Shepard Approximation of Fuzzy Input Fuzzy Output Functions

Barnabás Bede[1], Hajime Nobuhara[2], Imre J. Rudas[1], Kaoru Hirota[2]

[1]Department of Mechanical and System Engineering, Budapest Tech
Népszinház u. 8, H-1081 Budapest, Hungary,
e-mail: bbede@uoradea.ro

[2]Department of Computational Intelligence and Systems Science,
Tokyo Institute of Technology,
4259 Nagatsuta, Midoriku, Yokohama 226-8502, Japan,
e-mail: {nobuhara,hirota}@hrt.dis.titech.ac.jp

Abstract. Two extensions of classical Shepard operators to the fuzzy case are presented. We study Shepard-type interpolation/approximation operators for functions with domain and range in the fuzzy number's space and max-product approximation operators. Error estimates are obtained in terms of the modulus of continuity.

1 Introduction

In [16] it is proposed the problem of interpolating some fuzzy data. Since then many results in this sense are obtained. For the crisp input fuzzy output case (i.e. fuzzy-number-valued functions), in [7] and [10], the Lagrange interpolation polynomials are constructed, in [5] and [9] Bernstein approximation is studied and in [1] Jackson-type trigonometric polynomials are presented. Recently, in [2], Shepard approximation operators are extended to the case of fuzzy-number-valued functions, existence of best approximation and convergence of Lagrange interpolation polynomials are studied in the fuzzy case of crisp input fuzzy output case.

The idea of exploiting approximation capabilities of fuzzy systems in practice is present in the literature from the very beginning of fuzzy control. An important approach to fuzzy control is the interpolative control. Controllers of this type are the Kóczy-Hirota interpolators based on Shepard-type operators (see [8], [15]). Other types of interpolators can be found in e.g. [12], [6]. For these controllers/interpolators, the inputs and the outputs are both fuzzy sets (usually not necessarily fuzzy numbers). Approximation of fuzzy input fuzzy output functions by max t-norm compositions is another approach and it is used in fuzzy control (see [11]). Some results on this type of approximation can be found in the recent papers [13] and [3].

In many practical problems the inputs and/or the outputs are fuzzy numbers. Then naturally raises the problem of interpolating functions which have their range and/or domain in the set of fuzzy numbers. We study this problem in Section 2. An approach based on max-product compositions is presented in Section 3 and then some conclusions and further research topics are pointed out.

2 The Shepard approximation operator for fuzzy input fuzzy output functions

Firstly, let us recall some known concepts and results. Let $\mathbb{R}_{\mathcal{F}}$ be the space of fuzzy numbers (i.e. normal, convex, upper semicontinuous, compactly supported fuzzy sets of the real line). For $0 < r \leq 1$ and $u \in \mathbb{R}_{\mathcal{F}}$ we define $[u]^r = \{x \in \mathbb{R};\ u(x) \geq r\}$ and $[u]^0 = \overline{\{x \in \mathbb{R};\ u(x) > 0\}}$. Then it is well-known that for each $r \in [0,1]$, $[u]^r$ is a bounded closed interval, denoted by $[u]^r = [u_-^r, u_+^r]$, and for $u, v \in \mathbb{R}_{\mathcal{F}}$, $\lambda \in \mathbb{R}$, the sum $u \oplus v$ and the product $\lambda \cdot u$ are defined by $[u \oplus v]^r = [u]^r + [v]^r$, $[\lambda \cdot u]^r = \lambda[u]^r$, $\forall r \in [0,1]$, where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$.

Let $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$ defined by $D(u,v) = \sup_{r \in [0,1]} \max\{|u_-^r - v_-^r|, |u_+^r - v_+^r|\}$ the Hausdorff distance between fuzzy numbers. The following properties are well-known :

$D(u \oplus w, v \oplus w) = D(u,v)$, $\forall u, v, w \in \mathbb{R}_{\mathcal{F}}$,

$D(k \cdot u, k \cdot v) = |k| D(u,v)$, $\forall k \in \mathbb{R}$, $u, v \in \mathbb{R}_{\mathcal{F}}$,

$D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e)$, $\forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}$

and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

We denote by $B(u,r) = \{v \in K : D(u,v) < r\} \subset \mathbb{R}_{\mathcal{F}}$ the open ball having center $u \in \mathbb{R}_{\mathcal{F}}$ and radius r.

Let us recall the definition and some properties of the modulus of continuity (see e.g. [4]).

Definition 1 *(i) Let (X, d_1), (Y, d_2) be metric spaces and let $f : X \to Y$ be a continuous function. Then the function $\omega(f, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\omega(f, \delta) = \sup\{d_2(f(x), f(y));\ x, y \in X,\ d_1(x,y) \leq \delta\}$$

is called the modulus of continuity of f.

Theorem 2 *The following properties hold true*

i) $d_2(f(x), f(y)) \leq \omega(f, d_1(x,y))$ for any $x, y \in X$;

ii) $\omega(f, \delta)$ is nondecreasing in δ;

iii) $\omega(f, 0) = 0$;

iv) $\omega(f, \delta_1 + \delta_2) \leq \omega(f, \delta_1) + \omega(f, \delta_2)$ for any $\delta_1, \delta_2, \in \mathbb{R}_+$;

v) $\omega(f, n\delta) \leq n\omega(f, \delta)$ for any $\delta \in \mathbb{R}_+$ and $n \in \mathbb{N}$;

vi) $\omega(f, \lambda\delta) \leq (\lambda + 1) \cdot \omega(f, \delta)$ for any $\delta, \lambda \in \mathbb{R}_+$;

vii) If f is continuous then $\lim_{\delta \to 0} \omega(f, \delta) = 0$.

In our case we will consider that both the input space and the output space is $\mathbb{R}_{\mathcal{F}}$. So we propose the problem of approximation (interpolation) of a function $f : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ given a rule base, i.e. sampled data $(u_i, f(u_i)) \in \mathbb{R}^2_{\mathcal{F}}$, $i = 1, ..., n$. Usually the rule base appears in practice as a list of fuzzy IF-THEN rules of the form

$$\text{IF } u_i \text{ THEN } f(u_i),$$

which govern some process and they are used for the construction of a fuzzy controller. The fuzzy numbers u_i will be called inputs and the fuzzy numbers $f(u_i)$ will be called the outputs. The form of the Shepard approximation operator in this case is suggested by the form of crisp Shepard approximation operator and Kóczy-Hirota interpolators (see [14], [8], [15]).

For $\lambda \in \mathbb{R}$ we consider $Sh_n^\lambda : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ defined for any $u \in \mathbb{R}_{\mathcal{F}}$ as

$$Sh_n^\lambda(u) = \begin{cases} \sum_{i=1}^n \frac{\frac{1}{D(u,u_i)^\lambda}}{\sum_{i=1}^n \frac{1}{D(u,u_i)^\lambda}} \cdot f(u_i), \text{ if } u \notin \{u_1, ..., u_n\}, \\ f(u_i) \text{ if } u \in \{u_1, ..., u_n\}. \end{cases}.$$

Firstly we study some properties of the above defined approximation operator.

Proposition 3 *The following properties hold true:*
(i) $Sh_n^\lambda(u)$ is a fuzzy number for any $u \in \mathbb{R}_{\mathcal{F}}$;
(ii) Sh_n^λ is compatible with the rule base;
(iii) Sh_n^λ is continuous;
(iv) $[Sh_n^\lambda]^r = \sum_{i=1}^n \frac{\frac{1}{D(u,u_i)^\lambda}}{\sum_{i=1}^n \frac{1}{D(u,u_i)^\lambda}} \cdot [f(u_i)]^r$ for any $r \in [0,1]$ (here addition and multiplication denote standard interval operations);
(v) Sh_n^λ preserves the shape of fuzzy numbers in the sense that if all the outputs are triangular (or even $L-R$) fuzzy numbers, then $Sh_n^\lambda(u)$ is also a triangular ($L-R$) fuzzy number.

Proof. The proofs of (i), (ii) and (iv) are obvious by the definition of the operator Sh_n^λ.

For (iii) we observe that for $u \notin \{u_1, ..., u_n\}$ continuity of $Sh_n^\lambda(u)$ is obvious since the Hausdorff distance is a continuous function. Also it is easy to check that $\lim_{u \to u_i} Sh_n^\lambda(u) = f(u_i)$ and so it is continuous.

For (iv) we observe that a linear combination of triangular $(L-R)$ fuzzy numbers is again a triangular $(L-R)$ fuzzy number. ■

In order to obtain an estimate for the approximation error, the target function f will be considered on a compact subset of $\mathbb{R}_{\mathcal{F}}$. Then we get the following result.

Theorem 4 *Let $f : K \to \mathbb{R}_{\mathcal{F}}$, where $K \subset \mathbb{R}_{\mathcal{F}}$ is a compact. Let $(u_i, f(u_i)) \in \mathbb{R}^2_{\mathcal{F}}$, $i = 1, ..., N$ be a rule base and let $m \in \mathbb{N}$ be the greatest number such that the balls $B\left(u_i, \frac{1}{m}\right)$ cover K. Then for any $\lambda > 2$ there exists a rule base $(u_i, f(u_i)) \in \mathbb{R}^2_{\mathcal{F}}$, $i = 1, ..., n$ ($n \leq N$) selected from the original rule base, such that*

$$D(f(u), Sh_n^\lambda(u)) \leq (1+n) \cdot \omega\left(f, \frac{1}{m}\right).$$

Proof. Let $u \in K$ be a fixed element. Let $j_0 \in \{1, ..., N\}$ be such that $u \in B\left(u_{j_0}, \frac{1}{m}\right)$, i.e. $D(u, u_{j_0}) < \frac{1}{m}$. Let now $u_1, ..., u_n$ be the sequence of points wich is obtained from the original sequence $u_1, ...u_N$ by eliminating the points u_l, $l \in \{1, ..., N\}$, $l \neq j_0$ such that $D(u, u_l) < \frac{1}{m}$. Then for $i \neq j_0$, $i \in \{1, ..., n\}$ for the new sequence we have $D(u, u_i) \geq \frac{1}{m}$. Now let $Sh_n^\lambda : K \to \mathbb{R}_{\mathcal{F}}$,

$$Sh_n^\lambda(u) = \sum_{i=1}^{n} \frac{\frac{1}{D(u,u_i)^\lambda}}{\sum_{i=1}^{n} \frac{1}{D(u,u_i)^\lambda}} \cdot f(u_i).$$

In what follows we estimate the distance $D(Sh_n^\lambda(u), f(u))$. By using the properties of the Hausdorff distance we have

$$\begin{aligned} D(Sh_n^\lambda(u), f(u)) &= D\left(\sum_{i=1}^{n} \frac{\frac{1}{D(u,u_i)^\lambda}}{\sum_{i=1}^{n} \frac{1}{D(u,u_i)^\lambda}} \cdot f(u_i), \sum_{i=1}^{n} \frac{\frac{1}{D(u,u_i)^\lambda}}{\sum_{i=1}^{n} \frac{1}{D(u,u_i)^\lambda}} \cdot f(u)\right) \\ &\leq \sum_{i=1}^{n} \frac{\frac{1}{D(u,u_i)^\lambda}}{\sum_{i=1}^{n} \frac{1}{D(u,u_i)^\lambda}} \cdot D(f(u_i), f(u)). \end{aligned}$$

By the properties of the modulus of continuity we get

$$\begin{aligned} D(Sh_n^\lambda(u), f(u)) &\leq \sum_{i=1}^{n} \frac{\frac{1}{D(u,u_i)^\lambda}}{\sum_{i=1}^{n} \frac{1}{D(u,u_i)^\lambda}} \cdot \omega(f, D(u, u_i)) \\ &= \sum_{i=1}^{n} \frac{\frac{1}{D(u,u_i)^\lambda}}{\sum_{i=1}^{n} \frac{1}{D(u,u_i)^\lambda}} \cdot \omega\left(f, \frac{mD(u, u_i)}{m}\right) \\ &\leq \sum_{i=1}^{n} \frac{\frac{1}{D(u,u_i)^\lambda}}{\sum_{i=1}^{n} \frac{1}{D(u,u_i)^\lambda}} (1 + mD(u, u_i)) \cdot \omega\left(f, \frac{1}{m}\right) \\ &= \left(1 + m\frac{\sum_{i=1}^{n} \frac{1}{D(u,u_i)^{\lambda-1}}}{\sum_{i=1}^{n} \frac{1}{D(u,u_i)^\lambda}}\right) \cdot \omega\left(f, \frac{1}{m}\right). \end{aligned}$$

Since $D(u, u_{j_0}) < \frac{1}{m}$ we get

$$\begin{aligned} D(Sh_n^\lambda(u), f(u)) &\leq \left(1 + m\frac{\sum_{i=1}^{n} \frac{D(u,u_{j_0})^\lambda}{D(u,u_i)^{\lambda-1}}}{\sum_{i=1}^{n} \frac{D(u,u_{j_0})^\lambda}{D(u,u_i)^\lambda}}\right) \cdot \omega\left(f, \frac{1}{m}\right) \\ &\leq \left(1 + m\frac{D(u, u_{j_0}) + \sum_{i \neq j_0} \frac{D(u,u_{j_0})^\lambda}{D(u,u_i)^{\lambda-1}}}{1 + \sum_{i \neq j_0} \frac{D(u,u_{j_0})^\lambda}{D(u,u_i)^\lambda}}\right) \cdot \omega\left(f, \frac{1}{m}\right) \\ &\leq \left(1 + m\frac{\frac{1}{m} + \frac{1}{m^\lambda} \sum_{i \neq j_0} \frac{1}{D(u,u_i)^{\lambda-1}}}{1 + \sum_{i \neq j_0} \frac{D(u,u_{j_0})^\lambda}{D(u,u_i)^\lambda}}\right) \cdot \omega\left(f, \frac{1}{m}\right). \end{aligned}$$

Finally since for $i \neq j_0$ we have $D(u, u_i) \geq \frac{1}{m}$ and finally we get

$$D(Sh_n^\lambda(u), f(u)) \leq \left(1 + m\frac{\frac{1}{m} + \frac{1}{m^\lambda}(n-1)m^{\lambda-1}}{1 + \sum_{i \neq j_0} \frac{D(u,u_{j_0})^\lambda}{D(u,u_i)^\lambda}}\right) \cdot \omega\left(f, \frac{1}{m}\right)$$
$$\leq (1+n) \cdot \omega\left(f, \frac{1}{m}\right).$$

■

The existence of a rule base such that the conditions of the previous theorem are fulfilled is shown as follows. Since K is a compact subset of a complete metric space, it is also totally bounded and it follows that for any $m \in \mathbb{N}$ there exist points $u_1, ...u_k \in K$ such that $\bigcup_{i=1}^k B\left(u_i, \frac{1}{m}\right) = K$, where $B(u, r) = \{v \in K : D(u, v) < r\}$ denotes the open ball having center u and radius r in K.

If the inputs are crisp we obtain the approximation operators given in [2] for fuzzy-number-valued functions.

The error estimate in the previous theorem is not very practical since the number n is generally not known. Also, the estimate is not sharp, however in some situations it can be useful.

The condition that the target function is defined on a compact is not obviously satisfied, even by closed balls. Indeed, since fuzzy number's space endowed with the Hausdorff distance is not locally compact, even closed balls of it are not necessarily compact. However, for example, a closed ball in the set of triangular numbers is a compact set.

3 Shepard type max-product approximation operators

Another way of extending the Shepard approximation operators can be found in the recent paper [3]. Here max-product Shepard-type and exponential operators are defined and studied. In this case the target function is $f : K \to \mathcal{F}(Y)$, where $K \subset \mathbb{R}_\mathcal{F}$ is a compact set, (Y, d) is a compact metric space and $\mathcal{F}(Y)$ denotes the collection of all fuzzy subsets of Y. In this case $\mathcal{F}(Y)$ is endowed with the uniform distance, i.e. for $A, B \in \mathcal{F}(Y)$ the distance between A and B is considered $\|A - B\| = \sup_{y \in Y} |A(y) - B(y)|$. Let also $(u_i, f(u_i)) \in K \times \mathcal{F}(Y)$, $i = 1, ..., n$ be a rule base. We define

$$S_n^\lambda(u) = \bigvee_{i=0}^{n} \frac{\frac{1}{D(u,u_i)^\lambda}}{\bigvee_{i=0}^{n} \frac{1}{D(u,u_i)^\lambda}} \cdot f(u_i)$$

and

$$E_n^\lambda(A) = \bigvee_{i=0}^{n} \frac{e^{-\lambda D(u,u_i)}}{\bigvee_{i=0}^{n} e^{-\lambda D(u,u_i)}} \cdot f(u_i)$$

the Shepard-type and exponential max-product approximation operators (here the multiplication of a fuzzy set $B \in \mathcal{F}(Y)$ by a crisp real in $\alpha \in [0,1]$ interval is defined pointwise, i.e. $(\alpha \cdot B) \in \mathcal{F}(Y)$ is defined by $(\alpha \cdot B)(y) = \alpha \cdot B(y)$).

For a given $y \in Y$ we have

$$S_n^{\lambda}(u)(y) = \bigvee_{i=0}^{n} \frac{\frac{1}{D(u,u_i)^{\lambda}} \cdot f(u_i)(y)}{\bigvee_{i=0}^{n} \frac{1}{D(u,u_i)^{\lambda}}}$$

and

$$E_n^{\lambda}(u)(y) = \bigvee_{i=0}^{n} \frac{e^{-\lambda D(u,u_i)} \cdot f(u_i)(y)}{\bigvee_{i=0}^{n} e^{-\lambda D(u,u_i)}}.$$

In [3] the following error estimate is obtained

$$\left\| S_n^{\lambda}(u) - f(u) \right\| \leq \left(m \bigwedge_{i=0}^{n} D(u,u_i) + 1 \right) \omega \left(f, \frac{1}{m} \right),$$

for any $m \in \mathbb{N}$, which is more practical than the estimate in Theorem 4, but it is in the uniform distance which is not so distinctive as the Hausdorff distance. The max-product approximation operators are continuous and they are more general then the operators presented in Section 2, since the outputs in the first case need to be fuzzy numbers while in this second case the outputs can be fuzzy sets on an arbitrary compact metric space. Let us remark that in the case of both approximation operators presented in this paper the inputs are not restricted by additional conditions, while in the case of KH-interpolators the inputs must fulfill some additional condition.

4 Conclusions and further research

Two possible ways of extending Shepard operator for approximation of fuzzy input fuzzy output functions are presented and studied. These have both interpolation and approximation properties and are inspired by crisp Shepard operators and Kóczy-Hirota interpolators. In the first approach both the inputs and the outputs are fuzzy numbers while in the second case only the inputs are fuzzy numbers. Some approximation properties are studied for both cases.

For further research we propose the implementation of the above defined interpolators, numerical experiments on their effectiveness and also design of fuzzy controllers based on these operators.

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