Some Classes of Associative Binary Operations in Fuzzy Set Theory*

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Abstract: The main aim of this paper is to summarize recent advances on some new classes of associative operations (uninorms, nullnorms, t-operators) used in fuzzy set theory.

Keywords: associativity, uninorm, nullnorm, t-operator.

1 Introduction

Nowadays it is needless to define t-norms and t-conorms in papers related to theoretical or practical aspects of fuzzy sets and logic: researchers have learned the basics and these notions have became part of their everyday scientific vocabulary [13]. Nevertheless, from time to time it is necessary to summarize recent developments even in such a fundamental subject. This is the main aim of the present paper. Somewhat subjectively, we have selected topics where, on one hand, essential contributions have been made, and on the other hand, both theoreticians and practitioners may find these subjects interesting and useful.

In this paper we concentrate on associative operations that are more general than t-norms and t-conorms. These extensions are based on a flexible choice of the neutral (unit) element, or the absorbing element of an associative operation. The resulted classes are known as *uninorms* and *nullnorms* (in other terminology: *t-operators*), respectively.

2 Uninorms

Uninorms were introduced in [20] as a generalization of t-norms and t-conorms. For uninorms, the neutral element is not forced to be either 0 or 1, but can be any value in the unit interval.

Definition 1 ([20]**).** A uninorm U is a commutative, associative and increasing binary operator with a neutral element $e \in [0,1]$, i.e., for all $x \in [0,1]$ we have $U(x,e) = x$. □

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It is interesting to notice that uninorms U with a neutral element in $]0,1[$ are just those binary operators which make the structures $([0,1],\sup,U)$ and $([0,1],\inf,U)$ distributive semirings in the sense of Golan [7]. It is also known (see e.g. [10]) that in MYCIN-like expert systems *combining functions* are used to calculate the global degrees of suggested diagnoses. A careful study reveals that such combining functions are *representable uninorms* [3].

T-norms do not allow low values to be compensated by high values, while t-conorms do not allow high values to be compensated by low values. Uninorms may allow values separated by their neutral element to be aggregated in a compensating way. The structure of uninorms was studied by Fodor *et al.* [6].

For a uninorm U with neutral element $e \in]0,1]$, the binary operator T_U defined by

$$T_U(x,y) = \frac{U(e\,x, e\,y)}{e}$$

is a t-norm; for a uninorm U with neutral element $e \in [0,1[$, the binary operator S_U defined by

$$S_U(x,y) = \frac{U(e+(1-e)x, e+(1-e)y)-e}{1-e}$$

is a t-conorm. The structure of a uninorm with neutral element $e \in]0,1[$ (these are called *proper uninorms*) on the squares $[0,e]^2$ and $[e,1]^2$ is therefore closely related to t-norms and t-conorms. For $e \in]0,1[$, we denote by ϕ_e and ψ_e the linear transformations defined by $\phi_e(x) = \frac{x}{e}$ and $\psi_e(x) = \frac{x-e}{1-e}$. To any uninorm U with neutral element $e \in]0,1[$, there corresponds a t-norm T and a t-conorm S such that:

(i) for any $(x,y) \in [0,e]^2$: $U(x,y) = \phi_e^{-1}(T(\phi_e(x),\phi_e(y)))$;
(ii) for any $(x,y) \in [e,1]^2$: $U(x,y) = \psi_e^{-1}(S(\psi_e(x),\psi_e(y)))$.

On the remaining part E of the unit square it satisfies

$$\min(x,y) \le U(x,y) \le \max(x,y),$$

and could therefore partially show a compensating behaviour, i.e., take values strictly between minimum and maximum. Note that any uninorm U is either *conjunctive*, i.e., $U(0,1) = U(1,0) = 0$, or *disjunctive*, i.e., $U(0,1) = U(1,0) = 1$.

2.1 Representable Uninorms

In analogy to the representation of continuous Archimedean t-norms and t-conorms in terms of additive generators, Fodor *et al.* [6] have investigated the existence of uninorms with a similar representation in terms of a single-variable function. This search leads back to Dombi's class of *aggregative operators* [4]. This work is also closely related to that of Klement *et al.* on associative compensatory operators [12]. The result is as follows.

Theorem 1 ([6]). *Consider a uninorm U with neutral element $e \in]0,1[$, then there exists a strictly increasing continuous $[0,1] \to \overline{\mathbb{R}}$ mapping h with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$ such that*

$$U(x,y) = h^{-1}(h(x) + h(y))$$

for any $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ if and only if

(i) U is strictly increasing and continuous on $]0,1[^2$;
(ii) there exists a strong negation N with fixpoint e such that

$$U(x,y) = N(U(N(x),N(y))))$$

for any $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$. □

The strong negation corresponding to a representable uninorm U with additive generator h, as mentioned in condition (ii) above, is denoted N_U and is given by $N_U(x) = h^{-1}(-h(x))$.

Clearly, any representable uninorm comes in a conjunctive and a disjunctive version, i.e., there always exist two representable uninorms that only differ in the points $(0,1)$ and $(1,0)$. Representable uninorms are almost continuous, i.e., continuous except in $(0,1)$ and $(1,0)$.

2.2 Continuous Uninorms on the Open Unit Square

It is clear from [6] that a proper uninorm cannot be continuous on $[0,1]^2$. Therefore, Hu and Li [11] studied uninorms that are continuous on the *open* unit square $]0,1[^2$. Their results can be reinterpreted as follows.

Theorem 2 ([11]). *A uninorm with neutral element $e \in]0,1[$ is continuous on $]0,1[^2$ if and only if one of the following two conditions is satisfied:*

(a) There exists $a \in [0,e[$ so that

$$U(x,y) = \begin{cases} U^*(x,y) & \text{if } x,y \in [a,1] \\ \min(x,y) & \text{otherwise,} \end{cases}$$

where U^* is a representable uninorm with neutral element $a + (1-a) \cdot e$.

(b) There exists $b \in]e,1]$ so that

$$U(x,y) = \begin{cases} U^*(x,y) & \text{if } x,y \in [0,b] \\ \max(x,y) & \text{otherwise,} \end{cases}$$

where U^* is a representable uninorm with neutral element $b \cdot e$. □

3 Nullnorms (t-operators)

In [1] we studied two functional equations for uninorms. One of those required to introduce a new family of associative binary operations on $[0,1]$ as follows.

Definition 2 ([1]). A *nullnorm* V is a commutative, associative and increasing binary operator with an absorbing element $a \in [0,1]$ (i.e., $V(x,a) = a$ for all $x \in [0,1]$), and that satisfies

- $V(x,0) = x$ for all $x \in [0,a]$,
- $V(y,1) = y$ for all $y \in [a,1]$. □

On the other hand, in [2] the following notion was also defined, with the superfluous requirement of continuity [14].

Definition 3 ([2,14]). A *t-operator* is a two-place function $F : [0,1] \times [0,1] \to [0,1]$ which is associative, commutative, non-decreasing in each place and such that

- $F(0,0) = 0$; $F(1,1) = 1$;
- the sections $x \mapsto F(x,0)$ and $x \mapsto F(x,1)$ are continuous on $[0,1]$. □

It turns out that Definitions 2 and 3 yield exactly the same operations; that is, t-operators coincide with nullnorms. When $a = 1$ we obtain t-conorms, while $a = 0$ gives back t-norms. The basic structure of nullnorms is similar to that of uninorms, as we state in the following theorem.

Theorem 3 ([1]). *A binary operation V on $[0,1]$ is a nullnorm with absorbing element $a \in]0,1[$ if and only if there exists a t-norm T_V and a t-conorm S_V such that*

$$V(x,y) = \begin{cases} a \cdot S_V(\frac{x}{a}, \frac{y}{a}) & \text{if } x,y \in [0,a], \\ a + (1-a) \cdot T_V\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } x,y \in [a,1], \\ a & \text{otherwise.} \end{cases}$$

Thus, nullnorms are generalizations of the well-known *median* operator (when $T_V = \min$ and $S_V = \max$). Further important properties of nullnorms (such as duality and self-duality, classification) can be found in [14].

Recall that classes of uninorms and t-operators satisfying some algebraic properties (such as modularity, distributivity, reversibility) as well as their structure on finite totally ordered scales were studied in [18,15–17]. Note also that some families of associative operations based on distance (and are closely related to uninorms and nullnorms) can be found in [19].

Recently, the use of unipolar and bipolar scales, and their effect on the definition of associative operations on such scales were also studied in the literature. The interested reader can find important results in [8,5,9].

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