Generalized Fuzzy Operators

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Abstract- Information aggregation is one of the key issues in development of intelligent systems. Although fuzzy set theory provides a host of attractive aggregation operators for integrating the membership values representing uncertain information, the results do not always follow the modeled real phenomena. Researches on this area have shown that better results can be reached by using various aggregation operation families. In this paper some new approaches towards the generalizations of the conventional triangular norms are summarized. It is shown that by omitting and/or modifying some axioms in the axiom skeleton new generalized operation families can be obtained. The paper summarizes some new operations, including novel parametric generalized operations, absorbing-norms and distance-based operators.

Keywords: t-norms, t-conorms, fuzzy connectives, aggregation operators

1 Introduction

Many applications of fuzzy set theory, such as fuzzy logic control, fuzzy expert systems and fuzzy systems modeling involve the use of a fuzzy rule base to model complex and approximately or not well-known systems. Information aggregation is one of the key issues in these systems. Although fuzzy set theory provides a host of attractive aggregation operators for integrating membership values representing uncertain information the research on this area is still not completed.

Since the pioneering work of Zadeh the basic research was oriented towards the investigation of the properties of t-operators and also to find new ones satisfying the axiom system. As a result of this a great number (of various type) of t-operators have been introduced accepting the axiom system as a fixed, unchangeable skeleton. Until the last few years no strong efforts were devoted to generalize t-operators by modifying, "weakening" these axiom system.

In this paper a survey of some new approaches towards the generalization of t-operators is given. Some of the axioms is analyzed from the point of view of their necessity. First the questions of commutativity and associativity are discussed. And some examples of non-commutative and non-associative operations are given.

By modifying the neutral elements of t-norm and t-conorm with an arbitrary number from the unit interval, the concept of uninorm is received. As a kind of complement of uninorms a new, not necessarily monotone operation, the absorbing-norm is introduced, and their structure and properties are discussed.

The replacement of the neutral element with certain boundary conditions leads to the concepts of conjunction and disjunction operations and also their weakest forms, to the quasi- and pseudo-operations. The generation of these operations is outlined. Based on these theorems some new parametric classes of generalized operations are introduced.

The distance-based operations are generalized operations. Their properties and structures are discussed in the last part of the paper.

2 T-Operators

Definition 1 A mapping $T:[0,1]\times[0,1]\to[0,1]$ is a *t-norm* if it is commutative, associative, non-decreasing and $T(x,1)=x$, for all $x\in\left[0,1\right]$.

Definition 2 A mapping $S:[0,1]\times[0,1]\to[0,1]$ is a *t-conorm* if it is commutative, associative, non-decreasing and $S(x,0)=x$, for all $x\in\left[0,1\right]$.

Definition 3 A mapping $N:[0,1]\to[0,1]$ is a *negation*, if it is non-increasing and $N(0) = 1$ and $N(1) = 0$.

N is a *strict negation if* N is strictly decreasing and N is a continuous function. N is a *strong negation* if N is strict and $N(N(a)) = a$, that is, N is *involutive*.

Further it is assumed that T is a t-norm, S is a t-conorm and N is a strict negation.

T-norms and T-conorms can be obtained from each other as follows:

$$S(x, y) = N(T(N(x), N(y))) \text{ and } T(x, y) = N(S(N(x), N(y))). \qquad (1)$$

The simplest examples of T-norms and T-conorms mutually related by means of negation $N(x) = 1 - x$ are the followings

Minimum $T_M(x, y) = \min(x, y)$

Maximum $S_M(x, y) = \max(x, y)$

Product $T_P(x, y) = xy$

Probabilistic Sum $S_P(x, y) = x + y - xy$

Lukasievicz $T_L(x, y) = \max(x + y - 1, 0)$

Bounded Sum $S_L(x, y) = \min(x + y, 1)$

Weakest t-norm $T_W(x, y) = \begin{cases} x, \ \mathit{if}\ y = 1 \\ y, \ \mathit{if}\ x = 1 \\ 0, \ \mathit{otherwise} \end{cases}$

Strongest t-conorm $S_S(x, y) = \begin{cases} x, \ \mathit{if}\ y = 0 \\ y, \ \mathit{if}\ x = 0 \\ 1, \ \mathit{otherwise} \end{cases}$

Generally, for any T-norm T and T-conorm S

$$T_W(x, y) \le T(x, y) \le T_M(x, y) \le S_M(x, y) \le S(x, y) \le S_S(x, y) \qquad (2)$$

i.e. T-norms T_W and T_M are the minimal and the maximal boundaries for all T-norms, respectively. Also T-conorms S_M and S_S are the minimal and the maximal boundaries for all T-conorms, respectively. These inequalities are important from practical point of view as they establish the boundaries of the possible range of mappings T and S.

Parametric fuzzy connectives form also important class of t-operators. Three examples of parametric t-norms

$$T(x, y) = 1 - \sqrt[p]{(1-x)^p + (1-y)^p - (1-x)^p(1-y)^p} \qquad \text{(Schweizer and Sklar),}$$

$$T(x,y) = 1 - \min(1, \sqrt[p]{(1-x)^p + (1-y)^p}) \qquad \text{(Yager)},$$

$$T(x,y) = 1 \Big/ \left(1 + \left((1/x - 1)^p + (1/y - 1)^p \right)^{1/p} \right) \qquad \text{(Dombi)}.$$

3 The Commutative and Associative Axioms

One possible way of simplification of axiom skeletons of t-operators may not be requiring these operations to have the commutative and the associative properties. Non-commutative and associative operations are widely used in mathematics, so; Why do we restrict our investigations by keeping these axioms? What are the requirements of the most typical applications?

From theoretical point of view the commutative law is not required, while the associative law is necessary to extend the operation to more than two variables. In applications, like fuzzy logic control, fuzzy expert systems and fuzzy systems modeling fuzzy rule base and fuzzy inference mechanism are used, where the information aggregation is performed by operations. The inference procedures do not always require commutative and associative laws of the operations used in these procedures. These properties are not necessary for conjunction operations used in the simplest fuzzy controllers with two inputs and one output. For rules with greater amount of inputs and outputs these properties are also not required if the sequence of variables in the rules is fixed.

Moreover, a non-commutative T-norm may in fact be desirable for rules, because it gives us the possibility of taking into account the different character of influence of the error and the change in error on the output variable. So, if the commutative T-norm implies equality of rights of both operands, then the non-commutative operation with fixed positions of operands gives the possibility to build context dependent operations. Some examples for parametric non-commutative and non-associative operations will be given.

4 The Axiom of Neutral Element

For the generalization of triangular norms there are two ways. Uninorms, introduced by Yager and Rybalov in 1998 [5] are a kind of generalizations of t-norms and t-conorms where the neutral element can be any number from the unit interval. The other approach is the replacement of the neutral element with so called "boundary conditions".

4.1 Uninorms

Uninorms are such kind of generations of t-norms and t-conorms where the neutral element can be any number from the unit interval. The class of uninorms seems to play an important role both in theory and application [**5**], [**6**], [**7**].

Definition 4 [**5**] A *uninorm U* is a commutative, associative and increasing binary operator with a neutral element $e \in [0,1]$, i.e. $U(x,e) = x, \forall x \in [0,1]$.

The neutral element *e* is clearly unique. The case *e* = 1 leads to t-conorm and the case *e* = 0 leads to t-norm.

The first uninorms were given by Yager and Rybalov [**5**]

$$U_c(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in [e,1]^2 \\ \min(x,y) & \text{elsewhere} \end{cases} \tag{3}$$

and

$$U_d(x,y) = \begin{cases} \min(x,y), & \text{if } (x,y) \in [0,e]^2 \\ \max(x,y), & \text{elsewhere} \end{cases}. \tag{4}$$

U_c is a conjunctive right-continuous uninorm and U_d is a disjunctive left-continuous uninorm.

Regarding the duality of uninorms Yager and Rybalov have proved the following theorem [**5**].

Theorem 1 *Assume* U *is a uninorm with identity element e, then* $\overline{U}(x,y) = 1 - U(1-x, 1-y)$ *is also a uninorm with neutral element* $1-e$.

4.2 Nullnorms

Definition 5 [**6**] A mapping $U : [0,1] \times [0,1] \to [0,1]$ is *nullnorm, if there exists an absorbing element* $a \in [0,1]$, i.e., $V(x,a) = a, \forall x \in [0,1]$, V is *commutative*, V is *associative, non-decreasing and satisfies*

$$V(x,0) = x \text{ for all } x \in [0,a] \tag{5}$$

$$V(x,1) = x \text{ for all } x \in [a,1]. \tag{6}$$

4.3 Absorbing-norms

Absorbing-norms are generalizations of null norms introduced by Rudas [**8**].

Definition 6 Let *A* be a mapping $U:[0,1]\times[0,1]\rightarrow[0,1]$. *A* is an *absorbing-norm*, if it is commutative, associative and there exists *an absorbing element* $a \in [0,1]$, i.e., $A(x,a) = a, \forall x \in [0,1]$.

It is clear that *a* is an idempotent element $A(a,a) = a,$ hence the absorbing element is unique. If there would exist at least two absorbing elements $a_1, a_2, a_1, \neq a_2$ for which $A(a_1,a_2) = a_1,$ and $A(a_1,a_2) = a_2$, so thus $a_1, = a_2$.

T-operators are special absorbing-operators, namely for any t-norm *T*, $T(0,x) = 0, \forall \mathrm{x} \in [0,1]$ and for any t-conorm *S*, $S(1,x) = 1, \forall \mathrm{x} \in [0,1]$.

As a direct consequence of the definition we have if $x \le a$ then $A(x,a) = a = \max(x,a)$, if $x \ge a$ then $A(x,a) = a = \min(x,a)$.

These properties provide the background to define some simple absorbing-norms.

The trivial absorbing-norm $A_T : [0,1]\times[0,1]\rightarrow[0,1]$ *with absorbing element a is*

$$A_T : (\mathrm{x},\mathrm{y}) \rightarrow a, \forall (\mathrm{x},\mathrm{y}) \in [0,1]\times[0,1]. \tag{7}$$

Theorem 1 (Rudas [**8**]) The mapping $A_{\min} : [0,1]\times[0,1]\rightarrow[0,1]$ defined as

$$A_{\min}(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in [0,a]\times[0,a] \\ \min(x,y) & \text{elsewhere} \end{cases} \tag{8}$$

and the mapping $A_{\max} : [0,1]\times[0,1]\rightarrow[0,1]$ *defined as*

$$A_{\max}(x,y) = \begin{cases} \min(x,y), & \text{if } (x,y) \in [a,1]\times[a,1] \\ \max(x,y), & \text{elsewhere} \end{cases} \tag{9}$$

are absorbing-norms with absorbing element a.

Corollary 1 From the structure of $A_{\min}$ and $A_{\max}$ the following properties can be concluded:

$A_{\min}(0,0) = A_{\min}(0,1) = A_{\min}(1,0) = 0$, $A_{\min}(0,0) = 0$,

$A_{\max}(1,1) = A_{\max}(0,1) = A_{\max}(1,0) = 1$, $A_{\max}(1,1) = 1$.

With the combination of $A_{\min}$, $A_{\max}$ and A_T further absorbing-norms can be defined.

Theorem 2 (Rudas, [**8**]) *The mapping* $A_{\min}^a : [0,1]\times[0,1] \to [0,1]$ *defined as*

$$A_{\min}^a(x,y) = \begin{cases} a, & \text{if } (x,y)\in[0,a]\times[0,a] \\ \min(x,y), & \text{elsewhere} \end{cases} \tag{10}$$

and the mapping $A_{\max}^a : [0,1]\times[0,1] \to [0,1]$ *defined as*

$$A_{\max}^a(x,y) = \begin{cases} a, & \text{if } (x,y)\in[a,1]\times[a,1] \\ \max(x,y), & \text{elsewhere} \end{cases} \tag{11}$$

are absorbing-norms with absorbing element *a*.

Theorem 3 (Rudas [**8**]) *Assume that A is an absorbing-norm with absorbing element a. The dual operator of A* $\overline{A}(x,y) = 1 - A(1-x,1-y)$ *is an absorbing-norm with absorbing element* 1-*a.*

Definition 7

$$\overline{(A_{\min})}^{\max}(x,y) = \begin{cases} \min(x,y), & \text{if } (x,y)\in[0,a]\times[0,a] \\ \max(x,y) & \text{elsewhere} \end{cases} \tag{12}$$

$$\overline{(A_{\max})}^{\max}(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y)\in[a,1]\times[a,1] \\ \min(x,y), & \text{elsewhere} \end{cases} \tag{13}$$

We have received the first uninorms given by Yager and Rybalov.

Due to the constructions of these operators for the pairs $(A_{\min}, U_d)$ and $(A_{\max}, U_c)$ the laws of absorption and distributivity are fulfilled.

Let us define a kind of complements of $A_{\min}$ and $A_{\max}$ replacing the operator min with max and the max with min as follows.

Theorem 4. *For the pairs* $(A_{\min}, U_d)$ *and* $(A_{\max}, U_c)$ *the following hold*

1 Absorption laws

$$A_{\min}(U_d(x,z), x) = x \text{ for all } x\in[0,1], \tag{14}$$

$$U_d(A_{\min}(x,z), x) = x \text{ for all } x\in[0,1], \tag{15}$$

$$A_{\max}\left(U_c(x,z),x\right)=x \text{ for all } x\in[0,1] \;, \tag{16}$$

$$U_c\left(A_{\max}(x,z),x\right)=x \text{ for all } x\in[0,1] \;. \tag{17}$$

2 Distributive laws. For all $x\in[0,1]$

$$A_{\min}\left(x,U_d(y,z)\right)=U_d\left(A_{\min}(x,y),A_{\min}(x,z)\right) \;, \tag{18}$$

$$U_d\left(x,A_{\min}(y,z)\right)=A_{\min}\left(U_d(x,y),U_d(x,z)\right) \;, \tag{19}$$

$$A_{\max}\left(x,U_c(y,z)\right)=U_c\left(A_{\max}(x,y),A_{\max}(x,z)\right) \;, \tag{20}$$

$$U_c\left(x,A_{\max}(y,z)\right)=A_{\max}\left(U_c(x,y),U_c(x,z)\right) \;. \tag{21}$$

4.4 Generalized Conjunction and Disjunction Operations

The axiom systems of t-norms and t-conorms are very similar to each other except the neutral element, i.e. the type is characterized by the neutral element. If the neutral element is equal to 1 then the operation is a conjunction type, while if the neutral element is zero the disjunction operation is obtained. By using these properties we introduce the concepts of conjunction and disjunction operations. The followings are based on the work of Batyrshin et al. [9].

Definition 8 Let T be a mapping $T:[0,1]\times[0,1]\to[0,1]$. T is a *conjunction operation* if $T(x,1)=x$ for all $x\in[0,1]$.

Definition 9 Let S be a mapping $S:[0,1]\times[0,1]\to[0,1]$. S is a *disjunction operation* if $S(x,0)=x$ for all $x\in[0,1]$.

Conjunction and disjunction operations may also be obtained one from another by means of an involutive negation N

It can be seen easily, that conjunction and disjunction operations satisfy the following boundary conditions:

$$T(0,0) = T(0,1) = T(1,0) = 0, \tag{22}$$

$$T(1,1) = 1 \tag{23}$$

$$T(0,a) = \mathrm{T}(a,0) = 0 \tag{24}$$

$$S(0,1) = S(1,0) = S(1,1) = 1 \tag{25}$$

$$S(0,0) = 0. \tag{26}$$

$$S(1,a) = S(a,1) = 1 \tag{27}$$

By fixing these conditions new types of generalized operations are introduced.

Definition 10 Let T be a mapping $T:[0,1]\times[0,1]\to[0,1]$. T is a *quasi-conjunction operation* if

$T(0,0) = T(0,1) = T(1,0) = 0$, and

$T(1,1) = 1$. $S(x,y) = N(T(N(x),N(y)))$ and

$T(x,y) = N(S(N(x),N(y)))$.

Definition 11 Let S be a mapping $S:[0,1]\times[0,1]\to[0,1]$. S is a *quasi-disjunction operation* if $S(0,1) = S(1,0) = S(1,1) = 1$, and $S(0,0) = 0$.

It is easy to see that conjunction and disjunction operations are quasi-conjunctions and quasi-disjunctions, respectively, but the converse is not true.

Omitting $T(1,1) = 1$ and $S(0,0) = 0$ from the definitions further generalization can be obtained.

Definition 12 Let T be a mapping $T:[0,1]\times[0,1]\to[0,1]$. T is a *pseudo-conjunction operation* if $T(0,0) = T(0,1) = T(1,0) = 0$.

Definition 13 Let S be a mapping $S:[0,1]\times[0,1]\to[0,1]$. S is a *pseudo-disjunction operation* if $S(0,1) = S(1,0) = S(1,1) = 1$.

Classes of conjunction and disjunction operation can be generated by means of certain generator functions and pseudo-operations. The following discussion is based on the work of Batyrshin et al. [**9**], [**11**].

Theorem 5 *Suppose T_1, T_2 are conjunctions, S_1 and S_2 are functions $S_i:[0,1]\times[0,1]\to[0,1]$ $(i=1,2)$, are non-decreasing pseudo-disjunctions, h, g_1, g_2: $[0,1]\to[0,1]$ are non-decreasing functions such that $g_1(1) = g_2(1) = 1$; then the following functions*

$$T(x,y) = T_2(T_1(x,y), S_1(g_1(x),g_2(y))), \tag{28}$$

Comment: e9

$$T(x,y) = T_2(T_1(x,y), g_1(S_1(x,y))), \tag{29}$$

Comment: e10

$$T(x,y) = T_2(T_1(x,y), S_2(h(x),S_1(x,y))). \tag{30}$$

Comment: e11

are conjunction operations.

Disjunction operations may be generated dually or obtained from conjunctions by means of negation operation.

The conjunction operations given by (28), (29), (30) are commutative if T_1, T_2 and S_1 are commutative, $g_1 = g_2$ in (28), and $h(x) = c$, where c is a constant, $0 \le c \le 1$, in (30).

Some simple parametric conjunctions obtained this way with generators $g(x) = \max(1 - p(1-x), 0)$, $g(x) = x^q$, $h(x) = t$, $(p, q \geq 0,\ 0 \leq t \leq 1)$

are the following

$$T(x,y) = (xy)\max\left(1 - p(1-x), y^q\right), \tag{31}$$

$$T(x,y) = \min(x,y)\left(x^p + y^q - x^p y^q\right), \tag{32}$$

$$T(x,y) = \min(x,y)\max\left(x^p, y^q\right), \tag{33}$$

$$T(x,y) = \min(x,y)(x + y - xy)^p, \tag{34}$$

$$T(x,y) = (xy)\max\left(t, (x + y - xy)^p\right). \tag{35}$$

Comment: e12

Comment: e13

Comment: e14

Comment: e15

Comment: e16

It is easy to see that conjunctions (34) and (35) are commutative. The conjunctions (32) and (33) also became commutative when $p=q$. The permutation of x and y in the left hand sides of (32) and (33) will change the value of $T(x,y)$ when p is not equal to q. But, we can permute x^p and y^q in the right hand sides of conjunctions (32) and (33) and the value of correspondent conjunction $T(x,y)$ will be not changed. Such kind of "commutative law" of conjunction operations will be called generalized commutativity of parametric conjunctions and disjunctions. More exactly, we will say that conjunction (28) *satisfies the property of generalized commutativity* if T_1 and S_1 are commutative, $g(x,p)$ is a generator dependent on parameter p and $g_1(x) = g(x,p_x)$, $g_2(y) = g(y,p_y)$ where p_x and p_y are the values of parameter p. The conjunctions (32) and (33) satisfy the property of generalized commutativity, but the conjunction (31) does not satisfy this property.

A tuning of the parametric conjunctions satisfying the property of generalized commutativity may be started with equal values of the parameters p_x and p_y. In such a case, this conjunction will be commutative and will not depend on the order of its operands. After a separate tuning of the parameters, the values p_x and p_y reached will reflect the influence of these parameters on the performance of the fuzzy model. The example of function approximation based on the optimization of the parameters of the conjunction (32) satisfying the generalized commutativity conditions is considered in [**10**].

4.4.1 Generation of Quasi-Operations

Quasi conjunction and disjunction operations enable the building of simpler parametric conjunction operations, as it is shown by Batyrshin et al. in [9].

Theorem 6 Suppose N is a negation on [0,1] and T, S are some quasi-conjunction and quasi-disjunction operations, respectively; then the following relations define correspondingly quasi-disjunction and quasi-conjunction operations:

$$S_T(x,y) = N(T(N(x), N(y))) \quad T_S(x,y) = N(S(N(x), N(y))).$$

It follows from Theorem 6 that by means of any non-involutive negation it is possible to obtain some quasi-conjunction or quasi-disjunction from the conjunctions and the disjunctions considered in the previous section. However, such an approach does not result in construction of simple operations.

Corollary 2 If N is an involutive negation, then for any quasi-conjunction T and quasi-disjunction S= S_T and for any quasi-disjunction S and quasi-conjunction T= T_S the following De Morgan laws are fulfilled:

$$N(S(x,y)) = T(N(x), N(y)) \quad N(T(x,y)) = S(N(x), N(y))$$

Comment: t3

Theorem 7 *([9]) Suppose T_1 is a quasi-conjunction, S_1 is a quasi-disjunction and f, g, h:[0,1]→[0,1] are non-decreasing functions such that f(0)= g(0) = h(0) =0, f(1)= g(1) = h(1) = 1; then the functions*

$$T(x,y) = f(T_1(g(x),h(y))), \quad S(x,y) = f(S_1(g(x),h(y))),$$

are quasi-conjunction and quasi-disjunction respectively.

The functions *f*, *g* and *h* used in the generation of *quasi*-conjunctions and *quasi*-disjunctions are called generators of these operations. It is clear that *quasi*-conjunctions and *quasi*-disjunctions defined in Theorem 3 are commutative if $g = h$ and T_1, S_1 are respectively commutative. In a similar way as done to the conjunctions considered in the previous sections, we will say that these functions satisfy the property of generalized commutativity if T_1 and S_1 are commutative, $g_1(x,p)$ is a generator dependent on parameter *p* and $g(x) = g_1(x,p_x)$, $h(y) = g_1(y,p_y)$ where p_x and p_y are the values of parameter *p*.

Theorem 7 may be extended on *quasi*-connectives as follows.

Comment: t4

Theorem 8 *Suppose T_1, T_2 are quasi-conjunctions, S_1 and S_2 are functions S_i:[0,1]×[0,1]→[0,1] satisfying (2) and (3), h, g_1, g_2:[0,1]→[0,1] are non-decreasing functions such that $g_1(1)= g_2(1) = 1$; then the following functions*

$$T(x,y) = T_2(T_1(x,y), S_1(g_1(x),g_2(y))),$$

$$T(x,y) = T_2(T_1(x,y), g_1(S_1(x,y))),$$

$$T(x,y) = T_2(T_1(x,y), S_2(h(x),S_1(x,y))),$$

are quasi-conjunctions.

By the use of Theorems 7 and 8, the simplest parametric *quasi*-conjunction operations can be obtained as follows:

$$T(x,y) = x^p y^q, \tag{36}$$

$$T(x,y) = \min(x^p, y^q), \tag{37}$$

$$T(x,y) = (xy)^p (x + y - xy)^q, \tag{38}$$

Comment: e19

Comment: e20

Comment: e21

where $p,q \geq 0$.

The last quasi-conjunction is commutative and the first two ones satisfy the property of generalized commutativity. It is seen that the proposed definition of conjunction and disjunction operations enables one to build the simplest parametric classes of conjunction and disjunction operations. It is to be noted that another system of axioms for generalized connectives is considered in [12] where the fuzzy conjunction $(xy)^k$ is considered and its application to fuzzy modeling is discussed. This conjunction belongs to the parametric class of conjunctions (36) with $p = q$.

5 DISTANCE-BASED OPERATORS

Let e be an arbitrary element of the closed unit interval [0,1] and denote by $d(x,y)$ the distance of two elements x and y of [0,1]. The idea of definitions of distance-based operators is generated from the reformulation of the definition of the min and max operators as follows

$$\min(x,y) = \begin{cases} x, & \text{if } d(x,0) \leq d(y,0) \\ y, & \text{if } d(x,0) > d(y,0) \end{cases}, \quad \max(x,y) = \begin{cases} x, & \text{if } d(x,0) \geq d(y,0) \\ y, & \text{if } d(x,0) < d(y,0) \end{cases}$$

Definition 14 The *maximum distance minimum operator with respect to* $e \in [0,1]$ is defined as

$$\max_e^{\min}(x,y) = \begin{cases} x, & \text{if } d(x,e) > d(y,e) \\ y, & \text{if } d(x,e) < d(y,e) \\ \min(x,y), & \text{if } d(x,e) = d(y,e) \end{cases}. \tag{39}$$

Definition 15 The *maximum distance maximum operator with respect to* $e \in [0,1]$ is defined as

$$\max_e^{\max}(x, y) = \begin{cases} x, & \text{if } d(x,e) > d(y,e) \\ y, & \text{if } d(x,e) < d(y,e) \\ \max(x, y), & \text{if } d(x,e) = d(y,e) \end{cases}. \tag{40}$$

Definition 16 The *minimum distance minimum operator with respect to* $e \in [0,1]$ is defined as

$$\min_e^{\min}(x, y) = \begin{cases} x, & \text{if } d(x,e) < d(y,e) \\ y, & \text{if } d(x,e) > d(y,e) \\ \min(x, y), & \text{if } d(x,e) = d(y,e) \end{cases}. \tag{41}$$

Definition 17 The *minimum distance maximum operator with respect to* $e \in [0,1]$ is defined as

$$\min_e^{\max}(x, y) = \begin{cases} x, & \text{if } d(x,e) < d(y,e) \\ y, & \text{if } d(x,e) > d(y,e) \\ \max(x, y), & \text{if } d(x,e) = d(y,e) \end{cases}. \tag{42}$$

It can be proved by simple computation that the distance-based operators can be expressed by means of the min and max operators as follows.

$$\max_e^{\min} = \begin{cases} \max(x, y), & \text{if } y > 2e - x \\ \min(x, y), & \text{if } y < 2e - x \\ \min(x, y), & \text{if } y = 2e\text{-}x \end{cases} \tag{43}$$

$$\min_e^{\min} = \begin{cases} \min(x, y), & \text{if } y > 2e - x \\ \max(x, y), & \text{if } y < 2e - x \\ \min(x, y), & \text{if } y = 2e\text{-}x \end{cases} \tag{44}$$

$$\max_e^{\max} = \begin{cases} \max(x, y), & \text{if } y > 2e - x \\ \min(x, y), & \text{if } y < 2e - x \\ \max(x, y), & \text{if } y = 2e\text{-}x \end{cases} \tag{45}$$

$$\min_e^{\max} = \begin{cases} \min(x, y), & \text{if } y > 2e - x \\ \max(x, y), & \text{if } y < 2e - x \\ \max(x, y), & \text{if } y = 2e\text{-}x \end{cases} \tag{46}$$

6 Conclusions

In this paper some new approaches to the generalization of triangular operators are given. It is shown that by omitting and/or modifying some axioms from the axiom skeleton new generalized operation families can be obtained. Besides the theoretical discussions of the possible generalizations some concrete operator families are introduced.

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