satisfy $\xi^\top \zeta = 1$. Moreover, $\mathfrak{Re}(\lambda_k(C)) < 1, \forall\, k \neq 1$.*

Model 2 - Opinion Dynamics with Stubborn Agents

Friedkin-Johnsen algorithm

$$x_i(k+1) = \lambda_i \sum_{j=1}^{n} w_{ij} C x_j(k) + (1-\lambda_i) x_i(0)$$

Model:

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} C (x_j(t) - x_i(t)) + (C - I_n) x_i(t) + b_i(x_i(0) - x_i(t))$$

Stubbornness

Agreement:

$$\lim_{t \to \infty} x(t) = [I_{nd} + (\mathcal{L} - I_n) \otimes C + B \otimes I_d]^{-1} \times (B \otimes I_d) x(0)$$

Assumption:

Assumption 1 (The matrix C) *The matrix $C \in \mathbb{R}^{d \times d}$ is such that $\lambda_1(C) = 1$ is a simple eigenvalue with an associated right (respectively left) eigenvector ζ (respectively $\xi^\top$), such that the eigenvectors satisfy $\xi^\top \zeta = 1$. Moreover, $\mathfrak{Re}(\lambda_k(C)) < 1, \forall k \neq 1$.*

Model 2 - Opinion Dynamics with Stubborn Agents

Friedkin-Johnsen algorithm

$$x_i(k+1) = \lambda_i \sum_{j=1}^{n} w_{ij} C x_j(k) + (1-\lambda_i) x_i(0)$$

Model:

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} C (x_j(t) - x_i(t)) + (C - I_n) x_i(t) + b_i(x_i(0) - x_i(t))$$

Stubbornness

Assumption:

Assumption 1 (The matrix C) *The matrix $C \in \mathbb{R}^{d \times d}$ is such that $\lambda_1(C) = 1$ is a simple eigenvalue with an associated right (respectively left) eigenvector ζ (respectively $\xi^\top$), such that the eigenvectors satisfy $\xi^\top \zeta = 1$. Moreover, $\mathfrak{Re}(\lambda_k(C)) < 1, \forall k \neq 1$.*

Agreement:

$$\lim_{t \to \infty} x(t) = [I_{nd} + (\mathcal{L} - I_n) \otimes C + B \otimes I_d]^{-1} \times (B \otimes I_d) x(0)$$

➔ Overall influence of stubborn agents

Part-2: Analysis

Problem 3-Signed Matrices

(Separated positive coupling or negative coupling)

Model 3 - Inverse Proportional Couplings

$$\begin{pmatrix} \dot{x}_{i,1} \\ \dot{x}_{i,2} \\ \vdots \\ \dot{x}_{i,d} \end{pmatrix} = \sum_{j \in \mathcal{N}_i}^{n} \begin{bmatrix} a_{1,1}^{i,j} & a_{1,2}^{i,j} & \cdots & a_{1,d}^{i,j} \\ a_{2,1}^{i,j} & a_{2,2}^{i,j} & \cdots & a_{2,d}^{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,1}^{i,j} & a_{d,2}^{i,j} & \cdots & a_{d,d}^{i,j} \end{bmatrix} \begin{pmatrix} x_{j,1} - x_{i,1} \\ x_{j,2} - x_{i,2} \\ \vdots \\ x_{j,d} - x_{i,d} \end{pmatrix}$$

$$\triangleq \dot{\mathbf{x}}_i = \sum_{j \in \mathcal{N}_i}^{n} \mathbf{A}_{i,j}(\mathbf{x}_j - \mathbf{x}_i)$$

- The diagonal terms, i.e., $a_{k,k}^{i,j}$, if it is positive and as it increases, the agreement between $x_{j,k}$ and $x_{i,k}$ speeds up. Otherwise, if it is negative and as it increases in absolute value, the anti-agreement between $x_{j,k}$ and $x_{i,k}$ becomes significant.

Model 3 - Inverse Proportional Couplings

$$\begin{pmatrix} \dot{x}_{i,1} \\ \dot{x}_{i,2} \\ \vdots \\ \dot{x}_{i,d} \end{pmatrix} = \sum_{j \in \mathcal{N}_i}^{n} \begin{bmatrix} a^{i,j}_{1,1} & a^{i,j}_{1,2} & \cdots & a^{i,j}_{1,d} \\ a^{i,j}_{2,1} & a^{i,j}_{2,2} & \cdots & a^{i,j}_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ a^{i,j}_{d,1} & a^{i,j}_{d,2} & \cdots & a^{i,j}_{d,d} \end{bmatrix} \begin{pmatrix} x_{j,1} - x_{i,1} \\ x_{j,2} - x_{i,2} \\ \vdots \\ x_{j,d} - x_{i,d} \end{pmatrix}$$

$$\triangleq \dot{\mathbf{x}}_i = \sum_{j \in \mathcal{N}_i}^{n} \mathbf{A}_{i,j}(\mathbf{x}_j - \mathbf{x}_i)$$

- The diagonal terms, i.e., $a^{i,j}_{k,k}$, if it is positive and as it increases, the agreement between $x_{j,k}$ and $x_{i,k}$ speeds up. Otherwise, if it is negative and as it increases in absolute value, the anti-agreement between $x_{j,k}$ and $x_{i,k}$ becomes significant.

- The off-diagonal terms. For example, let us consider the effect of $a^{i,j}_{2,1}$. We can consider the following four cases

Model 3 - Inverse Proportional Couplings

$$\begin{pmatrix} \dot{x}_{i,1} \\ \dot{x}_{i,2} \\ \vdots \\ \dot{x}_{i,d} \end{pmatrix} = \sum_{j \in \mathcal{N}_i}^{n} \begin{bmatrix} a_{1,1}^{i,j} & a_{1,2}^{i,j} & \cdots & a_{1,d}^{i,j} \\ a_{2,1}^{i,j} & a_{2,2}^{i,j} & \cdots & a_{2,d}^{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,1}^{i,j} & a_{d,2}^{i,j} & \cdots & a_{d,d}^{i,j} \end{bmatrix} \begin{pmatrix} x_{j,1} - x_{i,1} \\ x_{j,2} - x_{i,2} \\ \vdots \\ x_{j,d} - x_{i,d} \end{pmatrix}$$

$$\triangleq \dot{\mathbf{x}}_i = \sum_{j \in \mathcal{N}_i}^{n} \mathbf{A}_{i,j}(\mathbf{x}_j - \mathbf{x}_i)$$

- The diagonal terms, i.e., $a_{k,k}^{i,j}$, if it is positive and as it increases, the agreement between $x_{j,k}$ and $x_{i,k}$ speeds up. Otherwise, if it is negative and as it increases in absolute value, the anti-agreement between $x_{j,k}$ and $x_{i,k}$ becomes significant.

- The off-diagonal terms. For example, let us consider the effect of $a_{2,1}^{i,j}$. We can consider the following four cases

Model 3 - Inverse Proportional Couplings

$$\begin{pmatrix} \dot{x}_{i,1} \\ \dot{x}_{i,2} \\ \vdots \\ \dot{x}_{i,d} \end{pmatrix} = \sum_{j \in \mathcal{N}_i}^{n} \begin{bmatrix} a_{1,1}^{i,j} & a_{1,2}^{i,j} & \cdots & a_{1,d}^{i,j} \\ a_{2,1}^{i,j} & a_{2,2}^{i,j} & \cdots & a_{2,d}^{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,1}^{i,j} & a_{d,2}^{i,j} & \cdots & a_{d,d}^{i,j} \end{bmatrix} \begin{pmatrix} x_{j,1} - x_{i,1} \\ x_{j,2} - x_{i,2} \\ \vdots \\ x_{j,d} - x_{i,d} \end{pmatrix}$$

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- The diagonal terms, i.e., $a_{k,k}^{i,j}$, if it is positive and as it increases, the agreement between $x_{j,k}$ and $x_{i,k}$ speeds up. Otherwise, if it is negative and as it increases in absolute value, the anti-agreement between $x_{j,k}$ and $x_{i,k}$ becomes significant.

- The off-diagonal terms. For example, let us consider the effect of $a_{2,1}^{i,j}$. We can consider the following four cases
  1) Case 1: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) \geq 0$

Model 3 - Inverse Proportional Couplings

$$\begin{pmatrix} \dot{x}_{i,1} \\ \dot{x}_{i,2} \\ \vdots \\ \dot{x}_{i,d} \end{pmatrix} = \sum_{j \in \mathcal{N}_i}^{n} \begin{bmatrix} a_{1,1}^{i,j} & a_{1,2}^{i,j} & \cdots & a_{1,d}^{i,j} \\ a_{2,1}^{i,j} & a_{2,2}^{i,j} & \cdots & a_{2,d}^{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,1}^{i,j} & a_{d,2}^{i,j} & \cdots & a_{d,d}^{i,j} \end{bmatrix} \begin{pmatrix} x_{j,1} - x_{i,1} \\ x_{j,2} - x_{i,2} \\ \vdots \\ x_{j,d} - x_{i,d} \end{pmatrix}$$

$$\triangleq \dot{\mathbf{x}}_i = \sum_{j \in \mathcal{N}_i}^{n} \mathbf{A}_{i,j}(\mathbf{x}_j - \mathbf{x}_i)$$

- The diagonal terms, i.e., $a_{k,k}^{i,j}$, if it is positive and as it increases, the agreement between $x_{j,k}$ and $x_{i,k}$ speeds up. Otherwise, if it is negative and as it increases in absolute value, the anti-agreement between $x_{j,k}$ and $x_{i,k}$ becomes significant.

- The off-diagonal terms. For example, let us consider the effect of $a_{2,1}^{i,j}$. We can consider the following four cases
  1) Case 1: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) \geq 0$ → +

Model 3 - Inverse Proportional Couplings

$$\begin{pmatrix} \dot{x}_{i,1} \\ \dot{x}_{i,2} \\ \vdots \\ \dot{x}_{i,d} \end{pmatrix} = \sum_{j \in \mathcal{N}_i}^{n} \begin{bmatrix} a^{i,j}_{1,1} & a^{i,j}_{1,2} & \cdots & a^{i,j}_{1,d} \\ a^{i,j}_{2,1} & a^{i,j}_{2,2} & \cdots & a^{i,j}_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ a^{i,j}_{d,1} & a^{i,j}_{d,2} & \cdots & a^{i,j}_{d,d} \end{bmatrix} \begin{pmatrix} x_{j,1} - x_{i,1} \\ x_{j,2} - x_{i,2} \\ \vdots \\ x_{j,d} - x_{i,d} \end{pmatrix}$$

$$\triangleq \dot{\mathbf{x}}_i = \sum_{j \in \mathcal{N}_i}^{n} \mathbf{A}_{i,j}(\mathbf{x}_j - \mathbf{x}_i)$$

- The diagonal terms, i.e., $a^{i,j}_{k,k}$, if it is positive and as it increases, the agreement between $x_{j,k}$ and $x_{i,k}$ speeds up. Otherwise, if it is negative and as it increases in absolute value, the anti-agreement between $x_{j,k}$ and $x_{i,k}$ becomes significant.

- The off-diagonal terms. For example, let us consider the effect of $a^{i,j}_{2,1}$. We can consider the following four cases
  1) Case 1: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) \geq 0$
  2) Case 2: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) < 0$ → -

Model 3 - Inverse Proportional Couplings

$$\begin{pmatrix} \dot{x}_{i,1} \\ \dot{x}_{i,2} \\ \vdots \\ \dot{x}_{i,d} \end{pmatrix} = \sum_{j \in \mathcal{N}_i}^{n} \begin{bmatrix} a_{1,1}^{i,j} & a_{1,2}^{i,j} & \cdots & a_{1,d}^{i,j} \\ a_{2,1}^{i,j} & a_{2,2}^{i,j} & \cdots & a_{2,d}^{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,1}^{i,j} & a_{d,2}^{i,j} & \cdots & a_{d,d}^{i,j} \end{bmatrix} \begin{pmatrix} x_{j,1} - x_{i,1} \\ x_{j,2} - x_{i,2} \\ \vdots \\ x_{j,d} - x_{i,d} \end{pmatrix}$$

$$\triangleq \dot{\mathbf{x}}_i = \sum_{j \in \mathcal{N}_i}^{n} \mathbf{A}_{i,j}(\mathbf{x}_j - \mathbf{x}_i)$$

- The diagonal terms, i.e., $a_{k,k}^{i,j}$, if it is positive and as it increases, the agreement between $x_{j,k}$ and $x_{i,k}$ speeds up. Otherwise, if it is negative and as it increases in absolute value, the anti-agreement between $x_{j,k}$ and $x_{i,k}$ becomes significant.

- The off-diagonal terms. For example, let us consider the effect of $a_{2,1}^{i,j}$. We can consider the following four cases
  1) Case 1: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) \geq 0$
  2) Case 2: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) < 0$
  3) Case 3: $(x_{j,2} - x_{i,2}) < 0$ and $(x_{j,1} - x_{i,1}) \geq 0$ → -

Model 3 - Inverse Proportional Couplings

$$\begin{pmatrix} \dot{x}_{i,1} \\ \dot{x}_{i,2} \\ \vdots \\ \dot{x}_{i,d} \end{pmatrix} = \sum_{j \in \mathcal{N}_i}^{n} \begin{bmatrix} a_{1,1}^{i,j} & a_{1,2}^{i,j} & \cdots & a_{1,d}^{i,j} \\ a_{2,1}^{i,j} & a_{2,2}^{i,j} & \cdots & a_{2,d}^{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,1}^{i,j} & a_{d,2}^{i,j} & \cdots & a_{d,d}^{i,j} \end{bmatrix} \begin{pmatrix} x_{j,1} - x_{i,1} \\ x_{j,2} - x_{i,2} \\ \vdots \\ x_{j,d} - x_{i,d} \end{pmatrix}$$

$$\triangleq \dot{\mathbf{x}}_i = \sum_{j \in \mathcal{N}_i}^{n} \mathbf{A}_{i,j}(\mathbf{x}_j - \mathbf{x}_i)$$

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  1) Case 1: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) \geq 0$
  2) Case 2: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) < 0$
  3) Case 3: $(x_{j,2} - x_{i,2}) < 0$ and $(x_{j,1} - x_{i,1}) \geq 0$
  4) Case 4: $(x_{j,2} - x_{i,2}) < 0$ and $(x_{j,1} - x_{i,1}) < 0$ → +

Model 3 - Inverse Proportional Couplings

$$\begin{pmatrix} \dot{x}_{i,1} \\ \dot{x}_{i,2} \\ \vdots \\ \dot{x}_{i,d} \end{pmatrix} = \sum_{j \in \mathcal{N}_i}^{n} \begin{bmatrix} a_{1,1}^{i,j} & a_{1,2}^{i,j} & \cdots & a_{1,d}^{i,j} \\ a_{2,1}^{i,j} & a_{2,2}^{i,j} & \cdots & a_{2,d}^{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,1}^{i,j} & a_{d,2}^{i,j} & \cdots & a_{d,d}^{i,j} \end{bmatrix} \begin{pmatrix} x_{j,1} - x_{i,1} \\ x_{j,2} - x_{i,2} \\ \vdots \\ x_{j,d} - x_{i,d} \end{pmatrix}$$

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- The off-diagonal terms. For example, let us consider the effect of $a_{2,1}^{i,j}$. We can consider the following four cases
  1) Case 1: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) \geq 0$ → +
  2) Case 2: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) < 0$ → -
  3) Case 3: $(x_{j,2} - x_{i,2}) < 0$ and $(x_{j,1} - x_{i,1}) \geq 0$ → -
  4) Case 4: $(x_{j,2} - x_{i,2}) < 0$ and $(x_{j,1} - x_{i,1}) < 0$ → +

For consensus!

Model 3 - Inverse Proportional Couplings

$$\begin{pmatrix} \dot{x}_{i,1} \\ \dot{x}_{i,2} \\ \vdots \\ \dot{x}_{i,d} \end{pmatrix} = \sum_{j \in \mathcal{N}_i}^{n} \begin{bmatrix} a_{1,1}^{i,j} & a_{1,2}^{i,j} & \cdots & a_{1,d}^{i,j} \\ a_{2,1}^{i,j} & a_{2,2}^{i,j} & \cdots & a_{2,d}^{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,1}^{i,j} & a_{d,2}^{i,j} & \cdots & a_{d,d}^{i,j} \end{bmatrix} \begin{pmatrix} x_{j,1} - x_{i,1} \\ x_{j,2} - x_{i,2} \\ \vdots \\ x_{j,d} - x_{i,d} \end{pmatrix}$$

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- The diagonal terms, i.e., $a_{k,k}^{i,j}$, if it is positive and as it increases, the agreement between $x_{j,k}$ and $x_{i,k}$ speeds up. Otherwise, if it is negative and as it increases in absolute value, the anti-agreement between $x_{j,k}$ and $x_{i,k}$ becomes significant.

- The off-diagonal terms. For example, let us consider the effect of $a_{2,1}^{i,j}$. We can consider the following four cases
  1) Case 1: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) \geq 0$ → +
  2) Case 2: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) < 0$ → -
  3) Case 3: $(x_{j,2} - x_{i,2}) < 0$ and $(x_{j,1} - x_{i,1}) \geq 0$ → -
  4) Case 4: $(x_{j,2} - x_{i,2}) < 0$ and $(x_{j,1} - x_{i,1}) < 0$ → +

For consensus!

→ *Cooperative opinion dynamics*

Model 3 - Inverse Proportional Couplings

$$\begin{pmatrix} \dot{x}_{i,1} \\ \dot{x}_{i,2} \\ \vdots \\ \dot{x}_{i,d} \end{pmatrix} = \sum_{j \in \mathcal{N}_i}^{n} \begin{bmatrix} a_{1,1}^{i,j} & a_{1,2}^{i,j} & \cdots & a_{1,d}^{i,j} \\ a_{2,1}^{i,j} & a_{2,2}^{i,j} & \cdots & a_{2,d}^{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,1}^{i,j} & a_{d,2}^{i,j} & \cdots & a_{d,d}^{i,j} \end{bmatrix} \begin{pmatrix} x_{j,1} - x_{i,1} \\ x_{j,2} - x_{i,2} \\ \vdots \\ x_{j,d} - x_{i,d} \end{pmatrix}$$

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- The off-diagonal terms. For example, let us consider the effect of $a_{2,1}^{i,j}$. We can consider the following four cases
  1) Case 1: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) \geq 0$ → + → −
  2) Case 2: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) < 0$ → − → +
  3) Case 3: $(x_{j,2} - x_{i,2}) < 0$ and $(x_{j,1} - x_{i,1}) \geq 0$ → − → +
  4) Case 4: $(x_{j,2} - x_{i,2}) < 0$ and $(x_{j,1} - x_{i,1}) < 0$ → + → −

Model 3 - Inverse Proportional Couplings

$$\begin{pmatrix} \dot{x}_{i,1} \\ \dot{x}_{i,2} \\ \vdots \\ \dot{x}_{i,d} \end{pmatrix} = \sum_{j \in \mathcal{N}_i}^{n} \begin{bmatrix} a_{1,1}^{i,j} & a_{1,2}^{i,j} & \cdots & a_{1,d}^{i,j} \\ a_{2,1}^{i,j} & a_{2,2}^{i,j} & \cdots & a_{2,d}^{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,1}^{i,j} & a_{d,2}^{i,j} & \cdots & a_{d,d}^{i,j} \end{bmatrix} \begin{pmatrix} x_{j,1} - x_{i,1} \\ x_{j,2} - x_{i,2} \\ \vdots \\ x_{j,d} - x_{i,d} \end{pmatrix}$$

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- The off-diagonal terms. For example, let us consider the effect of $a_{2,1}^{i,j}$. We can consider the following four cases
  1) Case 1: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) \geq 0$ → + → -
  2) Case 2: $(x_{j,2} - x_{i,2}) \geq 0$ and $(x_{j,1} - x_{i,1}) < 0$ → - → +
  3) Case 3: $(x_{j,2} - x_{i,2}) < 0$ and $(x_{j,1} - x_{i,1}) \geq 0$ → - → +
  4) Case 4: $(x_{j,2} - x_{i,2}) < 0$ and $(x_{j,1} - x_{i,1}) < 0$ → + → -

What happens?

→ *Non-cooperative opinion dynamics*

Model 3 - Inverse Proportional Couplings

Definition 2 (Structurally balanced): [1] A network $\mathcal{G}_k(\mathcal{V}, \mathcal{A}_k = [a^{i,j}_{k,k}], \mathcal{E})$ is said *structurally balanced* on topic k if it admits a bipartition of the nodes $\mathcal{V}_{1,k}, \mathcal{V}_{2,k}, \mathcal{V}_{1,k} \cup \mathcal{V}_{2,k} = \mathcal{V}, \mathcal{V}_{1,k} \cap \mathcal{V}_{2,k} = \emptyset$, such that $s^{i,j}_{k,k} \geq 0, \forall i, j \in \mathcal{V}_{l,k}$ $(l \in \{1, 2\})$, $s^{i,j}_{k,k} \leq 0, i \in \mathcal{V}_{l,k}, j \in \mathcal{V}_{m,k}, m \neq l,$ $(m, l \in \{1, 2\})$. It is said *structurally unbalanced* otherwise.

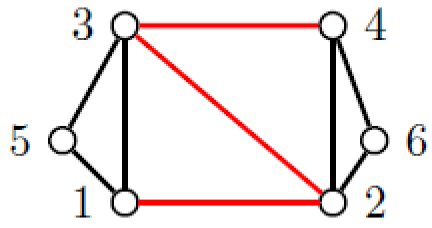

Fig. 1: Structural balanced graph: $\mathcal{V}_1 = \{1, 3, 5\}$ and $\mathcal{V}_2 = \{2, 4, 6\}$; (black) positive links; (red) negative links.

Model 3 - Inverse Proportional Couplings

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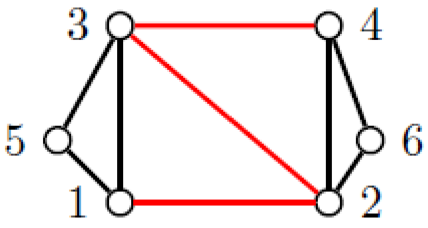
"With slight change of formulation…… "

C. Altafini, "Consensus problem of networks with antagonistic interactions," IEEE Trans. on Automatic Control, vol. 58. no. 4, pp. 935-946, 2013



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Model 3 - Inverse Proportional Couplings

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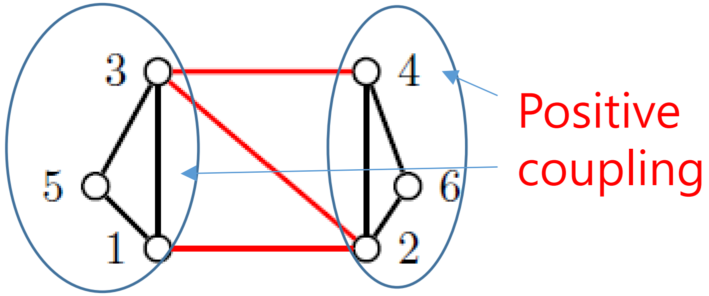

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Model 3 - Inverse Proportional Couplings

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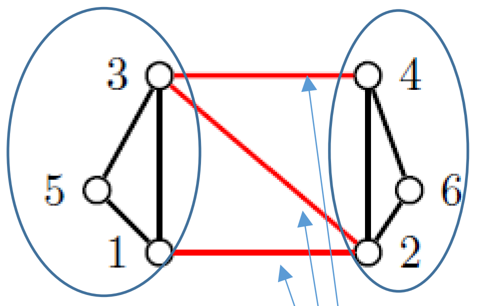

Fig. 1: Structural balanced graph: $\mathcal{V}_1 = \{1, 3, 5\}$ and $\mathcal{V}_2 = \{2, 4, 6\}$; (black) positive links; (red) negative links.

Negative coupling

Model 3 - Inverse Proportional Couplings

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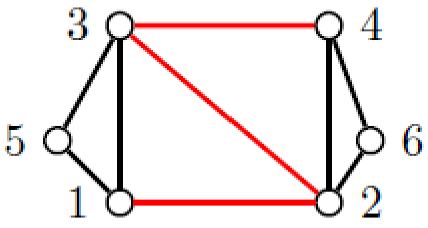

Fig. 1: Structural balanced graph: $\mathcal{V}_1 = \{1,3,5\}$ and $\mathcal{V}_2 = \{2,4,6\}$; (black) positive links; (red) negative links.

Theorem 1: There are some possible situations:

All positive couplings

1) For any topic $p \in \{1, \ldots, d\}$ if $s^{i,j}_{p,p} = 1, \forall (i,j) \in \mathcal{E}$ then it follows from the second term of (12) all agents reach consensus on topic p, i.e., $x_{i,p} = x_{j,p}, \forall i,j \in \mathcal{V}$.

Model 3 - Inverse Proportional Couplings

Definition 2 (Structurally balanced): [1] A network $\mathcal{G}_k(\mathcal{V}, \mathcal{A}_k = [a^{i,j}_{k,k}], \mathcal{E})$ is said *structurally balanced* on topic k if it admits a bipartition of the nodes $\mathcal{V}_{1,k}, \mathcal{V}_{2,k}, \mathcal{V}_{1,k} \cup \mathcal{V}_{2,k} = \mathcal{V}, \mathcal{V}_{1,k} \cap \mathcal{V}_{2,k} = \emptyset$, such that $s^{i,j}_{k,k} \geq 0, \forall i, j \in \mathcal{V}_{l,k}$ $(l \in \{1,2\})$, $s^{i,j}_{k,k} \leq 0, i \in \mathcal{V}_{l,k}, j \in \mathcal{V}_{m,k}, m \neq l,$ $(m, l \in \{1,2\})$. It is said *structurally unbalanced* otherwise.

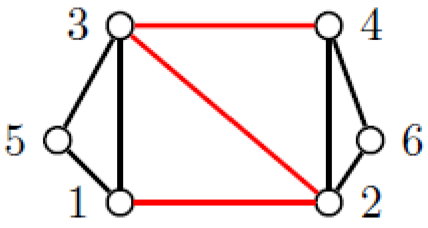

Fig. 1: Structural balanced graph: $\mathcal{V}_1 = \{1,3,5\}$ and $\mathcal{V}_2 = \{2,4,6\}$; (black) positive links; (red) negative links.

Theorem 1: There are some possible situations:

All positive couplings (cooperative dynamics)

1) For any topic $p \in \{1, \ldots, d\}$ if $s^{i,j}_{p,p} = 1, \forall (i,j) \in \mathcal{E}$ then it follows from the second term of (12) all agents reach consensus on topic p, i.e., $x_{i,p} = x_{j,p}, \forall i, j \in \mathcal{V}$.

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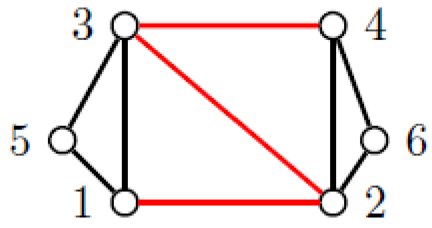

Fig. 1: Structural balanced graph: $\mathcal{V}_1 = \{1, 3, 5\}$ and $\mathcal{V}_2 = \{2, 4, 6\}$; (black) positive links; (red) negative links.

Theorem 1: There are some possible situations:

bipartite

2) $\mathcal{G}_p \triangleq \{\mathcal{V}, \mathcal{A}_p = [a^{i,j}_{p,p}], \mathcal{E}\}$ is structurally balanced then the system reaches *bipartite consensus* on topic p, i.e., agents in $\mathcal{V}_{1,p}$ and $\mathcal{V}_{2,p}$ reach consensus values which are in same absolute value but opposite signs.

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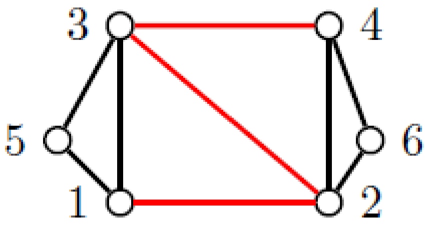


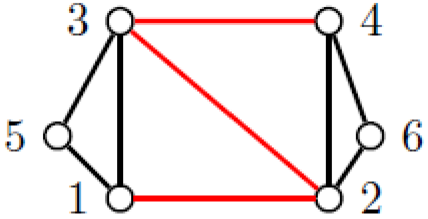
Fig. 1: Structural balanced graph: $\mathcal{V}_1 = \{1, 3, 5\}$ and $\mathcal{V}_2 = \{2, 4, 6\}$; (black) positive links; (red) negative links.

Theorem 1: There are some possible situations:

3) The interaction among agents in a single topic p, i.e., $\mathcal{G}_p = \{\mathcal{V}, \mathcal{A}_p = [a^{i,j}_{p,p}], \mathcal{E}\}$ is *structurally unbalanced*, $x_{i,p} = x_{j,p} = 0, \forall i, j \in \mathcal{V}$.

Model 3 - Inverse Proportional Couplings

Definition 2 (Structurally balanced): [1] A network $\mathcal{G}_k(\mathcal{V}, \mathcal{A}_k = [a_{k,k}^{i,j}], \mathcal{E})$ is said *structurally balanced* on topic k if it admits a bipartition of the nodes $\mathcal{V}_{1,k}, \mathcal{V}_{2,k}, \mathcal{V}_{1,k} \cup \mathcal{V}_{2,k} = \mathcal{V}, \mathcal{V}_{1,k} \cap \mathcal{V}_{2,k} = \emptyset$, such that $s_{k,k}^{i,j} \geq 0, \forall i,j \in \mathcal{V}_{l,k}$ $(l \in \{1,2\})$, $s_{k,k}^{i,j} \leq 0, i \in \mathcal{V}_{l,k}, j \in \mathcal{V}_{m,k}, m \neq l,$ $(m,l \in \{1,2\})$. It is said *structurally unbalanced* otherwise.



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Theorem 1: There are some possible situations:

4) Assuming that there exist anti-consensus on more than two distinct topics, i.e., 1 and 2, and the corresponding graphs $\mathcal{G}_1 = \{\mathcal{V}, [a_{1,1}^{i,j}], \mathcal{E}\}$ and $\mathcal{G}_2 = \{\mathcal{V}, [a_{2,2}^{i,j}], \mathcal{E}\}$ are *structural balanced*. If there exists two agents k and m such that $s_{1,1}^{k,m} < 0$ and $s_{2,2}^{k,m} < 0$, that is, anti-consensus couplings of two members k and m on topics 1 and 2, then either $x_{j,1} = 0$ or $x_{j,2} = 0, \forall j \in \mathcal{V}$.

Main references

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3. Minh Hoang Trinh, Mengbin Ye, Hyo-Sung Ahn, Brian D. O. Anderson, "Matrix-Weighted Consensus With Leader-Following Topologies," *Proc. of the 2017 Asian Control Conference*, Gold Coast Convention Centre, Australia, December 17-20, 2017

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Thank You!

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