Transitivity Preserving Aggregation of Preferences*

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Abstract: In this paper we summarize some of our results on aggregation of individual fuzzy preferences. We focus on aggregation functions that preserve some types of transitivity, by giving complete characterization and representation of two main aggregation classes. These are closely related to the weighted maximum and minimum operations. We also establish full characterization of these weighted forms via stability properties of the operations. These properties are expressed by functional equations. Solutions of these equations correspond exactly to the studied operations.

Keywords: individual and group preferences, Pareto principle, transitivity preserving aggregation rules, weighted maximum and minimum.

1 Introduction

Aggregation of binary preference relations is an important and challenging mathematical problem in such applied areas as operations research, social choice, group choice, multiple-criteria decision-making, etc. Perhaps, in its purest form this problem can be formulated in terms of group choice theory. Suppose A is a finite set of alternatives and $\Re = \langle R_1, \ldots, R_n \rangle$ is an ordered n-tuple of binary relations on A. Elements of $\Re$ are regarded as *individual preferences* and $\Re$ is called a *profile* of individual preferences on the set A. For a given A, an *aggregation rule* M assigns a *group preference* R to each profile $\Re$ of individual preferences on A (very often it is assumed that $n > 2$): $R = M(R_1, \ldots, R_n)$.

Depending on the application area, various restrictions are imposed on R. In the framework of group and social choice theories, the *Pareto principle* is considered to be a fundamental consistency property of any aggregation rule. According to [7], "... no one objects to the Pareto principle; on the contrary, procedures which violate it are considered unsatisfactory." In set-theoretic

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terms the Pareto principle can be written as

$$\bigcap_{i=1}^{n} R_i \subseteq R \subseteq \bigcup_{i=1}^{n} R_i \tag{1}$$

for any profile $\langle R_1, \ldots, R_n \rangle$ of individual preferences.

We denote $R_{\min} = \bigcap_i R_i$ and $R_{\max} = \bigcap_i R_i$. These binary relations are two extreme cases of aggregation rules satisfying the Pareto principle. The first one is very conservative: $aR_{\min}b$ if and only if aR_ib for all $i \in \{1, 2, \ldots, n\}$. The second is very liberal: $aR_{\max}b$ if and only if aR_ib for some $i \in \{1, 2, \ldots, n\}$. A reasonable aggregation rule should produce R somewhere "in the middle" [7].

Exactly the same can be said when the individual preferences are binary *fuzzy* relations on A. Moreover, there are some almost indispensable properties of preferences (like diverse forms of transitivity, completeness, etc) which are usually required.

Therefore, in the sequel we consider strongly complete and negatively transitive fuzzy preferences. We would like to find an aggregation rule that preserves these two properties. That is, let $R_1, \ldots, R_m$ be strongly complete and negatively transitive binary fuzzy relations on A. Find the class of all aggregation operators $M : [0,1]^m \to [0,1]$ such that $R = M(R_1, \ldots, R_m)$ is a strongly complete and negatively transitive binary fuzzy relation on A. We completely characterize this class, which turns out to be a generalization of the class of weighted maximum operators defined and investigated by Dubois and Prade [2]. The complete characterization of weighted maximum and minimum operations is established then. The results are obtained via stability properties of the operations.

2 Transitivity preserving aggregations

Transitivity of preferences expresses a kind of consistency. Extensions of classical forms for fuzzy relations, together with some other properties, can be formulated as follows.

Definition 1. A binary fuzzy relation R on A is called *antisymmetric* if $\min(R(a,b), R(b,a)) = 0$ for all $a, b \in A$; *transitive* if

$$R(a,b) \geq \min(R(a,c), R(c,b))$$

holds for all $a, b, c \in A$.

A binary fuzzy relation R on A is called *strongly complete* if we have for all $a, b \in A$ that $\max(R(a,b), R(b,a)) = 1$; *negatively transitive* if

$$R(a,b) \leq \max(R(a,c), R(c,b))$$

holds for all $a, b, c \in A$. □

Let $R_1, \dots, R_m$ be strongly complete and negatively transitive binary fuzzy relations on A. We would like to find an aggregation rule $M : [0,1]^m \to [0,1]$ such that $R = M(R_1, \dots, R_m)$ is a strongly complete and negatively transitive binary fuzzy relation on A.

One possible class of aggregation rules is given by

$$M(R_1, \dots, R_m) = \max_{j \in \overline{1,m}} h_j(R_j),$$

where h_j are non decreasing functions from $[0,1]$ to $[0,1]$, $\overline{1,m}$ is the index set $\{1, \dots, m\}$ and $h_j(1) = 1$ for some $j \in \overline{1,m}$, $h_j(0) = 0$ for all $j \in \overline{1,m}$.

The completeness of R directly follows from the property that $h_j(1) = 1$ for some j. We also have that M is monotonic and $M(0, \dots, 0) = 0$, $M(1, \dots, 1) = 1$.

The following theorem shows that the above class is the most general one with preserving the two properties.

Theorem 1 ([5])**.** *Suppose that $R_1, \dots, R_m$ are strongly complete and negatively transitive binary fuzzy relations on A. Then the aggregated relation R defined by a function $M : [0,1]^m \to [0,1]$ as*

$$R = M(R_1, \dots, R_m)$$

is a strongly complete and negatively transitive binary fuzzy relation on A if and only if

$$M(R_1, \dots, R_m) = \max_{j \in \overline{1,m}} h_j(R_j), \tag{2}$$

where h_j are non decreasing functions from $[0,1]$ to $[0,1]$ with $h_j(1) = 1$, for some $j \in \overline{1,m}$ and $h_j(0) = 0$, for all $j \in \overline{1,m}$. □

An important particular case corresponds to the weighted maximum we will study in the sequel. In that case we have

$$h_j(R_j) = \min(\omega_j, R_j) \text{ with } \max_{j \in \overline{1,m}} \omega_j = 1.$$

This theorem and the related proof (see [5] for more details and proofs) are very close to those obtained by Leclerc [6] in the framework of consensus functions on transitively valued relations.

The dual theorem for antisymmetric and transitive relations can be immediately deduced.

Theorem 2 ([5])**.** *Suppose that $R_1, \dots, R_m$ are antisymmetric and transitive binary fuzzy relations. The aggregated relation R defined as*

$$R = M'(R_1, \dots, R_m)$$

is an antisymmetric and transitive binary fuzzy relation if and only if

$$R = \min_{j \in J} f_j(R_j) \tag{3}$$

where f_j are non decreasing function from $[0,1]$ to $[0,1]$ with $f_j(0)=0$, for some $j \in \overline{1,m}$ and $h_j(1)=1$, for all $j \in \overline{1,m}$. □

An important particular case corresponds to the weighted minimum we will study in the sequel. In that case we have

$$f_j(R_j) = \max(\omega_j, R_j) \text{ with } \min_{j\in\overline{1,m}} \omega_j = 0.$$

3 Weighted maximum and minimum

Using the concept of possibility and necessity of fuzzy events [8,1], one can evaluate the possibility that a relevant goal is attained, and the necessity that all the relevant goals are attained by the help of the following formulas (see [2] for more details), where $x_1, x_2, \ldots, x_m \in [0,1]$ for $m \in \mathbb{N}$:

$$\max_{i=1,m}\{\min(w_i, x_i)\},\ w_i \in [0,1],\ \max_{i=1,m} w_i = 1 \quad \text{(weighted maximum)} \qquad (4)$$

and

$$\min_{i=1,m}\{\max(w_i, x_i)\},\ w_i \in [0,1],\ \min_{i=1,m} w_i = 0 \quad \text{(weighted minimum)}. \qquad (5)$$

The analogy between the weighted arithmetic mean and the weighted maximum is obvious: product corresponds to minimum, sum does to maximum. It is emphasized in [2] that weighted maximum and minimum operators can be calculated as medians, i.e., the qualitative counterparts of means. More formally, the following result is true (only the weighted maximum is recalled).

Proposition 1 ([2]). *Let $(a_1,\ldots,a_m) \in [0,1]^m$ and $(b_1,\ldots,b_m) \in [0,1]^m$ be such that $a_1 \le a_2 \le \ldots \le a_m$ and $1 = b_1 \ge b_2 \ge \ldots \ge b_m$. Then*

$$\max_{i=1,m}\{\min(a_i, b_i)\} = median(a_1,\ldots,a_m,b_2,\ldots,b_m).$$

□

It is easy to see that weighted maximum satisfies idempotency and monotonicity. Moreover, it fulfils also (with $T^{(m)} = M^{(m)}$)

- *stability for maximum* (SMAX):

  $$M^{(m)}(x_1 \vee t_1,\ldots,x_m \vee t_m) = M^{(m)}(x_1,\ldots,x_m) \vee T^{(m)}(t_1,\ldots,t_m)$$

  for all $(x_1,\ldots,x_m) \in [0,1]^m$, $(t_1,\ldots,t_m) \in [0,1]^m$.
- *stability for minimum with the same unit* (SMINU):

  $$M^{(m)}(r \wedge x_1,\ldots,r \wedge x_m) = r \wedge M^{(m)}(x_1,\ldots,x_m)$$

  for all $(x_1,\ldots,x_m) \in [0,1]^m$, $r \in [0,1]$.

In a sense, the converse is also true, as we state in the following theorem.

Theorem 3 ([5]). *Suppose that M is a nondecreasing function from $[0,1]^m$ to $[0,1]$ such that $M(0,\ldots,0)=0$ and $M(1,\ldots,1)=1$. Then M satisfies SMAX and SMINU if and only if there exist weights $w_1,\ldots,w_m \geq 0$ with $\max w_i = 1$ such that*

$$M(x_1,\ldots,x_m) = \max_{i=1,\ldots,m}\{\min(w_i,x_i)\}.$$

□

By duality, we can introduce the corresponding stability conditions in the case of the weighted minimum as follows:

- *stability for minimum* (SMIN):

$$M^{(m)}(x_1 \wedge t_1,\ldots,x_m \wedge t_m) = M^{(m)}(x_1,\ldots,x_m) \wedge T^{(m)}(t_1,\ldots,t_m)$$

  for all $(x_1,\ldots,x_m) \in [0,1]^m$, $(t_1,\ldots,t_m) \in [0,1]^m$.
- *stability for maximum with the same unit* (SMAXU):

$$M^{(m)}(r \vee x_1,\ldots,r \vee x_m) = r \vee M^{(m)}(x_1,\ldots,x_m)$$

  for all $(x_1,\ldots,x_m) \in [0,1]^m$, $r \in [0,1]$.

Obviously, the weighted minimum (5) satisfies both conditions. We state that the converse is also true in the following sense.

Theorem 4 ([5]). *Suppose that M is a nondecreasing function from $[0,1]^m$ to $[0,1]$ such that $M(0,\ldots,0)=0$ and $M(1,\ldots,1)=1$. Then M satisfies SMIN and SMAXU if and only if there exist weights $w_1,\ldots,w_m \geq 0$ with $\max w_i = 1$ such that*

$$M(x_1,\ldots,x_m) = \min_{i=1,\ldots,m}\{\max(w_i,x_i)\}.$$

□

4 Preference structures and aggregation

In this section we use an axiomatic construction of preference structures suggested by Fodor and Roubens [3,4]. We concentrate mainly on the strict preference relation P defined from R (as well as P_i defined from R_i).

If we want to determine the global credibility of the proposition "a is strictly better than b", two ways might be used to achieve this goal, each having two steps :

1a. Aggregate the monocriterion relations R_j to obtain a global outranking relation $R = M(R_1,\ldots,R_m)$.

1b. Define $P = p(R, R^{-1})$, according to axiomatics in [3,4].
2a. For each point of view j, define the strict preference $P_j = p'(R_j, R_j^{-1})$.
2b. Aggregate these strict preferences to obtain $P' = M'(P_1, \ldots, P_m)$.

We can illustrate these two procedures in Figure 1.

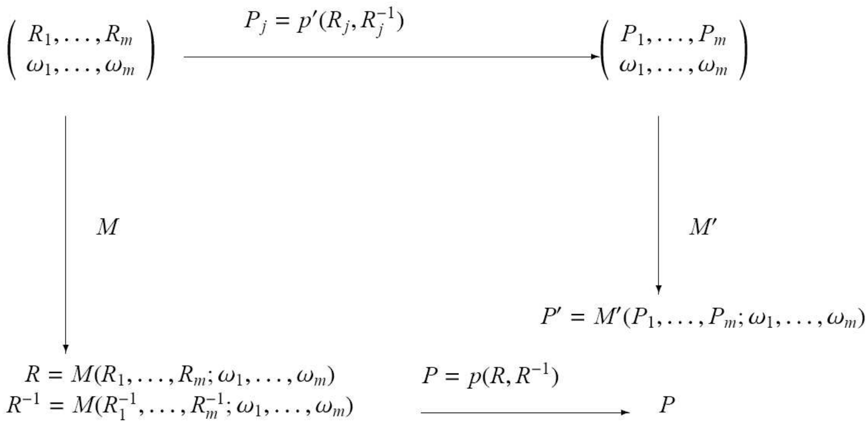

Fig. 1

We now examine conditions under which both procedures (1a,1b) and (2a, 2b) result in the same global strict preference relation.

Consider

$$R = M(R_1, \ldots, R_m) = \max_{j \in \overline{1,m}} h_j(R_j).$$

Due to the completeness of R, the axiomatical analysis of strict preference (see [3,4]) gives:

$$P = R^d = 1 - \max_{j \in \overline{1,m}} h_j(R_j^{-1})$$

which is an antisymmetric and transitive relation:

$$\min(P(a,b), P(b,a)) = 0$$

$$P(a,b) \geq \min(P(a,c), P(c,b)) \ , \quad \forall a, b, c \in A.$$

Using the same arguments, we define $P_j = R_j^d$ and we consider

$$P' = M'(P_1, \ldots, P_m) = M'(R_1^d, \ldots, R_m^d) = P.$$

We then obtain,

$$M'(1 - R_1^{-1}, \ldots, 1 - R_m^{-1}) = 1 - \max_{j \in \overline{1,m}} h_j(R_j^{-1})$$

and finally,

$$M'(x_1, \ldots, x_m) = \min_{j \in \overline{1,m}} (1 - h_j(1 - x_j))\,.$$

A convenient particular pair (M, M') to obtain $P = P'$ is thus given by the pair of weighted maximum and minimum:

$$M(R_1, \ldots, R_m; \omega_1, \ldots, \omega_m) = \max_{j \in \overline{1,m}} \min(\omega_j, R_j)$$
$$M'(P_1, \ldots, P_m; \omega_1, \ldots, \omega_m) = \min_{j \in \overline{1,m}} \max(1 - \omega_j, P_j).$$

We finally obtain a synthesis of the previous result in Figure 2:

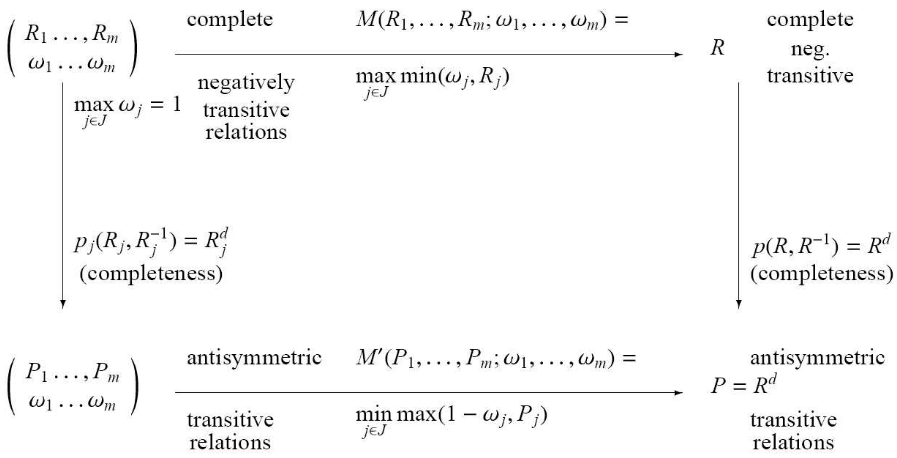

Fig. 2

Every α-cut of the valued relation P is a crisp antisymmetric and transitive relation, i.e., a crisp partial order.

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