Quantum Control Systems

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Overview

- Control systems
- Quantum systems
- Controllability problems

Control Systems

The mathematical structure of a control system is

$$\dot{x} = f(x, u, t), \qquad x|_{t=0} = x_0 \tag{1}$$

$$y = h(x, u, t) \tag{2}$$

where $x \in \mathcal{X}$ is the state, $u \in \mathcal{U}$ is the input, $y \in \mathcal{Y}$ is the output. This is called *state-space representation*, where the state-space $\mathcal{X}$ can be

- $\mathcal{X} \subset \mathbb{R}^n$: n-dimensional systems (classical mechanics)
- $\mathcal{X} \subset \mathrm{GF}(p)$: digital systems
- $\mathcal{X} \subset \mathcal{M}$: manifold (input affine non-linear systems)
- $\mathcal{X} \subset \mathrm{SU}(n)$: systems over Lie-groups

Quantum Systems

Each physical (quantum) system is associated with a (topologically) *separable complex Hilbert-space* H with inner product $\langle\psi|\phi\rangle$.

Physical observables are represented by densely-defined *self-adjoint operators* on H. The expected value of the observable A for the system in state represented by the unit vector $|\psi\rangle \in H$ is $\langle\psi|A|\psi\rangle$.

The states a qubit may be measured in are known as basis states (or vectors). As is the tradition with any sort of quantum states, *Dirac* (or bra-ket) *notation* is used to represent them. This means that the two computational basis states are conventionally written as $|0\rangle$ and $|1\rangle$.

Mathematical Representation

Classical mechanics: Hamilton-equation

Mathematical theory of dynamical systems:

- *State-space representation* (R. E. KALMAN)
- *Operator representation*: the system is represented as a linear operator, that maps the space of input functions onto the space of output functions
- *Statistical representation*: the theory of Hilbert-spaces

Basic concepts:

- controllability / (non)observability

Generalization: dynamical systems defined over Lie-groups

Qubit States

A pure qubit state is a linear superposition of those two states. This means that the qubit can be represented as a *linear combination* of $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are *probability amplitudes* and can in general both be complex numbers.
When we measure this qubit in the standard basis, the probability of outcome $|0\rangle$ is $|\alpha|^2$ and the probability that the outcome is $|1\rangle$ is $|\beta|^2$. Because one must measure *either* one state *or* the other, it follows that

$$|\alpha|^2 + |\beta|^2 = 1.$$

Qubit States

The state at any time t is given by:

$$\psi(t) = X(t)\psi(0),$$

where X is the so-called *evolution operator* (matrix), solution of the Schrödinger-equation

$$i\hbar\dot{X} = HX.$$

In quantum computers, the evolutionary operator X represents a (logic) operation to be performed on a quantum bit, i.e. the *reachability* question means: „can all operations be achieved on a quantum bit by opportunely shaping an input electro/magnetic field?" (Using finite energy.)

Controllability Problems

- Open-loop unconstrained controllability
- Switching system's controllability
- Constrained controllability

Open-loop Unconstrained Controllability

Reachability of Qubit States

Let $H = H_0 + \sum_{i=1}^{m} H_i u_i$, where H_i, $i = 0, 1, \ldots$ are Hermitian operators, then the Schrödinger-equation can be written as

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{m} B_i X(t) u_i(t) \tag{3}$$

where A, B_i are elements of the Lie algebra of 2×2 skew-Hermitian matrices with zero trace, which is denoted by su(2).

The solution of (3) with initial condition equals to identity varies in the Lie group associated to su(2), namely in the Lie group of 2×2 unitary matrices with determinant 1. This group is called the group of *special unitary matrices* and is denoted by SU(2).

Reachability of Qubit States

Definition. The set of *reachable states* $R(T)$ consists of all the possible values for $X(T)$ (solution of (3) at time T with initial condition equal to identity) obtained varying the controls $u_1, \ldots, u_m$ in the set of all the piecewise continuous functions defined in $[0, T]$.

Theorem. *Consider system (3) with* $3 \geq m \geq 2$ *and assume that* $B_1, \ldots, B_m$ *are linearly independent. Then, for any time* $T > 0$ *and for any desired final state* X_f *there exist a set of piecewise continuous control functions* $u_1, \ldots, u_m$ *driving the state of the system* X *to* $X(T) = X_f$ *at time* T*. This means that in this case* $R(T) = SU(2)$ *for every* $T > 0$.

Controllability of LTI Systems

The fundamental matrix for zero initial time is

$$\Phi(t) = e^{At} = \sum_{i=1}^{n} \psi_i(t) A^{i-1},$$

and the reachability subspace is

$$\mathcal{R} = \sum_{k=0}^{n-1} \operatorname{Im} A^k B.$$

Proposition. *It is possible to generate linearly independent functions* ψ_i, $i = 1, \ldots, n$ *if the Kalman-rank condition* $\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n$ *is satisfied.*

Controllability of Qubit States

Write the fundamental matrix (locally) as exponential function of the „coordinates of second kind" associated with the equation

$$\dot{x} = \sum_{i=0}^{N} \rho_i(t) A_i x.$$

Using the Wei–Norman equation:

$$\dot{g}(t) = \left(\sum_{i=1}^{K} e^{\Gamma_1 g_1} \cdots e^{\Gamma_{i-1} g_{i-1}} E_{ii}\right)^{-1} \rho(t), \qquad g(0) = 0,$$

where $\{\hat{A}_1, \ldots, \hat{A}_K\}$ is a basis of the Lie-algebra $\mathcal{L}(A_1, \ldots, A_N)$,

$$[\hat{A}_i, \hat{A}_j] = \sum_{l=1}^{K} \Gamma_{i,j}^{l} \hat{A}_l, \quad \Gamma_i = [\Gamma_{i,j}^{l}]_{j,l=1}^{K}.$$

Controllability of Qubit States

Generalized Kalman-rank condition. *For systems* $A(\rho), B(\rho)$ *the points attainable from the origin are those from the subspace spanned by the vectors*

$$\mathcal{R}_{(\mathcal{A},\mathcal{B})} := span \left\{ \prod_{j=1}^{K} A_{l_j}^{i_j} B_k \right\},$$

where $K \geq 0, l_j, k \in \{0, \cdots, N\}, i_j \in \{0, \cdots, n-1\}$, *i.e.*

$$\mathcal{R} \subset \mathcal{R}_{(\mathcal{A},\mathcal{B})}.$$

Controllability of Qubit States

Denote by $\mathcal{L}(A_0, \ldots, A_N)$ the finitely generated Lie-algebra containing the matrices $A_0, \ldots, A_N$, and let $\hat{A}_1, \ldots, \hat{A}_K$ be a basis of this algebra, then the points attainable from the origin are in the subspace

$$\mathcal{R}_{(\mathcal{A},\mathcal{B})} = \sum_{l=0}^{N} \sum_{n_1=0}^{n-1} \cdots \sum_{n_K=0}^{n-1} \mathrm{Im}\,(\hat{A}_1^{n_1} \cdots \hat{A}_K^{n_K} B_l).$$

The question is that under what condition is $\mathcal{R} = \mathcal{R}_{\mathcal{A},\mathcal{B}}$?

Controllability of Qubit States

The fundamental matrix can be written in exponential form:

$$\Phi(t) = \sum_{n_1=0}^{n-1} \cdots \sum_{n_K=0}^{n-1} \hat{A}_1^{n_1} \cdots \hat{A}_K^{n_K} \psi_{n_1,\dots,n_K}(t)$$

$$\Phi(t) = \sum_{\mathbf{j}\in\mathbf{J}} \hat{A}_{\mathbf{j}} \varphi_{\mathbf{j}}(t), \quad \hat{A}_{\mathbf{j}} := \hat{A}_1^{j_1} \cdots \hat{A}_K^{j_K}.$$

The subspace $\mathcal{R}_{\mathcal{A},\mathcal{B}}$ is the image space of the matrix $R_{\mathcal{A},\mathcal{B}} := [\,\hat{A}_{\mathbf{j}} B\,]_{\mathbf{j}\in\mathbf{J}}$. The controllability Grammian is given as

$$W(\sigma,\tau) = R_{\mathcal{A},\mathcal{B}} \left(\int_\sigma^\tau [\,\varphi_{\mathbf{j}}(s)\,]_{\mathbf{j}\in\mathbf{J}} [\,\varphi_{\mathbf{j}}(s)\,]_{\mathbf{j}\in\mathbf{J}}^* ds \right) R_{\mathcal{A},\mathcal{B}}^*.$$

Controllability of Qubit States

Theorem. *The quantum system is controllable, iff*

(i) *The generalized Kalman-rank condition is satisfied:*

$$\operatorname{rank} R_{\mathcal{A},\mathcal{B}} = \operatorname{rank}\, [\, \hat{A}_{\mathbf{j}} B \,]_{\mathbf{j} \in \mathbf{J}}$$

(ii) *The set of functions* $\{\varphi_{\mathbf{j}}(\sigma) \,|\, \mathbf{j} \in \mathbf{J}\}$ *contains* n *linearly independent functions.*

Switching System's Controllability

Hybrid Systems

Hybrid models characterize systems governed by continuous differential and difference equations and discrete variables. Such systems are described by several operating regimes (modes) and the transition from one mode to another is governed by the evolution of internal or external variables or events.

Depending on the nature of the events there are two big classes of hybrid systems that are considered in the control literature: *switching systems* and *impulsive systems*.

Switching Systems

A switching system is composed of a family of different (smooth) *dynamic modes* such that the switching pattern gives continuous, piecewise smooth trajectories. Moreover, it is assumed that one and only one mode is active at each time instant.

In a broader sense every time-varying system with measurable variations in time can be cast as a switching system, therefore it is usually assumed that the number of switching modes is finite and for practical reasons the possible switching functions (sequences) are restricted to be piecewise constant, i.e. only a finite number of transition is allowed on a finite interval.

Switching Systems

Formally, these systems can be described as:

$$\dot{x}(t) = f_{\sigma(t)}(x(t), u(t)),$$

$$y(t) = h_{\sigma(t)}(x(t), u(t)), \quad x(\tau^+) = \iota(x(\tau^-), u(\tau), \tau),$$

where $x \in^n$ is the state variable, $u \in \Omega \subset^m$ is the input variable and $y \in^p$ is the output variable.
The $\sigma :^+ \rightarrow S$ is a measurable switching function mapping the positive real line into $S = \{1, \cdots, s\}$. The impulsive effect can be described by the relation $(\tau, x(\tau^-)) \in \mathcal{I} \times \mathcal{A}$ with $\mathcal{I}$ a set of time instances and $\mathcal{A} \in^n$ a certain region of the state space.

Universality of Quantum Gates

Consider a bimodal system

$$\dot{X} = A_{\sigma(t)}X, \quad X(0) = I, \quad \sigma(t) : \mathbb{R}^+ \mapsto \{1, 2\}$$

and $A_1, A_2 \in \mathcal{U}(n)$ that is SU(n).
A set of gates is called *universal* if – by switching $\{A_1, A_2\}$ – it is possible to generate all (special) unitary evolutions.

Since A_1, A_2 generate the whole Lie-algebra $u(n)$ or $su(n)$, therefore almost every couple of skew-Hermitian matrices generate $u(n)$, i.e. almost every quantum gate is universal.

Constrained Controllability – Many Open Questions

Summary

Let $\Gamma = A + \mathbb{R}B \subset \mathcal{L}$, $\mathcal{L}$ is the Lie-algebra associated to G, i.e.

$$\dot{X} = AX + uBX, \quad X \in G, \quad u \in \mathbb{R}, \quad X(0) = I.$$

- Unconstrained controllability: $\mathrm{Lie}(\Gamma) = \mathcal{L}$. Apply the Lie-algebraic rank condition.
- Constrained controllability: $u \in \mathcal{U} \subset \mathbb{R}^+$. If $\mathrm{Lie}(\Gamma) = \mathrm{Lie}(-\Gamma)$ or $X \in \{\text{compact Lie-group}\} \Rightarrow LS\,\Gamma = \mathcal{L}$. Apply the Lie-algebraic rank condition.

Summary

Facts:

- $G = SO(3)$ then $\dot{X} = (A_1 + uA_2)X$, $u \in \mathcal{U} \subset \mathbb{R}$ is controllable for any $\mathcal{U}$ containing more than one element.
- $G = SO(3)$, controllability $\Leftrightarrow \mathcal{U}$ contains at least two distinct points.

Thank you for attention!