Possibility distributions: a normative view

Christer Carlsson
Institute for Advanced Management Systems Research,
Åbo Akademi University,
Lemminkäinengatan 14C, FIN-20520 Åbo, Finland

Robert Fullér *
Department of Operations Research,
Eötvös Loránd University,
Pázmány Péter sétány 1C, H-1147 Budapest, Hungary
e-mail: rfuller@mail.abo.fi

Péter Majlender
IAMSR/TUCS, Åbo Akademi University,
Lemminkäinengatan 14C, FIN-20520 Åbo, Finland

Abstract

In this paper we will summarize some normative properties of possibility distributions.

1 Probability and Possibility

In 2001 Carlsson and Fullér [1] introduced the possibilistic mean value, variance and covariance of fuzzy numbers. In 2003 Fullér and Majlender [4] introduced the notations of crisp weighted possibilistic mean value, variance and covariance of fuzzy numbers, which are consistent with the extension principle. In 2003 Carlsson, Fullér and Majlender [2] proved the possibilistic Cauchy-Schwartz inequality. Drawing heavily on [1, 2, 4, 5] we will summarize some normative properties of possibility distributions.

In probability theory, the dependency between two random variables can be characterized through their joint probability density function. Namely, if X

*Partially supported by OTKA T32412. Also at IAMSR.

and Y are two random variables with probability density functions $f_X(x)$ and $f_Y(y)$, respectively, then the density function, $f_{X,Y}(x,y)$, of their joint random variable (X,Y), should satisfy the following properties

$$\int_{\mathbb{R}} f_{X,Y}(x,t)dt = f_X(x), \quad \int_{\mathbb{R}} f_{X,Y}(t,y)dt = f_Y(y), \tag{1}$$

for all $x, y \in \mathbb{R}$. Furthermore, $f_X(x)$ and $f_Y(y)$ are called the the marginal probability density functions of random variable (X,Y). X and Y are said to be independent if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y),$$

holds for all x, y. The expected value of random variable X is defined as

$$E(X) = \int_{\mathbb{R}} x f_X(x)dx,$$

and if g is a function of X then the expected value of $g(X)$ can be computed as

$$E(g(X)) = \int_{\mathbb{R}} g(x)f_X(x)dx.$$

Furthermore, if h is a function of X and Y then the expected value of $h(X,Y)$ can be computed as

$$E(h(X,Y)) = \int_{\mathbb{R}^2} h(x,y)f_{X,Y}(x,y)dxdy.$$

Especially,

$$E(X+Y) = E(X) + E(Y),$$

that is, the the expected value of X and Y can be determined according to their individual density functions (that are the marginal probability functions of random variable (X,Y)). The key issue here is that the joint probability distribution vanishes (even if X and Y are not independent), because of the principle of 'falling integrals' (1).

Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a \leq b$, then the probability that X takes its value from $[a,b]$ is computed by

$$\mathrm{P}(X \in [a,b]) = \int_a^b f_X(x)dx.$$

The covariance between two random variables X and Y is defined as

$$\mathrm{Cov}(X,Y) = E\big((X - E(X))(Y - E(Y))\big) = E(XY) - E(X)E(Y),$$

and if X and Y are independent then $\mathrm{Cov}(X,Y) = 0$, since $E(XY) = E(X)E(Y)$. The covariance operator is a symmetrical bilinear operator and it is easy to see that $\mathrm{Cov}(\lambda, X) = 0$ for any $\lambda \in \mathbb{R}$.

The variance of random variable X is defined as the covariance between X and itself, that is

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \int_{\mathbb{R}} x^2 f_X(x)dx - \left(\int_{\mathbb{R}} x f_X(x)dx\right)^2.$$

For any random variables X and Y and real numbers $\lambda, \mu \in \mathbb{R}$ the following relationship holds

$$\mathrm{Var}(\lambda X + \mu Y) = \lambda^2 \mathrm{Var}(X) + \mu^2 \mathrm{Var}(Y) + 2\lambda\mu \mathrm{Cov}(X,Y).$$

A fuzzy set A in $\mathbb{R}$ is said to be a fuzzy number if it is normal, fuzzy convex and has an upper semi-continuous membership function of bounded support. The family of all fuzzy numbers will be denoted by $\mathcal{F}$. A γ-level set of a fuzzy set A in $\mathbb{R}^m$ is defined by $[A]^\gamma = \{x \in \mathbb{R}^m : A(x) \geq \gamma\}$ if $\gamma > 0$ and $[A]^\gamma = \mathrm{cl}\{x \in \mathbb{R}^m : A(x) > \gamma\}$ (the closure of the support of A) if $\gamma = 0$. If $A \in \mathcal{F}$ is a fuzzy number then $[A]^\gamma$ is a convex and compact subset of $\mathbb{R}$ for all $\gamma \in [0,1]$.

Fuzzy numbers can be considered as possibility distributions. Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a \leq b$, then the possibility that $A \in \mathcal{F}$ takes its value from $[a,b]$ is defined by [7]

$$\mathrm{Pos}(A \in [a,b]) = \max_{x \in [a,b]} A(x).$$

A fuzzy set B in $\mathbb{R}^m$ is said to be a joint possibility distribution of fuzzy numbers $A_i \in \mathcal{F}$, $i = 1, \dots, m$, if it satisfies the relationship

$$\max_{x_j \in \mathbb{R},\, j \neq i} B(x_1, \dots, x_m) = A_i(x_i),\ \forall x_i \in \mathbb{R}, i = 1, \dots, m.$$

Furthermore, A_i is called the i-th marginal possibility distribution of B, and the projection of B on the i-th axis is A_i for $i = 1, \dots, m$.

Let B denote a joint possibility distribution of $A_1, A_2 \in \mathcal{F}$. Then B should satisfy the relationships

$$\max_y B(x_1, y) = A_1(x_1), \quad \max_y B(y, x_2) = A_2(x_2),\ \forall x_1, x_2 \in \mathbb{R}.$$

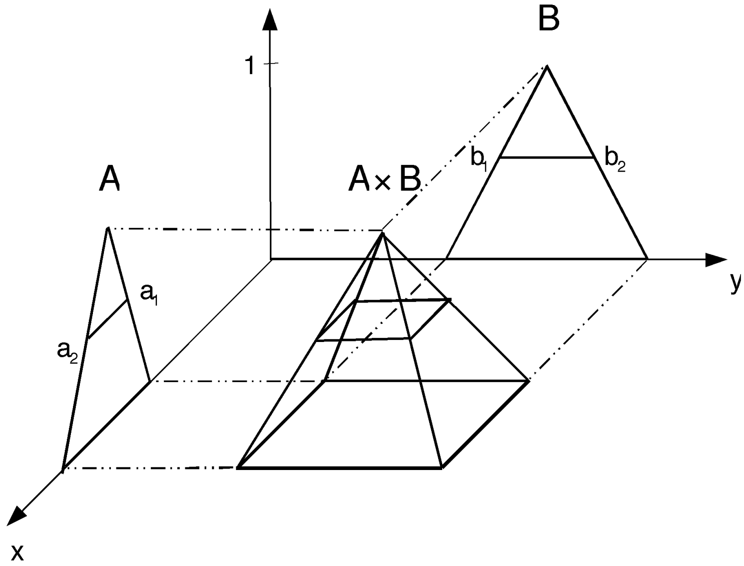

Figure 1: Independent possibility distributions.

If $A_i \in \mathcal{F}$, $i = 1, \ldots, m$, and B is their joint possibility distribution then the relationships $B(x_1, \ldots, x_m) \leq \min\{A_1(x_1), \ldots, A_m(x_m)\}$ and $[B]^\gamma \subseteq [A_1]^\gamma \times \cdots \times [A_m]^\gamma$, hold for all $x_1, \ldots, x_m \in \mathbb{R}$ and $\gamma \in [0, 1]$.

In the following the biggest (in the sense of subsethood of fuzzy sets) joint possibility distribution will play a special role among joint possibility distributions: it defines the concept of independence of fuzzy numbers.

Definition 1.1. *Fuzzy numbers $A_i \in \mathcal{F}$, $i = 1, \ldots, m$, are said to be independent if their joint possibility distribution, B, is given by*

$$B(x_1, \ldots, x_m) = \min\{A_1(x_1), \ldots, A_m(x_m)\},$$

or, equivalently, $[B]^\gamma = [A_1]^\gamma \times \cdots \times [A_m]^\gamma$, for all $x_1, \ldots, x_m \in \mathbb{R}$ and $\gamma \in [0, 1]$.

Marginal probability distributions are determined from the joint one by the principle of 'falling integrals' and marginal possibility distributions are determined from the joint possibility distribution by the principle of 'falling shadows'.

Let $A \in \mathcal{F}$ be fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$. A function $f\colon [0, 1] \to \mathbb{R}$ is said to be a weighting function [4] if f is non-negative, monotone increasing and satisfies the following normalization condition

$$\int_0^1 f(\gamma) d\gamma = 1. \tag{2}$$

2 Possibilistic expected value, variance, covariance

Definition 2.1. *[5] Let $A \in \mathcal{F}$ be a fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$. The central value of $[A]^\gamma$ is defined by*

$$\mathcal{C}([A]^\gamma) = \frac{1}{\int_{[A]^\gamma} dx} \int_{[A]^\gamma} x dx.$$

It is easy to see that the central value of $[A]^\gamma$ is computed as

$$\mathcal{C}([A]^\gamma) = \frac{1}{a_2(\gamma) - a_1(\gamma)} \int_{a_1(\gamma)}^{a_2(\gamma)} x dx = \frac{a_1(\gamma) + a_2(\gamma)}{2}.$$

Definition 2.2. *Let $A_1, \dots, A_n \in \mathcal{F}$ be fuzzy numbers, and let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then, $g(A_1, \dots, A_n)$ is defined by the sup–min extension principle [6] as follows*

$$g(A_1, \dots, A_n)(y) = \sup_{g(x_1, \dots, x_n) = y} \min\{A_1(x_1), \dots, A_n(x_n)\}.$$

Definition 2.3. *[5] Let $A_1, \dots, A_n \in \mathcal{F}$ be fuzzy numbers, let B be their joint possibility distribution and let $\gamma \in [0, 1]$. The central value of the γ-level set of $g(A_1, \dots, A_n)$ with respect to their joint possibility distribution B is defined by*

$$\mathcal{C}_B\left([g(A_1, \dots, A_n)]^\gamma\right) = \frac{1}{\int_{[B]^\gamma} dx} \int_{[B]^\gamma} g(x) dx,$$

where $g(x) = g(x_1, \dots, x_n)$.

Definition 2.4. *[5] Let $A, B \in \mathcal{F}$ be fuzzy numbers, let C be their joint possibility distribution, and let $\gamma \in [0, 1]$. The dependency relation between the γ-level sets of A and B is defined by*

$$\mathrm{Rel}_C([A]^\gamma, [B]^\gamma) = \mathcal{C}_C\Big(\big[(A - \mathcal{C}_C([A]^\gamma))(B - \mathcal{C}_C([B]^\gamma))\big]^\gamma\Big),$$

which can be written in the form,

$$\mathrm{Rel}_C([A]^\gamma, [B]^\gamma) =$$

$$\frac{1}{\int_{[C]^\gamma} dxdy} \int_{[C]^\gamma} xydxdy - \frac{1}{\int_{[C]^\gamma} dx} \int_{[C]^\gamma} xdx \times \frac{1}{\int_{[C]^\gamma} dy} \int_{[C]^\gamma} ydy.$$

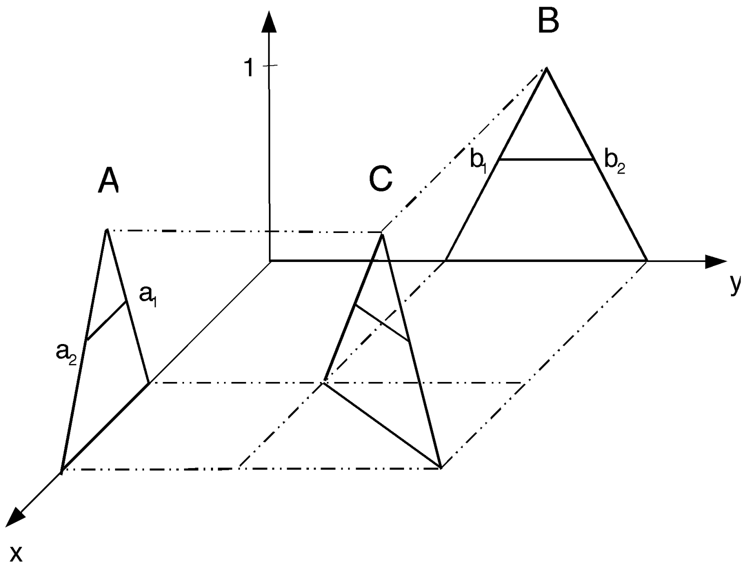

Figure 2: The case of $\rho_f(A, B) = 1$.

The covariance of A and B with respect to a weighting function f is defined as [5]

$$\begin{aligned}\mathrm{Cov}_f(A, B) &= \int_0^1 \mathrm{Rel}_C([A]^\gamma, [B]^\gamma) f(\gamma) d\gamma \\ &= \int_0^1 \big[\mathcal{C}_C([AB]^\gamma) - \mathcal{C}_C([A]^\gamma) \cdot \mathcal{C}_C([B]^\gamma)\big] f(\gamma) d\gamma.\end{aligned}$$

In [5] we proved that if $A, B \in \mathcal{F}$ are independent then $\mathrm{Cov}_f(A, B) = 0$. The variance of a fuzzy number A is defined as [5]

$$\mathrm{Var}_f(A) = \mathrm{Cov}_f(A, A) = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma.$$

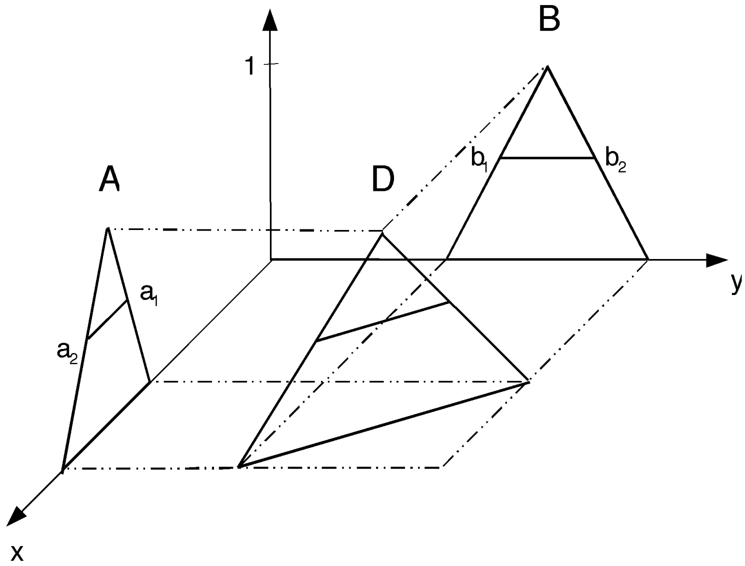

Figure 3: The case of $\rho_f(A,B) = -1$.

In [5] we proved that the that the 'principle of central values' leads us to the same relationships in possibilistic environment as in probabilitic one. It is why we can claim that the principle of 'central values' should play an important role in defining possibilistic dependencies.

Theorem 2.1. *[5] Let A, B and C be fuzzy numbers, and let $\lambda, \mu \in \mathbb{R}$. Then*

$$\mathrm{Cov}_f(\lambda A + \mu B, C) = \lambda \mathrm{Cov}_f(A, C) + \mu \mathrm{Cov}_f(B, C),$$

and

$$\mathrm{Var}_f(\lambda A + \mu B) = \lambda^2 \mathrm{Var}_f(A) + \mu^2 \mathrm{Var}_f(B) + 2\lambda\mu \mathrm{Cov}_f(A, B),$$

where all terms in this equation are defined through joint possibility distributions.

Furthermore, in [2] we have shown the following theorem.

Theorem 2.2. *Let $A, B \in \mathcal{F}$ be fuzzy numbers (with $\mathrm{Var}_f(A) \neq 0$ and $\mathrm{Var}_f(B) \neq 0$) with joint possibility distribution C. Then, the correlation* *coefficient between A and B, defined by*

$$\rho_f(A,B) = \frac{\mathrm{Cov}_f(A,B)}{\sqrt{\mathrm{Var}_f(A)\mathrm{Var}_f(B)}}.$$

satisfies the property

$$-1 \leq \rho_f(A,B) \leq 1.$$

for any weighting function f.

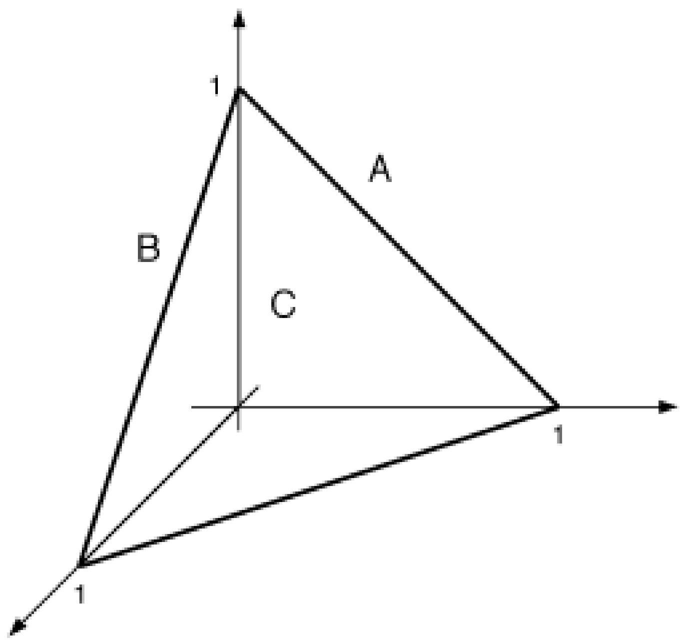

Figure 4: The case of $\rho_f(A,B) = 1/3$.

Let us consider three interesting cases. In [4] we proved that if A and B are independent, that is, their joint possibility distribution is $A \times B$ then $\rho_f(A,B) = 0$. Consider now the case depicted in Fig. 2. It can be shown [2] that in this case $\rho_f(A,B) = 1$. Consider now the case depicted in Fig. 3. It can be shown [2] that in this case $\rho_f(A,B) = -1$. Consider now the case depicted in Fig. 4. It can be shown that in this case $\rho_f(A,B) = 1/3$.

3 Summary

We have illustrated that by choosing appropriate operators we can establish probability-like theorems in possibilistic environment.

References

[1] C. Carlsson, R. Fullér, On possibilistic mean value and variance of fuzzy numbers, *Fuzzy Sets and Systems*, 122(2001) 315-326.

[2] C. Carlsson, R. Fullér and P. Majlender, On possibilistic Cauchy-Schwarz inequality *Fuzzy Sets and Systems*, (submitted).

[3] D. Dubois, H. Prade, The mean value of a fuzzy number, *Fuzzy Sets and Systems*, 24(1987) 279-300.

[4] R. Fullér and P. Majlender, On weighted possibilistic mean and variance of fuzzy numbers, *Fuzzy Sets and Systems*, (to appear).

[5] R. Fullér and P. Majlender, On possibilistic dependencies, *Fuzzy Sets and Systems*, (submitted).

[6] L. A. Zadeh, Fuzzy Sets, *Information and Control*, 8(1965) 338-353.

[7] L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, 1(1978) 3-28.