New Approach in Computational Cybernetics for Intelligent Adaptive Control of Non-linear Systems

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Abstract: In this paper possible extensions of the application of a new branch of Computational Cybernetics is considered. Originally the method was developed for the adaptive control of various, incompletely and inaccurately modeled mechanical systems described by the Euler-Lagrange equations. The systems investigated had only the "standardized" continuous non-linearities as the terms quadratic in the joint coordinate velocities and the dependence of the inertia matrix on these coordinates. Significant and practically occurring discontinuous non-linearities as non-viscous friction and backlash were missing in the early investigations. On the basis of simulation results it is shown that the method can work even under such conditions, too. Another potential field of application is the realm of chemical reactions. As a particular paradigm thermal decay of water is considered. The main pillars of the approach as the "Modified Renormalization Transformation" and the idea of "Complete Stability" also are discussed in the paper. For demonstration purposes simulation examples are presented for the control of a 3 DOF SCARA arm under external dynamic interactions, Stribeck-type friction and backlash unmodeled by the controller. In the chemical reaction example water is put in a high temperature thermal reservoir and the task is to provide a time-dependent nominal mole fraction of molecular hydrogen via changing the pressure of the system. It is concluded that the method is promising for a wide class of possible applications.

Keywords: Adaptive Control; Non-linear Systems; Modified Renormalization Transformation; Lie Groups; Complete Stability;

1 Introduction

In our days ‘Soft Computing -SC’ means a kind of integration of neural networks and fuzzy systems enhanced with high parallelism of operation and supported by several deterministic, stochastic or combined parameter-tuning methods (learning). SC can evade the development of complex analytical system models via identifying typical problem classes for the solution of which typical uniform architectures (e.g. multilayer perceptrons, Kohonen-networks, Hopfield-networks, Cellular Neural Networks, CNN Universal Machine, etc.) can be applied. Identification of the propr problem class and finding the appropriate stucture for dealing with it normally is easy. However, determining the necessary size of the structure and fitting its parameters via machine learning is far less easy. For neural networks certain solutions starts from a large initial network and apply dynamic pruning for getting rid of the "dead" nodes [1]. An alternative method starts with small network, and the number of nodes is increased step by step (e.g. [2-3]). Due to the possible existence of "local optima", for a pure "backpropagation training" inadequacy of a given number of neurons cannot be concluded simply. To evade this difficulty "learning methods", also including stochastic elements, were seruiously improved in the last decade (e.g. [4-7]). Like neural networks, fuzzy systems also use membership functions of typical (e.g. trapezoidal, triangular or step-like, etc.) shapes, and the fuzzy relations can also be utilized in a standardized way by using different, even parametric classes of fuzzy operators. This field till has very wide scale of possibilities to be utilized. For instance, in [8] a quite interesting generalization of the conjunction and disjunction operations are considered for the purposes of fuzzy control. Another important sub-field of fuzzy control in this line is function approximation based on the tuning of generalized connectives in [9]. Typical fields in which intelligent control is needed is the realm of Thermodynamics, especially Chemistry. Normally it is very difficult to obtain acceptably precise model of chemical reactions on analytical basis. Physical systems in chemical reaction are multiple input-multiple output (MIMO) systems with strong non-linear coupling the behavior of which can be “observed” and approximately described by fuzzy rules. In [10] e.g. the problem of identification of such processes is considered by the use of the technique of fuzzy clustering and Gaussian membership functions. The most significant problem in using fuzzy systems is that the number of the necessary fuzzy rules strongly increases with the degree of freedom and the intricacy of the problem. The same is true for the necessary neurons in a neural network approach. External dynamic interactions on which normally no satisfactory information is available influences the system's behavior in dynamic, time-dependent manner. Both the big size of the necessary structures, the huge number of parameters to be tuned, as well as the "goal" varying in time still mean serious problem.

While developing a “novel branch of soft computing” its inventors risked the supposition that "generality" and "uniformity" of the traditional SC structures

exclude the application of plausible simplifications which may be characteristic to a whole set of typical tasks. So consideration of narrower problem classes was initiated to take into account the possible simplifications in the uniform structures. The first steps in this direction were made in connection with Classical Mechanical Systems (CMSs) [11,12] to utilize the properties of Symplectic Geometry being an inherent internal symmetry of Hamiltonian Mechanics. The main point was to apply consecutive canonical transformations to map the 'observed' state drift to the rough model based 'expected' one. In connection with that the idea of the "Situation Dependent Partial System Identification" meant that all the effects of the model inaccuracy and the external dynamic interactions not modeled by the controller were taken into account in the properly defined Jacobians of these canonical tarnsformations. Later the problem was considered from a purely mathematical, formal point of view. It became clear that all the essential steps used in the control can be realized by other mathematical means than symplectic matrices. Symplectic Group can be replaced by other Lie groups defined in a similar manner via some "basic quadratic expression" [13]. In this approach the Lie group used in the control does not describe any internal physical symmetry of the system to be controlled. Regarding the convergence of the method satisfactory conditions were presented for SISO systems. A perturbation calculation based proof of satisfactory conditions of complete stability in the case of MIMO systems was given in [15]. In the sequel first the control problem will be considered in details from a general point of view.

2 The control algorithm

From purely mathematical point of view the control problem can be formulated as follows: there is given some *imperfect model of the system* on the basis of which some *excitation* is calculated for a desired input $\mathbf{i}^d$ as $\mathbf{e}=\varphi(\mathbf{i}^d)$. The system has its *inverse dynamics* described by the *unknown function* $\mathbf{i}^r=\psi(\varphi(\mathbf{i}^d))=f(\mathbf{i}^d)$ and resulting in a realized $\mathbf{i}^r$ instead of the desired one, $\mathbf{i}^d$. For the case of exact system model and available full information on the external dynamic interactions a fixed point of $f(\mathbf{i}^d)$ could be achieved. Normally we can obtain information via observation only on the "net" function $f()$ considerably varying in time. Furthermore, no any possibility exists for "manipulating" the nature of this function directly: we can manipulate or *deform* its actual input $\mathbf{i}^{d*}$ in comparison with the *desired one* in general. The aim is to achieve and maintain the $\mathbf{i}^d=f(\mathbf{i}^{d*})$ state. [We can directly manipulate only the nature of the *model function* $\varphi()$.] The Modified Renormalization Algorithm consists in seeking a series of linear transformations as

$$\begin{gathered}\mathbf{i}_0;\mathbf{S}_1\mathbf{f}(\mathbf{i}_0)=\mathbf{i}_0;\mathbf{i}_1=\mathbf{S}_1\mathbf{i}_0;\ldots;\mathbf{S}_n\mathbf{f}(\mathbf{i}_{n-1})=\mathbf{i}_0;\\ \mathbf{i}_{n+1}=\mathbf{S}_{n+1}\mathbf{i}_n;\mathbf{S}_n\xrightarrow[n\to\infty]{}\mathbf{I}\end{gathered} \tag{1}$$

(As is well known the original 'Renormalization Transformation' that can change the fixed point of a mapping was widely used in physics for describing the main properties of chaos e.g. by Feigenbaum [16-18].) It is evident that (1) does not unambiguously determine the possible quadratic $\mathbf{S}_n$ matrices. This ambiguity can be utilized via choosing "simple" linear transformations possibly having the properties as fast and computationally efficient invertibility and the ability to be as close to the identity transformation as possible. The proof of complete stability of such transformations was given in [15] independently of the particular algebraic solution chosen. In the sequel for MIMO systems the "Partially Stretched Orthogonal Transformations" as a computationally exceptionally efficient transformation introduced in [14] are used.

3 Simulation results for the mechanical system

At first the motion of an 3DOF SCARA arm is considered along a typical trajectory that can be used for assembly purposes. It is a straight line with respect to the workshop system of reference corresponding to the translation of the workpiece from the initial point to the point of insertion. Following that the workpiece must be pushed into the product under assembly along a vertical line. To obtain minimum time of operation, with respect to the Cartesian workshop frame the movement happens with constant (that is maximal) acceleration and deceleration. A simple “rod model” was chosen for describing the dynamics of the SCARA robot. The controller used an erroneous, rough approximation of this model. To represent external dynamic interactions the end-point of the arm was fixed to a damped dashpot of spring-constant *Spr=100 N/m*, viscous coefficient *Vis=100 Ns/m*, Coulomb Friction Force *Fc=100 N*, and Static Friction Force *Fs=200 N* with the exponent of *cs=0.1 m/s* according to the Stribeck model describing the frictio force as

$$F = Spr\left(L - L_0\right) + Vis \times v + sign(v)\left(F_c + F_s \exp\left(-v / c_s\right)\right) \qquad (2)$$

The dashpot's other end was asymmetrically fixed in the workshop. In the space of the joint coordinates a PID type control was prescribed. It could be exactly realized only in the possession of the perfect dynamic model of the system and that of the environmental interactions. It was supposed that the electric drives of this system are "perfect" ones, that is they immediately deliver the commanded torque without any delay or internal dynamics.

In Fig. 1 a)-c) the phase trajectories of the nominal and the simulated motion, the joint coordinate errors and the position error of the arm's end-point are given. (In the phase space it can well be seen that q_2 starts with positive, q_1 with negative value, and each generalized coordinate increases during the 1st phase of the motion, while in the insertion phase q_3 decreases to achieve its initial value.) Fig. 1 e)-d) describes the external disturbances not modeled by the controller. The

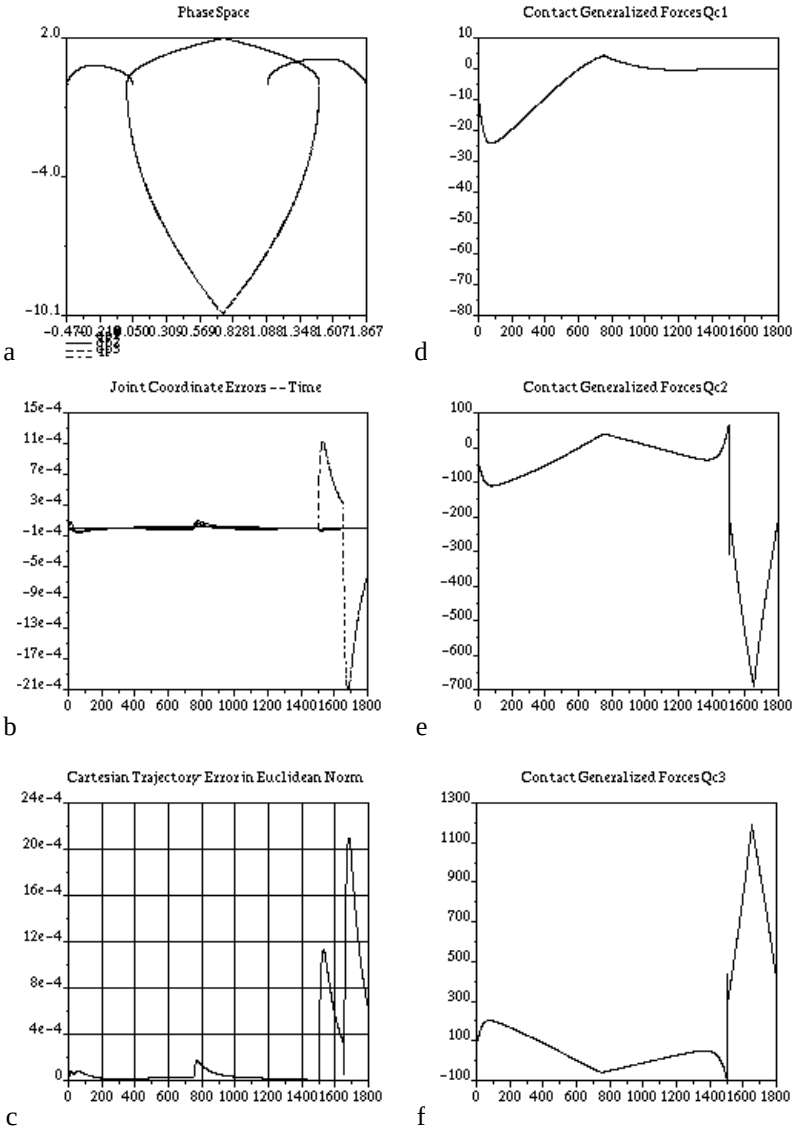

Fig. 1. The nominal and simulated phase trajectories (*rad, rad/s for* q_1 *and* q_2*, m, m/s for* q_3) a); the joint coordinate errors versus time (in *rad*, *m*, *and ms*, *respectively*) b); the position error of the end-point (*m*) c);. the generalized force components exerted by the dashpot (*Nm* for Q_{c1} and Q_{c2}, and *N* for Q_{c3}) versus time (in *ms* units) e)-f), respectively.

jump in Q_{c2} and Q_{c3} at *t=1500 ms* well describes the Coulomb and the static friction, while in the 'triangular' parts the viscous friction dominates. It can well be

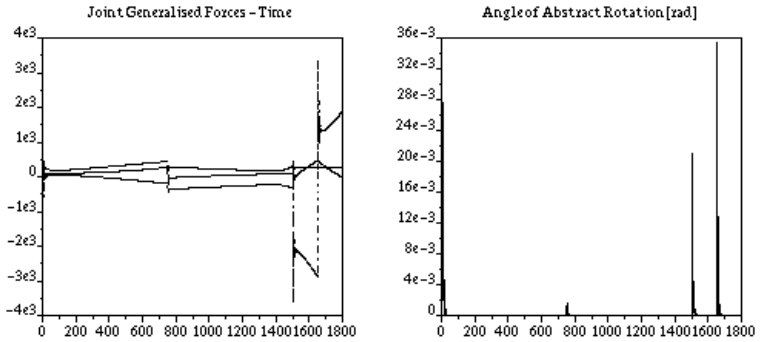

Fig. 2. Generalized force components exerted by the drives (*Nm* for Q_1, Q_2, *N* for Q_3) versus time (in *ms*), and the angle of the abstract rotation of the adaptive control (in *rad*)

seen that the lateral position error is very small while the horizontal one is about 0.8 *mm* at the end of the act of the insertion, in spite of the quite considerable external influences.

For comparison in Fig. 2 the generalized forces of the robot drives, and the angle of abstarct rotation of the adaptive control are given. At the first jump of the components deceleration replaces acceleration in the part moving phase. The jump at *t=1500 ms* corresponds to the start of the insertion, and at about *t=1650 ms* the deceleration replaces acceleration in the insertion phase. The very little angles in the abstract space of transformation indicate that the appropriate transformations are very close to the identity operator as it was required for the convergence of the method. It can well be seen that in spite of the quite considerable modeling errors and external perturbations the adaptive control works well for the mechanical system.

4 Chemical reactions

As possible field of application a single degree of freedom chemical reaction taking place in the mixture of ideal gases placed in a heat reservoir, namely the decay/formation of water is considered. In general in a chemical reaction the change in the mole numbers of the constituents of the mixture is constrained by the stoichiometric rules and can be described by the "reaction equation" of the form $\sum_i \nu_i A_i \Leftrightarrow 0$. the $\{\nu_i\}$ stoichiometric coefficients can be rational or integer numbers. In accordance with that the change in the mole numbers of the constituents can be described by the change in the mole number of a given (arbitrarily chosen) constituent called as the „*fraction of proportionality*", ξ:

$dN_i = \nu_i d\xi$. If e.g. hydrogen is burned in oxygen the chemical reaction and the appropriate constraints can be expressed as follows

$$\begin{aligned} &2H_2 + 1O_2 \Leftrightarrow 2H_2O, \text{ that is} \\ &dN_{H_2} = -dN_{H_2O}, \qquad . \\ &dN_{O_2} = -\frac{1}{2} dN_{H_2O} \end{aligned} \tag{3}$$

In the case of ideal gases in the thermal equilibrium of the mixture at pressure *p* and temperature *T* the mole fractions cannot take arbitrary value. They have to satisfy the mass action law

$$K(T) = p^{\sum_i \nu_i} \prod_i \left(\frac{N_i}{N} \right)^{\nu_i} \tag{4}$$

in which $N=\Sigma N_i$ denotes the mole number of the mixture, while the $K(T)$ function is referred to as the *equilibrium constant* of the given reaction [e.g. 19]. The equilibrium constant can be experimentally determined and tabulated according to (4) without calorimetric measurements. The mass action law has great practical importance: by using it the composition of different mixtures can be controlled either by changing the temperature, the pressure, or both, or by subtracting one or more constituents from the vessel of the reaction. Normally, (4) establishes a strongly non-linear coupling between *T*, *p* and the mole fractions $x_i:=N_i/N$. Consider for instance 2 mole water confined in a vessel at low temperature and then heated to 2000 °K at different pressures. Let us determine the chemical composition of the system in the equilibrium! First choose the „*fraction of proportionality*" to be used. Let it be the mole number of the H_2 molecules, $\xi = N_{H_2}$. The rational stoichiometric coefficients of the reaction are: $0=H_2+(1/2)O_2-H_2O$, therefore $\nu_{H_2} = 1, \nu_{O_2} = 1/2, \nu_{H_2O} = -1$. If we have ξ mole H_2 then we must have $N_{H_2} = \xi, N_{H_2O} = 2-\xi, N_{O_2} = \xi/2$ moles of the other constituents, the total mole number of the system is $N = \xi + 2 - \xi + \xi/2 = 2 + \xi/2$, respectively. So the „*mass action law*" states that

$$\prod_i x^{\nu_i} = \frac{\prod N_i}{N^{\sum \nu_i}} = \frac{\xi^1 (\xi/2)^{1/2} (2-\xi)^{-1}}{(2+\xi/2)^{1+1/2-1}} = Kp^{-\sum \nu_i} = Kp^{-(1+1/2-1)} \tag{5}$$

that is $\frac{\xi^1 (\xi/2)^{1/2} (2-\xi)^{-1}}{(2+\xi/2)^{1/2}} = Kp^{-1/2}$. Solution of this equation can be transformed into seeking the roots of a polynomial according to ξ by putting this equation to the 2nd order then rearranging it as $0 = \xi^3 - 2K^2 p^{-1} (2+\xi/2)(2-\xi)^2, \quad \xi \in [0,2]$. The appropriate solution was numerically obtained by MS EXCEL's Solver starting from $\xi=0$. The results for the mole fraction of the molecular hydrogen are described in Fig. 3 in which the 6th order polynomial trendline indicates that an accurate and smooth approximation for it is possible by a 6th order polynomial

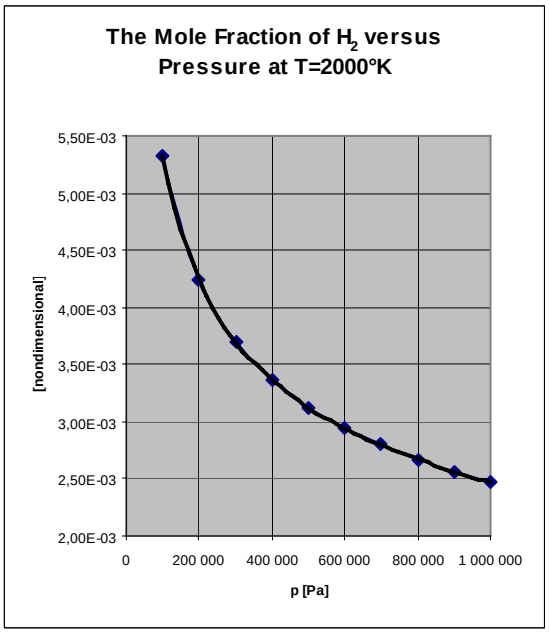

Fig. 3. The mole fraction of molecular hydrogen in the thermal decay of water

function of p. This polynomial was fitted and used for control purposes by the use of EXCEL, too.

From the point of view of a real-time control the above procedure is too costly. Depending on the particular circumstances the mass action law alone is not satisfactory for modeling the system. For instance, whenever the mixture flows through a pressure decreasing throttle valve the full enthalpy of the mixture has to remain constant, that is together with the presure after the valve, (4), and the restriction $H(T,\xi)=H_{initial}$ determines the compostion. For this purpose modeling of the enthalpy of the mixture also becomes inevitable. This may require calorimetric measurements at final temperatures the results of which have to be extrapolated to T=0 °K, too. Such an extrapolation can never be “exact”. Further numerical problem arises from the fact that while the enthalpy of the mixture is relatively a smooth function of its variables, in (4) K can very strongly depend on T. So in general a not very precise model can be worked out by ample numerical calculations via Solver applying the method of the Lagrange multipliers and the reduced gradient for the mass action law in the “goal cell”, and the constant value of the enthalpy as a constraint with automatic recognition of the sensitivity reagrding the different variables of the problem.

It is evident that the new approach can be very fruitfully used in this field. While the thermodynamic tables can be used for developing a rough approximate model

of the process on the basis of which the non-adaptive variant of the control can be designed, the adaptive version can learn the exact behavior of the actual system in real-time.

In the case of the present paradigm the control task is given as follows. Some water is confined in a vessel inserted into a thermal reservoir stabilizing its temperature at 2000 °K. It is prescribed a "nominal" mole fraction of molecular hydrogen as $x^{Nom}(t)$ as a function of time to be obtained from the decay of water molecules. The task is to find a proper $p(t)$ function which results in a realized $x^R(t)$ –supposed to be directly measurable by appropriate spectroscopic means-- at least asymptotically convergent to $x^{Nom}(t)$. The solution based on a rough, imprecise system model $p^M(x)$ and linear control: Let $\alpha>0$ be a feedback parameter and let us try to describe for the desired asymptotic convergence the control law

$$\dot{x}^{Des}(t) = \dot{x}^{Nom}(t) - \alpha\left(x^R(t) - x^{Nom}(t)\right) \tag{6}$$

For this purpose at time t try to apply the 'speed of change in the pressure', and estimate the pressure at the next control step in $t+\Delta t$ as

$$\dot{p}(t) = \frac{\partial p^M\left(x^R(t)\right)}{\partial x^R}\dot{x}^{Des}(t), \quad p(t+\Delta t) = p(t) + \dot{p}(t)\Delta t \tag{7}$$

As a rough system model, on the basis of the calculations the approximate constant value $\frac{\partial p^M\left(x^R(t)\right)}{\partial x^R} = -3 \;\; MPa$ was used, and Δt=1 s was applied supposing that within this time the thermal equilibrium is set in the hot gas. This control results in considerable error in the mole fractions because the asymptotic behaviour prescribed cannot be well approximated due to the very rough approximation $\frac{\partial p^M\left(x^R(t)\right)}{\partial x^R} = -3 \;\; MPa$. To improve the quality of control in the n^{th} step the ratio of the $s(n) := \dot{x}^{Des}(n\Delta t) / \dot{x}^R(n\Delta t)$ ratio is observed which is obtained by applying the modified setting of the speed in changing the pressure as $\dot{p}(n\Delta t) = \frac{\partial p^M\left(x^R(n\Delta t)\right)}{\partial x^R} s(n-1)s(n-2)...s(1)\dot{x}^{Des}(n\Delta t)$ in which the initial factor is $s(1)$=1. This means a set of a kind of "cumulative" corrections in a learning control comparing to each other the expected and the observed behaviour of the controlled system at step "n", and utilizing this result in a "forecasted behaviour" just in the next step. However, in this form the control cannot be applied because when a zero factor occurs it results in a fixed pressure by making any further cahneg zero. Division by zero also can make a problem. To avoid that calculation of the factors was modified as follows:

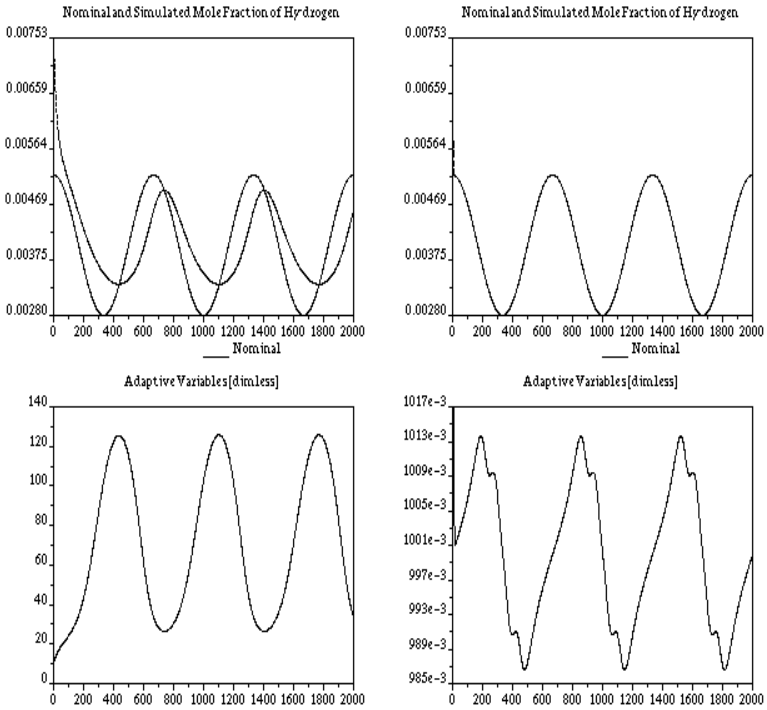

Fig. 4. The results of the non-adaptive and the adaptive control for the thermal decay of H_2O, and the control signal

$$s_{sign} := sign\left(\dot{x}^{Des}\left(n\Delta t\right)\right) \times sign\left(\dot{x}^{R}\left(n\Delta t\right)\right)$$
$$s(n) = \begin{cases} 1 \quad if \quad s_{sign} = 0 \\ s_{sign} \times \left(\left|\dot{x}^{Des}\left(n\Delta t\right)\right| + 10^{-25}\right) / \left(\left|\dot{x}^{R}\left(n\Delta t\right)\right| + 10^{-25}\right) \end{cases} \tag{8}$$

This avoids occurring 0 in the series $\{s(n)\}$, and causes negligible distortion for finite numerator and denominator, furthermore avoids the 0/0 problem, that whenever the control does not have useful information to determine 'what to do' it makes the factor $s(n)$ approach 1. It is worth noting that according to [13] in the case of single input-single output systems the proof of complete stability does not require perturbation calculation, in this case the concept of monotonity of a single variable function can be utilized instead.

In Fig. 4 the nominal and the «simulated» mole fraction of the molecular hydrogen is given for the non-adaptive and the adaptive control for the same α value. It is evident that adaptivity considerably improves the quality of the control. To reveal the explanation of the improvement, the control parameter $s(t)$ also is given in Fig. 4 for the non-adaptive (in this case it is only calculated without any further

utilization) and the adaptive control, together with the appropriate $p(t)$ functions. It is clear that in the case of the simple linear control the rough system model results is very bad estimation for the necessary speed of change in the pressure. This is substantiated by the very big $s(t)$ ratios in the case of the non-adaptive control. In contrast to that in the adaptive control $s(t)$ soon approaches and remains in the close vicinity of 1, that is the adaptively forecasted and learned behavior of the system are very close to each other.

5 Conclusions

In this paper an extension of the application of a new branch of Computational Cybernetics is reported. Computer simulation was used to verify and exemplify the operation of the method for a 3 DOF SCARA paradigm having realistic Stribeck friction terms and backlash. As another example of application controlling chemical reactions as the realm of strong non-linear coupling with inaccurate models were briefly analysed. As a paradigm a one degree of freedom chemical reaction, the thermal decay of water molecules was considered and successfully controlled by the method. Research aiming at further potential fields of applications as the control of chemical reactions of more than one degree of freedom and unstable systems seems to be expedient in the future.

6 Acknowledgment

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7 References

[1] R. Reed: "Pruning Algorithms - A Survey", IEEE Transactions on Neural Networks, 4., pp.- 740-747, 1993.

[2] S. Fahlmann, C. Lebiere: "The Cascade-Correlation Learning Architecture", Advances in Neural Information Processing Systems, 2, pp. 524-532, 1990.

[3] T. Nabhan, A. Zomaya: "Toward Generating Neural Network Structures for Function Approximation", Neural Networks, 7, pp. 89-9, 1994.

[4] G. Magoulas, N. Vrahatis, G. Androulakis: "Effective Backpropagation Training with Variable Stepsize" Neural Networks, 10, pp. 69-82, 1997.

[5] C. Chen, W. Chang: "A Feedforward Neural Network with Function Shape Autotuning", Neural Networks, 9, pp. 627-641, 1996.
[6] W. Kinnenbrock: "Accelerating the Standard Backpropagation Method Using a Genetic Approach", Neurocomputing, 6, pp. 583-588, 1994.
[7] A. Kanarachos, K. Geramanis: "Semi-Stochastic Complex Neural Networks", IFAC-CAEA '98 Control Applications and Ergonomics in Agriculture, pp. 47-52, 1998.
[8] Batyrshin I., Kaynak O., Rudas I.: "Generalized Conjunction and Disjunction Operations for Fuzzy Control", in: Proceeding 6th European Congress on Intelligent Technignes & Soft Computing, EUFIT'98, Aachen, Germany, 1998, Vol. 1, 52-57.
[9] Batyrshin I., Bikbulatov A., Kaynak O., Rudas I.: "Functions Approximation Based on the Tuning of Generalized Connectives", Proc. of EUROFUSE - SIC '99, Budapest, Hungary, 1999, pp. 556-561.
[10] J. Abonyi J.K. Tar, F. Szeifert: "Identification of MIMO Processes by Fuzzy Clustering", In the Proc. of the 2001 IEEE International Conference on Intelligent Engineering Systems (INES 2001), Helsinki, Finland, September 16-18, 2001, pp.65-70. (ISBN 952 15-0689-X).
[11] J.K. Tar, I.J. Rudas, J.F. Bitó: "Group Theoretical Approach in Using Canonical Transformations and Symplectic Geometry in the Control of Approximately Modeled Mechanical Systems Interacting with Unmodelled Environment", Robotica, Vol. 15, pp. 163-179, 1997.
[12] J.K. Tar, I.J. Rudas, J.F. Bitó, O.M. Kaynak: "A New Utilization of the Hamiltonian Formalism in the Adaptive Control of Mechanical Systems Under External Perturbation" Intelligent Automation and Soft Computing, Vol.5, Issue:4, 1999, pp. 303-312.
[13] J.K. Tar, M. Rontó: "Adaptive Control Based on the Application of Simplified Uniform Structures and Learning Procedures", Zbornik Radova, Vol. 24 No. 2, 2000, pp. 174-194. (ISSN: 0351-1804)
[14] Yahya El Hini: "Comparison of the Application of the Symplectic and the Partially Stretched Orthogonal Transformations in a New Branch of Adaptive Control for Mechanical Devices", Proc. of the 10th International Conference on Advanced Robotics", August 22-25, Budapest, Hungary, pp. 701-706, ISBN 963 7154 05 1.
[15] J. K. Tar, Imre J. Rudas, János F. Bitó, K. Kozlowski: "A New Approach in Computational Cybernetics Based on the Modified Renormalization Algorithm Guaranteeing Complete Stability in the Control of a Wide Class of Physical Systems", accepted for publication in 6th International Conference Systems 2002 (INES 2002), May 26-28, 2002, Opatija, Croatia, pp. 19-24, ISBN 953-6071-17-7.
[16] M.J. Feigenbaum, J. Stat. Phys. 19, 25, 1978;
[17] M.J. Feigenbaum, J. Stat. Phys. 21, 669, 1979;
[18] M.J. Feigenbaum, Commun. Math. Phys. 77, 65, 1980;
[19] Herbert B. Callen: „Thermodynamics and Introduction to Thermostatistics", John Wiley & Sons, 1985, (ISBN 0-471-86256-8).