# Risk-Sensitive Control

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*Abstract: The purpose of the paper is to present a method for risk-sensitive minimum variance control of stochastic discrete-time linear systems with the gaussian system and measurement noise models, which connects control pair (**u**(i),**x**(i)) conditional density optimization, optimal control law solving and risk-neutral control gain specification. The basic idea consist in determination of the join covariance matrix **T** of the mentioned control pair, combined with the risk-parameter setting, which represents the amount of risk in the control policy.*

*Keywords: LQ control methods, minimum variance control, risk-neutral methods, risk-sensitive control.*

## 1 Introduction

For linear systems, the optimal control problem reduces to a linear quadratic regulator (LQ) problem, whose solution can be obtained by solving an algebraic Riccati equation. It is well-known fact that linear quadratic optimal control yields a stable closed-loop system and the minimal value of the performance index. The disadvantage, however, is that the performance of this controllers is degraded in the presence of system parameter deviations.

The approach, to be presented in this paper, translates the control problem into an risk-sensitive generalization of minimum variance optimal control with emphasis on optimal control gain selection. The solution to the optimal control problem is then a solution to the optimized conditional join density function of control pair (**u**(i),**x**(i)) under system and measurement noise and discrete-time Kalman filter realization. This is motivated by the need for robustness in the widely used (risk-neutral) minimum variance control, including adaptive control and risk neutral control is presented as the risk sensitive parameter approaches zero. An interesting point is that presented control has the essential structure of a standard LQ controller.

## 2 Problem Formulation

Many different methods can be used in the control design of stochastic linear multi-variable systems with the discrete-time state-space description

$$\mathbf{x}(i+1) = \mathbf{F}\mathbf{x}(i) + \mathbf{G}\mathbf{u}(i) + \mathbf{v}(i) \tag{1}$$

$$\mathbf{y}(i) = \mathbf{C}\mathbf{x}(i) + \mathbf{o}(i) \tag{2}$$

where $\mathbf{x}$(i), $\mathbf{u}$(i) and $\mathbf{y}$(i) are the state, input and output vectors, respectively, **F, G** and **C** are matrices of appropriate dimensions, $\mathbf{v}$(i), $\mathbf{o}$(i) are gaussian noises. If couple (**A, B**) controllable, there exist several approaches to the determination of the control matrix **K** in static output feedback

$$\mathbf{u}(i) = -\mathbf{K}\mathbf{x}_e(i) \tag{3}$$

subject to (1), (2), with a state-variable vector estimate at time point *i* in dependence on all the values {$\mathbf{y}$(j) : j = 0,1,2, … ,i }of signal and noise up to time point *i*.

It is supposed in the next that the actual estimate error

$$\mathbf{e}(i) = \mathbf{x}(i) - \mathbf{x}_e(i) \tag{4}$$

is gaussian, with expectation E{$\mathbf{e}$(i)}= **0**, covariance $\mathbf{P}$(i), and density function

$$f_{\mathbf{X},\mathbf{X}_e}(\mathbf{e}(i)) = K_e \exp(-\frac{1}{2}(\mathbf{x}(i) - \mathbf{x}_e(i))^T \mathbf{F}^T \mathbf{P}(i)\mathbf{F}(\mathbf{x}(i) - \mathbf{x}_e(i))) \tag{5}$$

and the density functions of the system noise sequence {$\mathbf{v}$(j) : j = 0,1,2, … } and the measurement noise sequence {$\mathbf{o}$(j) : j = 0,1,2, } are

$$f_{\mathbf{V}}\mathbf{v}(i) = K_v \exp(-\frac{1}{2}(\mathbf{v}^T(i)a^{-1}\mathbf{v}(i)) \tag{6}$$

$$f_{\mathbf{O}}(\mathbf{o}(i)) = K_o \exp(-\frac{1}{2}(\mathbf{o}^T(i)a^{-1}\mathbf{o}(i)) \tag{7}$$

respectively.

The risk-sensitive minimum variance control design task is, in general, for a system described by (1), (2) to determine such control law (3) that minimizes the cross-covariance of the join density function

$$f_{(\mathbf{W})}(\mathbf{w}(i)) = f_{\mathbf{Y}}(\mathbf{y}(i+1))\, f_{(\mathbf{X},\mathbf{X}_e)}(\mathbf{e}(i))\, f_{\mathbf{V}}(\mathbf{v}(i))\, f_{(\mathbf{O})}(\mathbf{o}(i)) \tag{8}$$

where

$$f_{\mathbf{Y}}(\mathbf{y}(i+1)) = K_Y \exp(\frac{b}{2}(\mathbf{y}^T(i+1)\mathbf{y}(i+1)) \tag{9}$$

Here, b < 0 is a real number and represents the amount of risk in the control policy.

This problem is a special case of one-step-ahead prediction considered as the risk-sensitive generalization of minimum variance control.

# 3 Measured noise Orthogonalisation

The optimal control condition (8) can be solved analytically in the linear model case. Thus considering the model (1), (2) the density function (9) can be rewritten as

$$\begin{gathered} f_{\mathbf{Y}}(\mathbf{y}(i+1)) = f_{\mathbf{Z}}(\mathbf{z}(i)) = \\ = K_y \exp(\frac{b}{2}(\mathbf{C}(\mathbf{Fx}(i)+\mathbf{Gu}(i)+\mathbf{v}(i)+\mathbf{o}(i))^T(\mathbf{C}(\mathbf{Fx}(i)+\mathbf{Gu}(i)+\mathbf{v}(i)+\mathbf{o}(i)) \end{gathered} \tag{10}$$

where joint vector variable is a vector $\mathbf{z}$(i) with components formally arranged in the form

$$\mathbf{z}^T(i) = [\mathbf{u}^T(i) \quad \mathbf{x}^T(i) \quad \mathbf{v}^T(i) \quad \mathbf{o}^T(i)] \tag{11}$$

Using this notation the criterion (8) can be reformulated as

$$f_{\mathbf{W}}(\mathbf{w}(i)) = K_w \exp(\frac{1}{2}c_w(i)) \tag{12}$$

where

$$\mathbf{w}^T(i) = [\mathbf{u}^T(i) \quad \mathbf{x}_e^T(i) \quad \mathbf{x}^T(i) \quad \mathbf{v}^T(i) \quad \mathbf{o}^T(i)] \tag{13}$$

$$K_w = K_y K_e K_v K_o \tag{14}$$

$$\begin{gathered} c_w(i) = -(\mathbf{x}(i)-\mathbf{x}_e(i))^T\mathbf{F}^T\mathbf{P}(i)\mathbf{F}(\mathbf{x}(i)-\mathbf{x}_e(i))) - \mathbf{v}^T(i)a^{-1}\mathbf{v}(i) - \mathbf{o}^T(i)a^{-1}\mathbf{o}(i) + \\ +b(\mathbf{C}(\mathbf{Fx}(i)+\mathbf{Gu}(i)+\mathbf{v}(i)+\mathbf{o}(i))^T(\mathbf{C}(\mathbf{Fx}(i)+\mathbf{Gu}(i)+\mathbf{v}(i)+\mathbf{o}(i)) \end{gathered} \tag{15}$$

As a consequence of gaussian properties it follows that

$$
\begin{aligned}
c_w(i) &= (\mathbf{Fx}(i) + \mathbf{Gu}(i) + \mathbf{v}(i))^T (c\mathbf{C}^T\mathbf{C})(\mathbf{Fx}(i) + \mathbf{Gu}(i) + \mathbf{v}(i)) \\
&(\mathbf{Fx}(i) + \mathbf{Gu}(i) + \mathbf{v}(i))^T b\mathbf{C}^T \mathbf{o}(i) + b\mathbf{o}^T (i)\mathbf{C}(\mathbf{Fx}(i) + \mathbf{Gu}(i) + \mathbf{v}(i)) \\
&-(\mathbf{x}(i) - \mathbf{x}_e(i))^T \mathbf{F}^T \mathbf{P}(i)\mathbf{F}(\mathbf{x}(i) - \mathbf{x}_e(i))) - \mathbf{v}^T (i)a^{-1}\mathbf{v}(i) - \mathbf{o}^T (i)(b - a^{-1})\mathbf{o}(i)
\end{aligned} \tag{16}
$$

and

$$
\mathbf{W}_{oo} = (b - a^{-1}), \quad \mathbf{W}_{ox1} = b\mathbf{C}^T, \quad \mathbf{W}_{x1x1} = b\mathbf{C}^T\mathbf{C} \tag{17}
$$

The marginal density function of measurement noise is then

$$
f_{\mathbf{O}m}(\mathbf{o}(i)) = K_{om} \exp(\frac{1}{2}\mathbf{o}^T (i)(b - a^{-1})\mathbf{o}(i)) \tag{18}
$$

and the conditional density function of reduced random vector variable can be constructed as

$$
\begin{aligned}
f_{\mathbf{X1}}(\mathbf{x_1}(i) \mid \mathbf{o}(i)) &= K_{X1} \exp(\frac{1}{2}\mathbf{x}_1^T \mathbf{V}\mathbf{x}_1(i)) = \\
&= K_{x1} \exp(\frac{1}{2}\mathbf{x}_1^T (i)(b\mathbf{C}^T\mathbf{C} - b\mathbf{C}^T (b - a^{-1})^{-1} b\mathbf{C})\mathbf{x}_1(i))
\end{aligned} \tag{19}
$$

where

$$
\mathbf{x}_1(i) = \mathbf{Fx}(i) + \mathbf{Gu}(i) + \mathbf{v}(i) \tag{20}
$$

$$
\mathbf{V} = b\mathbf{C}^T\mathbf{C}(1 - b(b - a^{-1})^{-1}) \tag{21}
$$

# 4 System noise Orthogonalisation

To solve the problem of minimal covariance the conditional expectation can be accepted as zero and thus, by combining (17) and (18), optimized join density function criterion (12) can be modified as

$$
f_{\mathbf{W}}(\mathbf{w}(i)) = K_{w1} \exp(\frac{1}{2}c_{w1}(i)) f_{\mathbf{O}m}(\mathbf{o}(i)) \tag{22}
$$

where

$$
\mathbf{w}_1^T(i) = [\mathbf{u}^T(i) \quad \mathbf{x}_e^T(i) \quad \mathbf{x}^T(i) \quad \mathbf{v}^T(i)] \tag{23}
$$

$$
K_{w1} = K_{x1}K_eK_vK_{om} \tag{24}
$$

$$c_{w1}(i) = -(\mathbf{x}(i) - \mathbf{x}_e(i))^T \mathbf{F}^T \mathbf{P}(i)\mathbf{F}(\mathbf{x}(i) - \mathbf{x}_e(i))) - \mathbf{v}^T(i)a^{-1}\mathbf{v}(i) + \\ +(\mathbf{Fx}(i) + \mathbf{Gu}(i) + \mathbf{v}(i))^T \mathbf{V}(\mathbf{Fx}(i) + \mathbf{Gu}(i) + \mathbf{v}(i)) \tag{25}$$

Thus, applying completing arguments, as in derivative of measurement noise orthogonality, the value is

$$c_{w1}(i) = (\mathbf{Fx}(i) + \mathbf{Gu}(i))^T \mathbf{V}(\mathbf{Fx}(i) + \mathbf{Gu}(i)) \\ (\mathbf{Fx}(i) + \mathbf{Gu}(i))^T \mathbf{Vv}(i) + \mathbf{v}^T(i)\mathbf{V}(\mathbf{Fx}(i) + \mathbf{Gu}(i)) \\ -(\mathbf{x}(i) - \mathbf{x}_e(i))^T \mathbf{F}^T \mathbf{P}(i)\mathbf{F}(\mathbf{x}(i) - \mathbf{x}_e(i))) - \mathbf{v}^T(i)a^{-1}\mathbf{v}(i)$$

(26)

and

$$\mathbf{W}_{vv} = \mathbf{V} - a^{-1}\mathbf{I}, \quad \mathbf{W}_{vx0} = \mathbf{V}, \quad \mathbf{W}_{x0x0} = \mathbf{V} \tag{27}$$

The marginal density function of measurement noise is then

$$f_{\mathbf{V}m}(\mathbf{v}(i)) = K_{vm} \exp(\frac{1}{2}\mathbf{v}^T(i)(\mathbf{V} - a^{-1}\mathbf{I})\mathbf{v}(i)) \tag{28}$$

and the conditional density function of reduced random vector variable can be constructed as

$$f_{\mathbf{X}0}(\mathbf{x_0}(i) \mid \mathbf{v}(i), \mathbf{o}(i)) = K_{x1} \exp(\frac{1}{2}\mathbf{x}_1^T \mathbf{W}\mathbf{x}_1(i)) = \\ = K_{x1} \exp(\frac{1}{2}\mathbf{x}_0^T(i)(\mathbf{V} - \mathbf{V}(\mathbf{V} - a^{-1}\mathbf{I})^{-1}\mathbf{V})\mathbf{x}_0(i)) \tag{29}$$

where

$$\mathbf{x}_0(i) = \mathbf{Fx}(i) + \mathbf{Gu}(i) \tag{30}$$

$$\mathbf{W} = \mathbf{V} - \mathbf{V}(\mathbf{V} - a^{-1}\mathbf{I})^{-1}\mathbf{V} \tag{31}$$

# 5 System state Orthogonalisation

By the same way as in the last two sections the joint density function (22) can be reformulated as

$$f_{\mathbf{W}}(\mathbf{w}(i)) = K_{w2} \exp(\frac{1}{2}c_{w2}(i)) f_{\mathbf{V}m}(\mathbf{v}(i)) f_{\mathbf{O}m}(\mathbf{o}(i)) \tag{32}$$

where

$$\mathbf{w}_2^T(i) = [\mathbf{u}^T(i) \quad \mathbf{x}_e^T(i) \quad \mathbf{x}^T(i)] \tag{33}$$

$$K_{w2} = K_{x0}K_eK_{vm}K_{om} \tag{34}$$

$$\begin{gathered} c_{w2}(i) = -(\mathbf{x}(i) - \mathbf{x}_e(i))^T \mathbf{F}^T \mathbf{P}(i)\mathbf{F}(\mathbf{x}(i) - \mathbf{x}_e(i))) + \\ +(\mathbf{Fx}(i) + \mathbf{Gu}(i))^T \mathbf{W}(\mathbf{Fx}(i) + \mathbf{Gu}(i)) \end{gathered} \tag{35}$$

The value (35) can be expanded as

$$\begin{gathered} c_{w2}(i) = \mathbf{x}^T(i)(\mathbf{F}^T\mathbf{WF} - \mathbf{F}^T\mathbf{P}(i)\mathbf{F})\mathbf{x}(i) + \\ +\mathbf{u}^T(i)\mathbf{G}^T\mathbf{WGu}(i) + \mathbf{u}^T(i)\mathbf{G}^T\mathbf{WFx}(i) + \mathbf{x}^T(i)\mathbf{F}^T\mathbf{WGu}(i) + \\ +\mathbf{x}^T(i)\mathbf{F}^T\mathbf{P}(i)\mathbf{Fx}_e(i) + \mathbf{x}_e^T(i)\mathbf{F}^T\mathbf{P}(i)\mathbf{Fx}(i) - \mathbf{x}_e^T(i)\mathbf{F}^T\mathbf{P}(i)\mathbf{Fx}_e(i) \end{gathered} \tag{36}$$

thus

$$\begin{gathered} \mathbf{W}_{xx} = \mathbf{F}^T\mathbf{WF} - \mathbf{F}^T\mathbf{P}(i)\mathbf{F}, \quad \mathbf{W}_{x_e s} = \mathbf{F}^T\mathbf{P}(i)\mathbf{F}, \quad \mathbf{W}_{x_e x_e} = -\mathbf{F}^T\mathbf{P}(i)\mathbf{F}, \\ \mathbf{W}_{ux} = \mathbf{G}^T\mathbf{WF}, \quad \mathbf{W}_{ux_e} = -\mathbf{G}^T\mathbf{WF}, \quad \mathbf{W}_{uu} = \mathbf{G}^T\mathbf{WG} \end{gathered} \tag{37}$$

The marginal density function of system state is then

$$f_{\mathbf{X}m}(\mathbf{x}(i)) = K_{xm}\exp(\frac{1}{2}\mathbf{x}^T(i)(\mathbf{F}^T\mathbf{WF} - \mathbf{F}^T\mathbf{P}(i)\mathbf{F})\mathbf{x}(i)) \tag{38}$$

and the conditional variances are

$$\mathbf{W}_{u|x} = \mathbf{G}^T\mathbf{WG} - \mathbf{G}^T\mathbf{WF}(\mathbf{F}^T\mathbf{WF} - \mathbf{F}^T\mathbf{P}(i)\mathbf{F})^{-1}\mathbf{F}^T\mathbf{P}(i)\mathbf{G} \tag{39}$$

$$\mathbf{W}_{x_e|x} = -\mathbf{F}^T\mathbf{P}(i)\mathbf{F} - -\mathbf{F}^T\mathbf{P}(i)\mathbf{F}(\mathbf{F}^T\mathbf{WF} - \mathbf{F}^T\mathbf{P}(i)\mathbf{F})^{-1}\mathbf{F}^T\mathbf{P}(i)\mathbf{F} \tag{40}$$

$$\mathbf{W}_{ux_e|x} = -\mathbf{G}^T\mathbf{WF}(\mathbf{F}^T\mathbf{WF} - \mathbf{F}^T\mathbf{P}(i)\mathbf{F})^{-1}\mathbf{F}^T\mathbf{P}(i)\mathbf{F} \tag{41}$$

respectively.

Thus, the conditional density function of new vector variables is

$$f_{\mathbf{X}d}(\mathbf{x}_d(i) \mid \mathbf{x}(i), \mathbf{v}(i), \mathbf{o}(i)) = K_{xd}\exp(\frac{1}{2}\mathbf{x}_d^T(i)\mathbf{Tx}_d(i)) \tag{42}$$

$$\mathbf{x}_d^T(i) = [\mathbf{u}^T(i) \quad \mathbf{x}_e^T(i)] \tag{43}$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{W}_{ux|x} & \mathbf{W}_{ux_e|x} \\ \mathbf{W}_{ux_e|x}^T & \mathbf{W}_{x_e|x} \end{bmatrix} \tag{44}$$

# 5 Control Optimality

Assumption that the system state variable vector is fully filtered implies that the joint density function (32}) can be transformed to the next form

$$f_{\mathbf{W}}(\mathbf{w}(i)) = K_{w3} \exp(\frac{1}{2} c_{w3}(i))\, f_{\mathbf{X}m}(\mathbf{x}(i))\, f_{\mathbf{V}m}(\mathbf{v}(i))\, f_{\mathbf{O}m}(\mathbf{o}(i)) \tag{45}$$

where

$$K_{w3} = K_{xd} K_{xm} K_{vm} K_{om} \tag{46}$$

$$c_{w3} = [\mathbf{u}^T(i)\ \mathbf{x}_e^T(i)] \begin{bmatrix} \mathbf{W}_{ux|x} & \mathbf{W}_{ux_e|x} \\ \mathbf{W}_{ux_e|x}^T & \mathbf{W}_{x_e|x} \end{bmatrix} \begin{bmatrix} \mathbf{u}(i) \\ \mathbf{x}_e(i) \end{bmatrix} \tag{47}$$

The value of exponent for fully filtered system state vector is characterized by the relation

$$\frac{\partial c_{w3}}{\partial \mathbf{u}^T(i)} = [\mathbf{1}\ \ \mathbf{0}] \begin{bmatrix} \mathbf{W}_{ux|x} & \mathbf{W}_{ux_e|x} \\ \mathbf{W}_{ux_e|x}^T & \mathbf{W}_{x_e|x} \end{bmatrix} \begin{bmatrix} \mathbf{u}(i) \\ \mathbf{x}_e(i) \end{bmatrix} = \mathbf{0} \tag{48}$$

thus

$$\mathbf{W}_{ux|x}\mathbf{u}(i) + \mathbf{W}_{ux_e|x}\mathbf{x}_e(i) = \mathbf{0} \tag{49}$$

Once the matrices are determined, the optimal control law (3) is completely described by next equation

$$\mathbf{u}(i) = -\mathbf{W}_{ux|x}^{-1}\mathbf{W}_{ux_e|x}\mathbf{x}_e(i) = -\mathbf{K}\mathbf{x}_e(i) \tag{50}$$

where

$$\mathbf{K} = -(\mathbf{G}^T\mathbf{W}\mathbf{G} - \mathbf{G}^T\mathbf{W}(\mathbf{W}-\mathbf{P}(i))^{-1}\mathbf{W}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{W}(\mathbf{W}-\mathbf{P}(i))^{-1}\mathbf{P}(i)\mathbf{F} \tag{51}$$

$$\mathbf{W} = \mathbf{V} - \mathbf{V}(\mathbf{V} - a^{-1}\mathbf{I})^{-1}\mathbf{V} \tag{52}$$

$$\mathbf{V} = b\mathbf{C}^T\mathbf{C}(1 - b(b - a^{-1})^{-1}) \tag{53}$$

$a$ is the value of variance of the system noise and measurement noise and b < 0 is a real number representing the amount of risk in the control policy.

## Concluding Remarks

The paper gives some background material on formulation of risk-sensitive minimal variance control, using join density function of control signal, system state, estimation error, system noise and measurement noise. The basic idea consist in determination of the join conditional covariance matrix **T** of the control vector and estimated system state vector, combined with the risk-parameter setting, which represents the amount of risk in the control policy.

The importance of selecting an information-state-oriented performance index has been demonstrated, where present study has indicated, that the used approach is robust in risk-sensitive sense and nominal in the risk-neutral sense. In the small noise limit the control is optimum for a solution with worst case noise.

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