Possibility of Control of Hybrid Systems

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Abstract:
This paper presents a method for optimal control of hybrid systems. An inequality of Bellman type is considered and every solution to this inequality gives a lower bound on the optimal value function. A discretization of this "hybrid Bellman inequality" leads to a convex optimization problem in terms of finitedimensional linear programming. From the solution of the discretized problem, a value function that preserves the lower bound property can be constructed. An approximation of the optimal feedback control law is given and tried on some examples.

Keywords: hybrid systems, optimal control, dynamic programming.

1. Introduction

Hybrid systems are systems that involve interaction between discrete and continuous dynamics. Such systems have been studied with growing interest and activity in recent years. Very often, the same phenomenon can be described either by a discrete model or a continuous one, depending on the context and purpose of the model [1]. One reason for the interest is that modeling and simulation of a complex system often require a combination of mathematical models from a variety of engineering disciplines. Practical control systems typically involve switching between several different modes, depending on the range of operation. Basic aspects of hybrid systems were treated in [6],[7]. For stability analysis, see [3] and references therein. Related methods were discussed for discrete systems in [2] and on an abstract level for hybrid systems in [4]. This paper presents a novel computational approach to optimal control of hybrid systems, based on ideas from dynamic programming and convex optimization. Discretization of Bellman's inequality gives a lower bound on the optimal cost in terms of linear programming.

2. Problem Formulation

Define a hybrid system as

$$\begin{cases} \dot{x}(t) = f_{q(t)}(x(t), u(t)) \\ q(t) = \nu(x(t), q(t^-), \mu(t)) \end{cases} \tag{1}$$

where $x(t) \in X \subset R^n$ is the state vector, $u(t) \in \Omega_u \subset R^m$ is a continuous input signal of the system.
There is also a discrete input, $\mu(t) \in \Omega_\mu$, which allows for the selection between *N* different system modes, $q(t) \in Q = \{1,2,...,N\}$. , The notation $q(t^-)$ is used for the left-hand limit of *q* at *t*. *Sq,,r* is a set (parameterized by *q* and *)* such that switching from mode *q* to *r* is possible when $x \in S_{q,r} \subseteq X$. The time argument, *t*, will often be omitted in the sequel for readability.
The optimal control problem is to minimize the cost function

$$J(x_0, q_0) = \int_{t_0}^{t_f} l_q(x,u)dt + \sum_{k=1}^{M} s(x(t_k), q(t_k^-), q(t_k^+)) \tag{2}$$

subject to (1) while bringing the system from an initial state (x_0, q_0) at time t_0, to a final state (x_f, q_f) at time t_f, where the end time, t_f, is free. Here, *M* is an arbitrary finite number of switches occurring at times $t_0 < t_1 < t_2 < ... < t_M < t_f$ and $s(q,r,t) > 0$ is an associated cost for switching from discrete state q to r, the continuous part being x just before the switch. Note that $s(\cdot) > 0$ removes the problem of infinitely many jumps in a finite interval. The framework developed in this paper would also allow the number of continuous states to vary with the discrete mode according to
$\dot{x}_q(t) = f_{q(t)}(x_q(t), u_q(t))$,
where $x_q(t) \in X_q \subset R^{n(q)}, u_q(t) \in \Omega_{u_q} \subset R^{m(q)}$. The usage of the system description H1I, however, will hopefully prevent the reader from getting stuck on details.

3. Lower Bounds on Optimal Cost

Let $V_q : X \mapsto R, q = 1,2,..., N$ be a set of continuous, piecewise functions C^1 that satisfy

$$0 \leq \frac{\partial V_q(x)}{\partial x} f_q(x,u) + l_q(x,u) \qquad (3)$$
$$\forall x \in X, u \in \Omega_u, q \in Q$$

$$0 \leq V_r(x) - V_q(x) + s(x,r,q) \qquad (4)$$
$$\forall x \in S_{q,r}; q,r \in Q : q \neq r$$

$$0 = V_{q_f}(x_f) \qquad (5)$$

where $f_q(x,u)$ gives the dynamics of a hybrid system, $l_q(x,u)$ and $s(x,r,q)$ define a cost function for the system. Then, for every $(x_0, q_0), V_{q_0}(x_0)$ gives a lower bound on the cost for optimally bringing the system from (x_0, q_0) to $(x_f, q_f), x(t) \in X \;\; \forall t \in [t_0, t_f]$.

Evidence: Let $\hat{u}(\cdot)$ and $\hat{\mu}(\cdot)$ be control signals that drive the system from (x_0, q_0) at time t_0 to (x_f, q_f) at time $t_f \equiv t_{M+1}$. Let $\hat{q}(t)$ denote the mode trajectory resulting from $\hat{\mu}(t)$ and define $x_k = x(t_k), x_k^- = x(t_k^-),$ and $\hat{q}_k = \hat{q}(t), \; t_k \leq t \leq t_{k+1}$.
Then

$$J(x_0, \hat{q}_0) = \sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} l_{\hat{q}_k}(x, \bar{u})dt + \sum_{k=1}^{M} s(x_k^-, \bar{q}_{k-1}, \bar{q}_k) \geq \sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} -\frac{\partial V_{\bar{q}_k}(x)}{\partial x} f_{\bar{q}_k}(x, \bar{u})dt +$$
$$+ \sum_{k=1}^{M} \left\{ V_{\bar{q}_{k-1}}(x_k^-) - V_{\bar{q}_k}(x_k^-) \right\} = \sum_{k=0}^{M} \left\{ V_{\bar{q}_k}(x_k) - V_{\bar{q}_k}(x_{k+1}) \right\} + \sum_{k=1}^{M} \left\{ V_{\bar{q}_{k-1}}(x_k) - V_{\bar{q}_k}(x_k) \right\} =$$
$$= V_{\bar{q}_0}(x_0) - V_{\bar{q}_M}(x_{M+1}) = V_{\bar{q}_0}(x_0)$$

The optimal value function, $V_q^*(x)$ will meet the the constraints (3)-(5) also, under appropriate interpretation of $\partial V_q(x) / \partial x$. So therefore the inequalities do not introduce any conservatism in the lower bound.

4. Discretization

Impose a computer to solve (3)-(5) for a specific control problem, a approach is to grid the state space to require the disparities to be met at a set of evenly distributed points in *X*. In the case of a two-dimensional continuous state space, introduce the notation

$$x_{jk} = x_f + jhe_1khe_2$$
$$X^{jk} = \{x_{jk} + \theta_1 he_1 + \theta_2 he_2 : 0 \le \theta_i \le 1\}$$

$$\hat{X}^{jk} = \{x_{jk} + \theta_1 he_1 + \theta_2 he_2 : -1 \le \theta_i \le 1\}$$
$$(\underset{-q}{f}^{jk})_i = \min_{x \in \hat{X}^{jk}, u \in \Omega_u} (f_q(x,u))_i$$
$$(\bar{f}_q^{jk})_i = \max_{x \in \hat{X}^{jk}, u \in \Omega_u} (f_q(x,u))_i$$
$$(\underset{-q}{l}^{jk})_i = \min_{x \in \hat{X}^{jk}, u \in \Omega_u} (l_q(x,u))_i$$
$$V_q^{jk} = V_q(x_{jk})$$
$$\Delta_i V_q^{jk} = (V_q(x_{jk} + he_i) - V_q(x_{jk}))/h$$
$$\Delta_{-i} V_q^{jk} = (V_q(x_{jk}) - V_q(x_{jk} - he_i))/h$$

where e_1 and e_2 are unit vectors along the coordinate axes, and h is the grid size.

Introduce new vector variables, $\lambda_q^{jk} \in R^n$ for (*j*, *k*, *q*) such that $x_{jk} \in X, q \in Q$. The disparities (3)-(5) can then be replaced by

$$0 \le (\lambda_q^{jk})_1 + (\lambda_q^{jk})_2 + \underset{-q}{l}^{jk} \qquad (6)$$
$$(\lambda_q^{jk})_{|i|} \le (\underset{-q}{f}^{jk})_{|i|} \Delta_i V_q^{jk} \qquad i = -2,-1,1,2 \qquad (7)$$

$$(\lambda_q^{jk})_{|i|} \le (\bar{f}_q^{jk})_{|i|} \Delta_i V_q^{jk} \qquad i = -2,-1,1,2 \qquad (8)$$
$$0 \le V_r^{jk} - V_q^{jk} + s(x_{jk}, q, r) \qquad \forall x_{jk} \in S_{q,r} \qquad (9)$$
$$0 = V_{qf}^{00} \qquad (10)$$

where (6)-(8) form a combination of backward and forward difference approximations of (3).

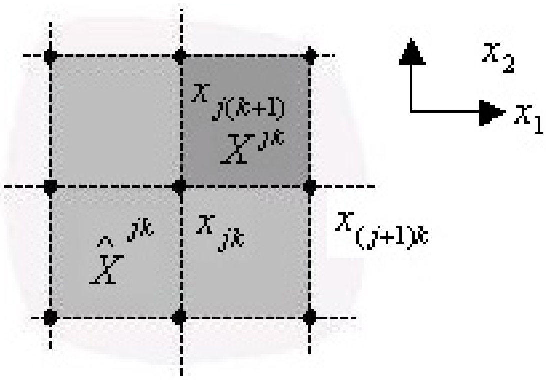

Figure 1: Illustration of X^{jk} and $\hat{X}^{jk}$.

For $x = x_{jk} + \theta_1 he_1 + \theta_2 he_2 \in X^{jk}$, define the interpolating function

$$V_q(x) = (1-\theta_1)(1-\theta_2)V_q^{jk} + \theta_1(1-\theta_2)V_q^{(j+1)k} + \\ + (1-\theta_1)\theta_2 V_q^{j(k+1)} + \theta_1\theta_2 V_q^{(j+1)(k+1)} \tag{11}$$

4.1 Discretization in R^2

If V_q^{jk} satisfy (6)-(10) for all $q \in Q$ and for all grid points $x_{jk} \in X \subset R^2$ such that X^{jk} intersects X , then the interpolating function V_q defined by (11) satisfies (3)-(5) and, for every $(x_0, q_0), V_{q_0}(x_0)$ is a lower bound of $J(x_0, q_0)$.

The gradient of V_q is given [???] by

$$\frac{\partial V_q}{\partial x} = \begin{bmatrix} (1-\theta_2)\Delta_1 V_q^{jk} + \theta_2 \Delta_1 V_q^{j(k+1)} \\ (1-\theta_1)\Delta_2 V_q^{jk} + \theta_1 \Delta_2 V_q^{(j+1)k} \end{bmatrix}^T \tag{12}$$

The disparity (4) is met since V_q is a convex combination of grid points that all meet (9), and (5) is the same condition as (10).

Note a special case in which the computational load of the local optimizations in Discretization in R^2 is lightened: if $f_q(x,u) = h_q(x) + g_q(x)u$ and $l_q(x,u) = o_q(x) + m_q(x)u$ while $\Omega_u = [-1,1]$, then u can be entirely eliminated from (6)-(8) by replacing $\underset{-q}{f}^{jk}$, $\overset{-jk}{f}_q$, and $\underset{-q}{l}^{jk}$ with $\underset{-q}{h}^{jk} \pm \underset{-q}{g}^{jk}$, $\overset{-jk}{h}_q \pm \overset{-jk}{g}_q$, and $\underset{-q}{o}^{jk} \pm \underset{-q}{m}^{jk}$ respectively. This will double the set of equations (6)-(8), but the functions h_q, g_q, o_q and m_q are optimized over $\hat{X}^{jk}$ solely.

5. Computing the Control Law

Provided that the lower bound, V_q, is a good enough approximation of the optimal cost, the optimal feed-back control law can be calculated as

$$\begin{cases} \hat{u}(x,q) = \underset{u \in \Omega_u}{\arg\min} \left\{ \dfrac{\partial V_q}{\partial x} f_q(x,u) + l_q(x,u) \right\} \\ \hat{\mu}(x,q) = \underset{\mu \in \Omega_u | x \in S_{q,v}}{\arg\min} \left\{ V_v(x) + s(x,q,v) \right\} \end{cases} \tag{13}$$

where $v = v(x,q,\mu)$. Thus, the continuous input, $\hat{u}$, is computed in a standard way. The discrete input, $\hat{\mu}$, is chosen such that switching occur whenever there exist a discrete mode for which the value function has a lower value than the cost of the value function for the current mode minus the cost for switching there. Consider the true optimal value function, V_q^*. For those (x,q,r) where the optimal trajectory requires mode switching, the inequality (3) will turn to equality i.e. $V_q^* = V_r^* + s(x,q,r)$ (this will be shown in example. A consequence of this is that for (13) to describe correct switching between the modes, $s(x,q,q)$ has to be defined as $s(x,q,q) = \varepsilon > 0$ (rather than the real cost $s(x,q,q) = 0$. For V_q^*, the proper control law is achieved as ε approaches 0^+. A small value of ε suffices, however, for numerical computations. Integration of (2) along a simulated trajectory based on (13) will provide an upper bound on the optimal cost. The better the control law, the better the estimate.

6. Example

Consider the system

$$\begin{cases} x_1' = x_2 \\ x_2' = g_q(x_2)u; q = 1,2; |u| \leq 1 \end{cases} \tag{14}$$

where $g_q(x)$ is plotted in Fig. 2. This system represented A car with two gears. This could be seen as a crude model of a car, u being the throttle, $g_q(x)$ the efficiency for gear number q.

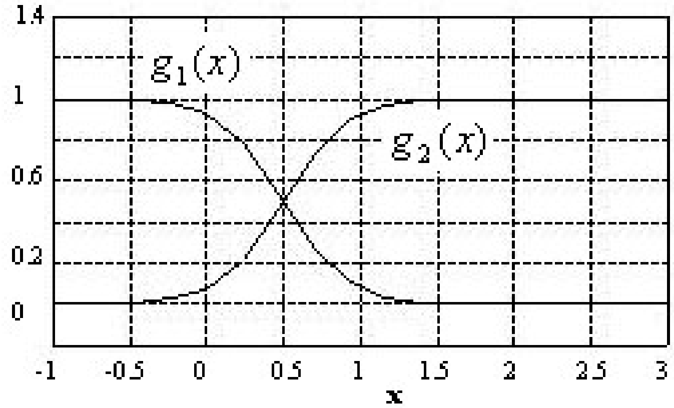

Figure 2: Gear efficiency at various speeds.

The problem is to bring (14) from $x_i = (-5,0), q_i = 1$ to $x_f = (0,0), q_f = 1$ in minimum time. Torque losses when using the clutch calls for an additional penalty for gear changes. Thus, the components of (2) have been chosen as $l_1(x,u) = l_2(x,u) = 1, s(x,2,1) = 0.5$.

The problem is plugged into the machinery of Section 4 and $V_q(x)$ is maximized over a region $-5.5 \leq x_1 \leq 1, -0.5 \leq x_2 \leq 3$. Figure 2 reveals that the first gear is almost useless for high speeds, leading to $V_1 = V_2 + 0.5$ for $x_2 > 1$. This is the cost for using the second gear optimally after a gear switch.

Studying Fig. 3, where $V_1 - V_2$ is plotted, the strategy for changing gears is even more obvious: there is only one discrete mode allowed under optimal control when the difference hits its maximum distance. In conformity with previous reasoning, $V_1 - V_2 = 0.5$ for $x_2 > 1$, indicating the need for a change of gears when using the first gear at high speed. Analogously, the second gear should be avoided, starting with zero speed.

A simulation of the controlled system is shown in Fig. 4, where the initial point is marked with a square. The state trajectory coincides with the one of a professional rally-driver with lousy brakes.

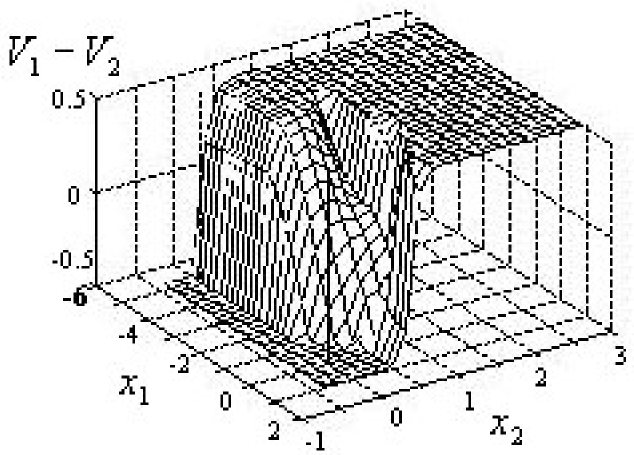

Figure 3: The difference between *V*1 and *V*2.

In the beginning, maximum throttle is used on the first gear (solid line). When the speed roughly reaches the point of equal efficiency between the gears $(x_2 = 0.5)$, they are switched in favor of the second gear (dashed line). At half the distance, the gas pedal is lightened to use the braking force of the engine. In the end, the first gear is used again before the origin is hit. As seen in the figure, the granularity of the discretization grid $(h = 0.18)$ prevents the solution from hitting the exact origin.

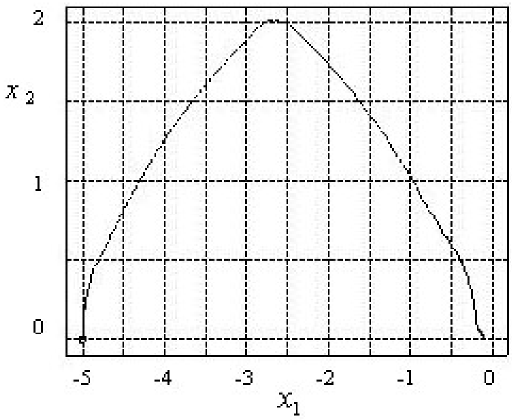

Figure 4: Phase portrait of a simulation. The solid line shows where gear number one has been used, the dashed line shows the second gear. The initial point is marked with a square.

Conclusion

Hybrid systems combine discrete and continuous dynamics. The An extended version of Bellman's inequality was discretized in this paper to compute a lower bound on the optimal cost function, using linear programming. analysis should therefore contain techniques that are well suited for computer science as well as control theory. The emphasis in this paper is on the continuous part, the discrete part consisting of a few system modes. At the other end of the hybrid spectrum, where purely discrete systems are found, X will reduce to a single point. The first inequality of proposition 1 will then be superfluous. The set of inequalities given by (4), possibly large depending on Q, should be met for $S_{q,r} = \{x_f\}$.

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