ON A WAY TO ALGEBRA OF PETRI NETS: Algebra of PN Terms

Štefan Hudák, Slavomír Šimoňák
Department of Computers and Informatics
FEI TU 04011 Košice, Letná 9, Slovak Republic,
e-mail:stefan.hudak@tuke.sk, e-mail:slavomir.simonak@tuke.sk

Abstract. The present work is devoted to the study of a process algebra oriented towards processes in *Place/Transition Nets (PTN in short)*. The processes are described by special kind of terms, termed here as *PN terms.* Since the time we have been dealing with Petri Nets some of the myths have survive. One of the myth is that Petri Nets are not very well suited for to be treated in de/compositional way [4]. Another myth is connected with an absence of non interleaved semantics of Petri Nets. Our treatment of Petri Nets in this work is aimed at disproving the myths. In the paper we introduce syntax and semantics of PN terms; we define the semantics by PT nets, and based on that two equivalence relations are defined on PN terms. The latter make a provision to validate some algebraic laws to hold on PN terms. Finally the algebra APT is defined.

Keywords: Place/Transition Nets, PN terms, E-(B-)terms, Process Algebra, E-open terms, B-open terms.

1 Place/Transition Nets[1]

By Place/Transition Net(PTN in short) we mean in this work a kind of Petri Net with multiple arcs, tokens without individualities and with unbounded capacities of places. We define PTN N to be a 4-tuple

$$N = (P, T, pre, post) \tag{1}$$

and

P is a finite set of places

T is a finite set of transitions

pre:P $\times$ T $\longrightarrow$ $\mathbb{N}$ $-$ *preset function*
$post : P\times$ T $\longrightarrow$ $\mathbb{N}$ $-$ *postset function*
all define a structure on the set $P \cup T$. It is very common to represent the Petri Net by the oriented bipartite graph (Fig. 1).

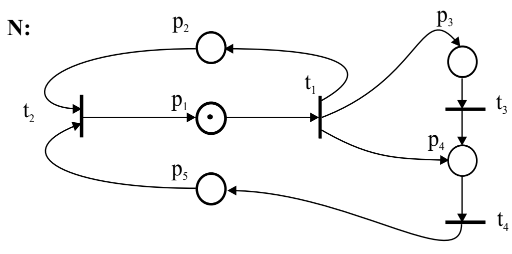

Figure 1: A Petri Net

The following useful notations can be defined:

[1]This work was supported by VEGA Grant No.1/0377/03 of the VEGA- The Scientific Grant Agency of Slovakia

$$\begin{aligned} {}^\bullet t \;&=\; \{p \mid pre(p,t) \;\neq\; 0\} \text{ the set of preconditions of } t \\ t^\bullet \;&=\; \{p \mid post(p,t) \;\neq\; 0\} \text{ the set of postconditions of } t \\ p^\bullet \;&=\; \{t \mid pre(p,t) \;\neq\; 0\} \\ {}^\bullet p \;&=\; \{t \mid post(p,t) \;\neq\; 0\} \end{aligned}$$

By the marking of PTN

$$N \;=\; (P, T, pre, post)$$

we mean a totally defined function

$$m: \qquad P \longrightarrow \mathbb{N} \tag{2}$$

where $\mathbb{N} \;= 0, 1, 2, \ldots$ is the set of natural numbers. Now we may define *marked PTN*. More formally we say that marked PTN $N_0 = (P, T, pre, post, m_0)$, where m_0 is *an initial marking.*

We use a marking m to describe the situation or configuration in the PTN N_0.

Namely, we say the condition represented by the place p in PTN N_0 holds iff $m(p) \;\neq 0$. Without loss of generality we assume that P and T have k and m elements respectively. i.e. $P \;=\; \{p_1, p_2, , \ldots, p_k\}$, $T \;=\; \{t_1, t_2, , \ldots, t_m\}$ and we fix some ordering of both, places and transitions from now on. Using the ordering of places we can consider m to be the k-dimensional nonnegative integer vector, i.e. $\vec{m} \in \mathbb{N}^k$, or we can express that explicitly as

$$\vec{m} = \;(m(p_1), m(p_2), \ldots, m(p_k))$$

and $m(p_i)$ is the value of m in p_i , $i \;=\; 1, 2, \ldots, k$, according to (2).

In our example (Fig. 1) $m(p_1) \;=\; 1$ and $m(p_i) \;=\; 0$ for $i \;=\; 2, 3, 4$, or alternatively $\vec{m} = \;(1, 0, 0, 0, 0)$. For simplicity we will use the denotation m for either interpretations of the marking m when it doesn't cause any troubles. We say t is enabled in m, and denote it m $\vdash\!\!\overset{t}{\text{—}}$, iff for every $p \;\in^\bullet t,\; m(p) \;\geq\; pre(p,t)$. In Fig. 1 t_1 is enabled in $m \;=\; (1, 0, 0, 0, 0)$ because ${}^\bullet t_1 \;=\; \{p_1\}$ and $m(p_1) \;=\; 1$, and $pre(p_1, t_1\;) = \;1$. In general, given PTN N, a marking m of N, several transitions from T can be enabled in m. Once the transition t is enabled it can fire. The effect of the firing t in m is the creation of a new marking m' that depends on m and t. We use a denotation

$$m \vdash\!\!\overset{t}{\text{—}} m'$$

and m' is defined in the following way:

$$\forall p \in P: \quad m'(p) \;=\; \begin{cases} m(p) - \quad pre(p,t) & p \;\in\; {}^\bullet t \;\setminus t^\bullet \\ m(p) + \quad post(p,t) & p \;\in\; t^\bullet \setminus^\bullet t \\ m(p) - \quad pre(p,t) \;+\; post(p,t) & p \;\in\; {}^\bullet t \;\cap t^\bullet \\ m(p) & \textit{otherwise} \end{cases}$$

We say the sequence of transitions $\sigma \;=\; t_1 t_2 \ldots t_r$ is *admissible firing sequence* in PTN N_0, , provided a sequence of markings $m_0, m_1, \ldots, m_r$ exists such that $m_{i-1} \vdash\!\!\overset{t_i}{\text{—}} m_i$, $i \;=\; 1, 2, \ldots, r$. In that case we write $m_0 \vdash\!\!\overset{\sigma}{\text{—}} m_r$, or simply $m_0 \vdash\!\!\overset{*}{\text{—}} m_r$, when σ is immaterial. The marking m_r is to be called *the reachable marking* in N, from m_0 (via $\sigma \in T^*$); the latter means that a marking m_0 is reachable marking. We fix the marking m_0 to be *the initial marking* of PTN $N \;=\; (P, T, pre, post)$ and we denote it $N_0 \;=\; (N, m_0)$ or $N_0 \;=\; (P, T, pre, post, m_0)$.

Given PTN $N_0 \;=\; (P, T, pre, post, m_0)$ we define the set of reachable markings

$$\mathcal{R}(N_0) \;=\; \{m \mid m_0 \vdash\!\!\overset{\sigma}{\text{—}} m\, \sigma \in T^*\},$$

Now we may give some interpretation to transitions of PTN N_0 by using some *labelling mapping* $\ell : T^* \to \Sigma^*$, where Σ is some alphabet serving the purpose of an interpretation of transitions of N_0. The mapping ℓ is chosen in such a way to satisfy the following propereties :

- $\forall t \in T : \ell(t) = a,\ a \in \Sigma \cup \{\lambda\}$, where λ is the empty symbol. The latter means that some transitions may not be labelled . For the empty symbol λ it holds that $\forall x \in \Sigma^* : x\lambda = \lambda x = x$.
- $\forall x \in T^*, t \in T : \ell(x \cdot t) = \ell(x) \cdot \ell(t)$ that means that ℓ is a *homomorphism*.

By that virtue we define *labelled PTN*

$$N_0 \ = \ (P, T, pre, post, m_0, \ell)$$

We can define the language of the labelled PTN N_0

$$L(N_0) \ = \ \{w \ \in \ \Sigma^* \mid \exists \sigma \in T^* : m_0 \overset{\sigma}{\vdash} m\},$$

$w = \ell(\sigma)$

In the case $\Sigma = T$, in that case ℓ is the identity mapping, and we obtain that

$$L(N_0) \ = \ \{\sigma \ \in \ T^* \mid m_0 \overset{\sigma}{\vdash} m\},$$

and we call it *PTN language*.

1.1 Analysis of PTNs

Let us assume that we are given some PTN $N_0 = (P, T, pre, post, m_0)$ and we fix the net for what follows. An example of such a net can serve PTN in Fig. 1.

We would like to do an analysis of PTN N_0 in a systematic way with respect to (wrt) *computational processes* PTN N_0 can allow or can perform, and also what *markings* can be *reachable* in N_0. To do so, first we are going to introduce some concepts. Namely, given any place $p \in P$ we are going to characterize p by means of two notions.

By *prefix(language) of p – B(p)* we mean simply the set of all sequences of *transition firings* that take PTN N_0 from initial marking m_0 to some marking m such that *m(p)≠0.*

By *suffix(language) of p – E(p)* we mean simply the set of all sequences of *transition firings* that take PTN N_0 from some marking m, such that *m(p)=1* , to some reachable marking.

To define $B(p)$ and $E(p)$ more formally we will use a structural approach based on the structure of PTNs and *the relevant vicinity of the place p* . The situation is clarified by Fig. 2.

For the vicinity of the place p we may write.

Ia) $B(p) = B(q)u$

Ib) $B(p) = (B(q_1)||B(q_2)...||B(q_m))u$

Ic) $B(p) = B(q_1)u_1 + B(q_2)u_2 + ... + B(q_m)u_m$

Id) $B(p) = (B(q_1)||B(q_2))u_1 + (B(q_2)||B(q_3))u_2$

Ie) $B(p) = B(p)u$

Equations Ia) – Ie) express the structure of the prefix language of the place p in dependence of the places and transitions which comprise the vicinity of p. For example in a) the statement $B(p) = B(q)u$ should be understood that any firing sequence in $B(p)$will be composed of some firing sequence contained in $B(q)$followed by u. The case c) says that any firing sequence in $B(p)$will be of the form of

$B(q_1)u_1$ or $B(q_2)u_2$ or $\ldots B(q_m)u_m$. The *or clause is exclusive or* . Some words are needed to clarify cases b) and d). In b) case $B(p)$can be thought of as to consist of *concurrent* firing sequences which are

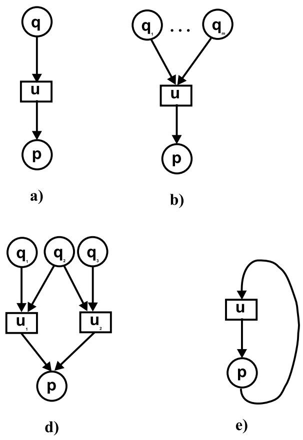

Figure 2: Pre-vicinity of the place p

synchronized by transition u in the sense that *all* of them should be finished prior to the transition u fires. So firing sequence

$$(u_1||u_2||\cdots||u_m)u$$

will be typical of $B(p)$, provided

$$u_1 \in B(q_1), u_2 \in B(q_2), \cdots u_m \in B(q_m)$$

.

We can also define suffix (language) of the place $p - E(p)$ based on the vicinity of the place p as it is depicted in Fig. 3.

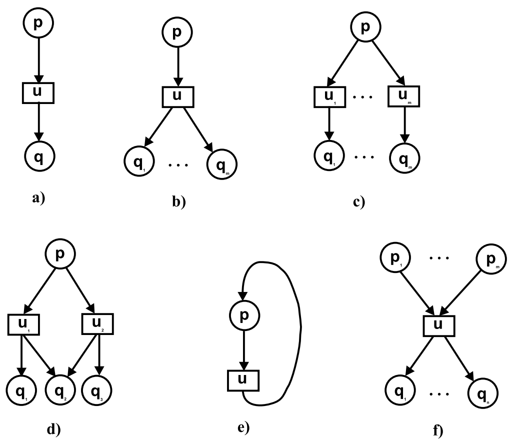

Figure 3: Post-vicinity of the place p

For the vicinity of the place p we may write.

IIa) $E(p) = uE(q)$

IIb) $E(p) = u(E(q_1)||E(q_2)...||E(q_m))$

IIc) $E(p) = u_1E(q_1) + u_2E(q_2) + ... + u_mE(q_m)$

IId) $E(p) = u_1(E(q_1)||E(q_2)) + u_2(E(q_2)||E(q_3))$

IIe) $E(p) = uE(p) + \delta$, where δ stands here for *do nothing* or *empty process*

IIf) $E(p_1)||E(p_2)||\cdots||E(p_m)|| = u(E(q_1)||E(q_2)||\cdots||E(q_n))$

The notion of *equality (=)* here means that terms on both sides are *the same* (identical) i.e. denotes syntactically identical sets of strings.

Several words are worth to be said as far as the interpretations of the equations IIa) – IIf) are concerned. Equation IIa) $E(p) = uE(q)$ can be given two interpretations:
the first will claim that E(p) stands here for a set of *partial firing words (partial firing sequences)* each of them beginning with u and followed by a partial firing word from E(q);
the second interpretation will pertain *the condition holding* [7, 8]in PTN in question and it will simply means that one condition holding in the place p will be replaced by a condition holding in the place q and the shift in condition holding will be caused by firing of transition with the label u.

The two interpretations will apply accordingly in equations IIb) – IIf) as well. First we have to introduce *the multiset denotation* for markings. In any place p of PTN $N = (P, T, pre, post, m_0)$ $m(p)$ may take on any value from the set of natural numbers $\mathbf{N}$, i.e, $m(p) \in \mathbf{N}$, and thus $m(p) = n$ expresses a situation of *'multiple condition holding'* in the place p.
Using multiset notation [10] we will represent the marking m as

$$m = (a_1\mathbf{p_1}, a_2\mathbf{p_2}, ..., a_k\mathbf{p_k}) \tag{3}$$

In (3) we say that in the place p_1 there is a_1 tokens , in p_2 there is a tokens , and so on. In what follows we are going to use a simplified version of (3)

$$m = (a_1p_1 + a_2p_2 + ... + a_kp_k) \tag{4}$$

According to (4) we will use the denotation in representation of sets of PTN processes as well. We will write

$$E(m) = E(a_1p_1 + a_2p_2 + ... + a_kp_k) \tag{5}$$

$$B(m) = B(a_1p_1 + a_2p_2 + ... + a_kp_k) \tag{6}$$

provided m is as in (4) .

Now we may go back to the interpretations of equations IIa) - IIf).
Equation IIb) in terms of the second interpretation can be rewritten as

$$E(p) = u(E(q_1 + q_2 + ... + q_m)) \tag{7}$$

By $E(q_1+q_2+...+q_m)$ here we mean a shorthand notation for suffix term denoting the set of processes in PTN $N = (Pl, T, pre, post, m)$ with the marking m such that

$$m(q_1) = 1, m(q_2) = 1, \cdots m(q_m) = 1,$$

So we may write

E(m)=E(q1)$||E(q_2)|| \cdots ||E(q_m) = E(q_1 + q_2 + ... + q_m)$ (8)

We will use the notation $E(q_1)||E(q_2)|| \cdots ||E(q_m)$ or $E(q_1 + q_2 + ... + q_m)$ to describe *the set of all processes (computations)* in PTN $N = (P, T, pre, post, m)$, i.e. the set of all processes that are possible in PTN N in the marking $m = q_1 + q_2 + \cdots + q_m$.

We may generalize the notation for any marking m such that

$$m(q_1) = a_1, m(q_2) = a_2, \cdots, m(q_m) = a_m,$$

for any $a_i \in \mathbf{N}$, $i = 1, 2, ..., m$.

In that case we will use the multiset denotation for the marking , i.e.

$$m = a_1 \cdot q_1 + a_2 \cdot q_2 + ... + a_m \cdot q_m \tag{9}$$

Based on that we will use alternatively a denotation $E(N)$ for the set of all processes that are possible in PTN N in the marking $m = a_1 \cdot bfq_1 + a_2 \cdot q_2 + ... + a_m \cdot q_m$. Put it in another way,

$$E(N) = E(a_1 \cdot q_1 + a_2 \cdot q_2 + ... + a_m \cdot q_m$$

or

$$E(N) = E(a_1 \cdot q_1)||E(a_2 \cdot q_2)|| \cdots ||E(a_m \cdot q_m)$$

In what follows we are going to deal mostly with *suffix terms,* however the results achieved for the suffix terms can be obtained in a similar way and in somewhat similar form for *prefix terms* as well.

On the way to do analysis of PTN $N_0 = (P, T, pre, post, m_0)$we have achieved first goal: we are now able to characterize PTN by a set of equations of type IIa) – IIf) with respect to $E(p)$. As an example we introduce the characterization of PTN of Fig.1 via *suffix terms* and here we go.

$$\begin{aligned} E(p_1) &= t_1 \cdot (E(p_2)||E(p_3)||E(p_4)) \\ E(p_2) &= t_2 \cdot E(p_1) \\ E(p_3) &= t_3 \cdot E(p_4) \\ E(p_4) &= t_4 \cdot E(p_5) \\ E(p_5) &= t_2 \cdot E(p_1) \end{aligned}$$

2 Processes of PTNs and algebra of PTN processes

We start this section with a definition of the universe of processes that expresses computations in Petri Nets, e.g. in *Place/Transition Nets (PTN).*

Assume we have an alphabet V that consists of a finite number of elements, i.e. $V = \{a_1, a_2, ..., a_n\}$.

We denote by $V^i = \{x | x = b_1 b_2 ... b_i, b_j \in V, j = 1, 2, ..., i\}$ the set of all strings of the length i over the alphabet V , i.e. that consist of i symbols. By V^* we denote *the set of all strings over the alphabet* V *, i.e.* $V^* = \bigcup_{i=0}^{\infty} V^i$

Notice $V^0 = \{\lambda\}$, where λ stands for *the empty string.* There is only one operation defined On V^* , namely it is *concatenation* operation, which is denoted by '$\cdot$' . The notation $x \cdot y$ stands here for *concatenating* x and y, that means that either x will take the place and after that y will follow.

We introduce two other operations: $+$ and $||$. The first ($+$) stands here for *alternative*, that in the case of $x + y$ means that either x or y will take place as processes, where *either ... or* has meaning as *exclusive or.* The second operation $||$ will denote *parallelism* (concurrency), that in the case of $x||y$ means that x and y will take place *in parallel* i.e.(concurrently).

Using the operations we define the notion $V^{|i|}$ inductively as follows:

$$V^{|0|} = \{\lambda\}$$

$$V^{|1|} = \{x_1||x_2||....||x_s | x_i \in V^1, s \geq 1, 1 \leq i \leq s\}$$

$$V^{|i+1|} = \{uv | u \in V^{|i|}, v \in V^{|1|}\}$$

We define further

$$V^{|\omega|} = \{V^{|0|}||V^{|1|}||V^{|2|}||...||V^{|i|}||...\} = ||_{i=0}^{\infty} V^{|i|}$$

where $A||B = \{(a||b) | a \in A, b \in B\}$, and A, B are sets of m-strings.

$$V^{|\omega(i+1)|} = V^{|\omega i|} \cdot V^{|\omega|}$$

Finally we define

$$V^{|*|} = \bigcup_{i=0}^{\infty} V^{|\omega i|} \tag{10}$$

We assume that $V^{|\omega 0|} = \{\lambda\}$.

The expression $V^{|*|}$ stands here for the universe of *generalized strings*, which we call *m- strings* that will denote all legal computations in PTNs.

We define *PN terms* as follows:

1. δ, λ are PN terms,
2. any $u \in V^*$ is PN term,
3. for any α , β PN terms $\alpha \cdot \beta$, $\alpha + \beta$, and $\alpha || \beta$ are PN terms.

We will call the PN terms defined in 1. – 3. as *closed PN terms* (shortly closed terms) .

To proceed further we have to introduce some notions. First we fix some PTN $N = (P, T, pre, post)$, and we assume that PTN N has k places, i.e. $P = \{p_1, p_2, ..., p_k\}$.

Now we define so called $E-$ and *B- open PN terms.*

Let u, v be closed terms, q, p places of PTN N and *E(q), E(p)* be variables with the domain of *Post_processes* and *B(q), B(p)* be variables with the domain of *Pre_processes.*
We say that

4. $E(q), u \cdot E(q)$ are *E-open terms,*
5. let $u \cdot E(q)$ and $v \cdot E(p)$ be open terms,
   then $u \cdot E(q) + v \cdot E(p)$
   and $u \cdot E(q) || v \cdot E(p)$
   are *E-open terms*

Similarly we can define so called *B-open terms.*
We say that

6. $B(q), B(q) \cdot u$ are *B-open terms,*
7. let $B(q) \cdot u$ and $B(q) \cdot v$ be open terms, then
   then $B(q) \cdot u + B(p) \cdot v$
   and $B(q) \cdot u || B(p) \cdot v$
   are *B-open terms*

When it is clear what kind of *openess* we are dealing with we will use the term *open term* for short.

We define the notion *equality of two PN terms.* Specifically, we are going to use *three* notions of equality on PN terms: *syntactical equality* which we denote by $=$, *linguistic equality* which we denote by $=_L$, and *reachability equality* which we denote by $=_r$.

So, when we write

$$E(q) = u \cdot E(q) + v \cdot E(p) \tag{11}$$

we mean by that that expressions on both sides of (11) are of *the same form*, i.e. they are *identical.* To define the other two equalities we need some notions to be added.

To define $=_L$ we associate some PT nets with PN terms and we make the association according to syntax of PN terms.

Assume we have an alphabet A and PN terms defined over A. The association of PN terms and PT nets is shown in Fig. 4, and Fig. 5.

Several remarks are in the order as far as the above association is concerned.

The association assumes a mapping, say

$$\eta : PNTerm \rightarrow PTN$$

has been defined. We assume further that *PNTerm* stands here for the set of all terms defined for fixed alphabet of labels A and the domain of place names *Places.* In what follows we will assume that we deal with *labelled* PTN, which we define as

$$N = (Pl, T, pre, post, m_0, lab)$$

where $lab : T \rightarrow A \cup \{\lambda\}$ is a partial labelling mapping that assigns a label to some transitions of T, and some transitions bear label λ, which will stand for *the empty label.*

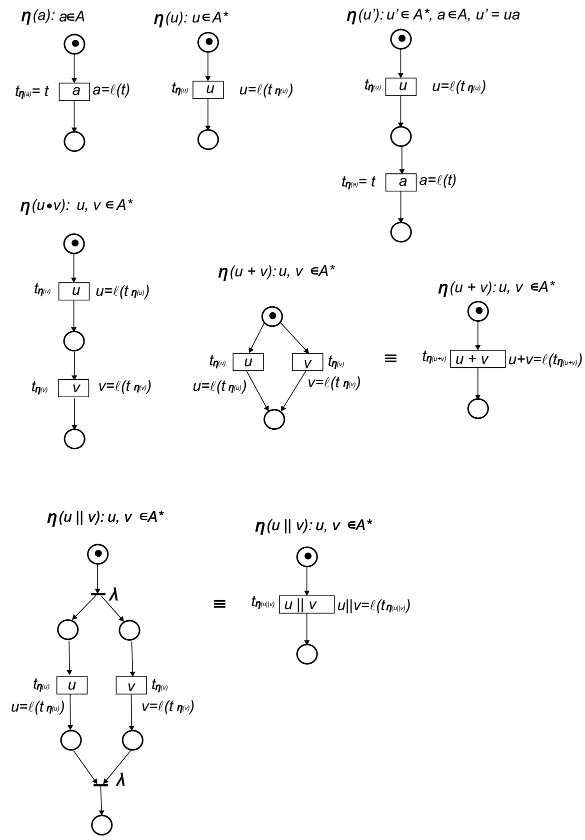

Figure 4: Association of PN terms and PT nets- 1.part

In the net $\eta(a)$ associated with term a, we put a inside the frame denoting transition $t_{\eta(a)}$, that a actually denotes (i.e. $a = \ell(t_{\eta(a)})$), to stress the association defined in that way.

In the case of the net $\eta(u)$ associated with particular term u, we use the same approach by introducing a notion of *generalized transition* $t_{\eta(u)}$. The construction of $t_{\eta(u)}$ is clear from Fig. 4, and Fig. 5. So, we still put u inside the frame standing for generalized transition $t_{\eta(u)}$, that u actually denotes (i.e. $u = \ell(t_{\eta(u)})$), to stress the association defined in that way. We apply the generalization to all terms coming from $A^{|*|}$. The graph notation introduced for PT nets associated with particular term α could add to complexity of presentation, and could look awkward in some cases. For the reason of simplicity we will omit from the transition denotation tripple

$$< t_{\eta(\alpha)},\ \alpha\ , \ell(t_{\eta(\alpha)}) >$$

the first, and the third part, and leaving just α for denotation of the generalized transition of PT net $\eta(\alpha)$). That adds significantly to the simplicity and clarity of presentation, as it can be seen from Fig. 7.

We can consider the nets assigned to PN terms as a kind of *semantics* for PN terms, which we name as *net semantics.*

Based on the mapping *lab* we can define *PTN language* of PTN $\eta(u)$, denoted as $L(\eta(u))$, assigned to PN term u. The language $L(\eta(u))$ will be a subset of $V^{|*|}$, i.e. $L(\eta(u)) \subseteq V^{|*|}$. The PTN language is defined in the manner which is depicted in Fig. 6.

The process semantics of the PN term u, we will denote it as $[\![u]\!]$, and will be defined by the process language of the PTN $\eta(u)$, i.e. $[\![u]\!] = L(\eta(u))$. Now we are going to define the process semantics inductively as follows:

$$[\![\lambda]\!] = \{\lambda\}, \quad [\![\delta]\!] = \phi$$

$$\forall a \in V : [\![a]\!] = \{a\}$$

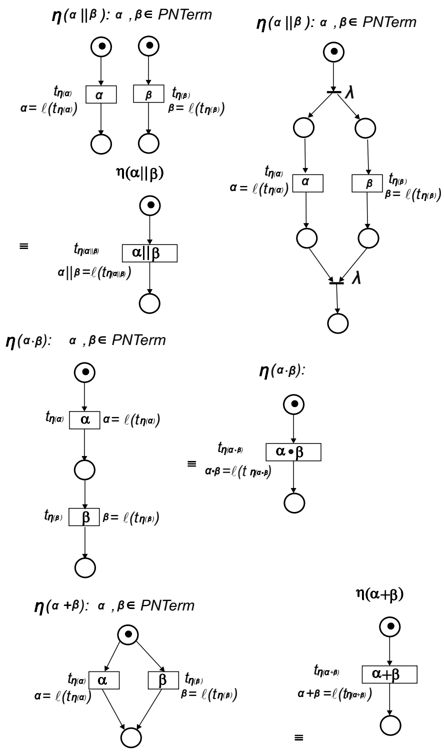

Figure 5: Association of PN terms and PT nets- 2.part

$$\forall u \in V^{|*|} : [\![u]\!] = \{u\}$$
$$\forall u, v \in V^{|*|} : [\![u \cdot v]\!] = \{(u \cdot v)\}$$
$$\forall u, v \in V^{|*|} : [\![u + v]\!] = \{u, v\}$$
$$\forall u, v \in V^{|*|} : [\![u||v]\!] = \{(u||v)\}$$
$$\forall \alpha, \beta \in PNTerms : [\![\alpha \cdot \beta]\!] = [\![\alpha]\!] \cdot [\![\beta]\!]$$
$$\forall \alpha, \beta \in PNTerms : [\![\alpha + \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!]$$
$$\forall \alpha, \beta \in PNTerms : [\![\alpha||\beta]\!] = \{(u||v)| \\ u \in [\![\alpha]\!], v \in [\![\beta]\!]\}$$

Notice, that *the process semantics domain* is the universe $V^{|*|}$, that represents all processes of PNs. We define now the relation $=_L$

$$u =_L v \Leftrightarrow_{df} [\![u]\!] = [\![v]\!] \tag{12}$$

That is clear that the relation $=_L$ is *equivalence relation.*

To define the relation $=_r$ we will use the notion of *reachable marking* caused by computation u in PTN $\eta(u)$ with the initial marking as it is depicted in Fig. 4. We will use the denotation $[u]$ for *reachability semantics* of PN term u. To be more precise we define

$$[u] = \mathcal{R}(\eta(u))$$

where $\mathcal{R}(\eta(u))$ denotes the set of reachable markings of the PTN $\eta(u)$. In this case we will use a special form of PTN for $\alpha||\beta$ and $\alpha + \beta$ terms that is depicted in Fig. 7. We call the PNs in that form as *B-sharpened PNs.* The reason for introducing the PNs in the form of *B-sharpened* PNs (BPTN for short) is to be able to compare two nets with different set of *'output'* places .

Now we define the relation $=_r$. We will write

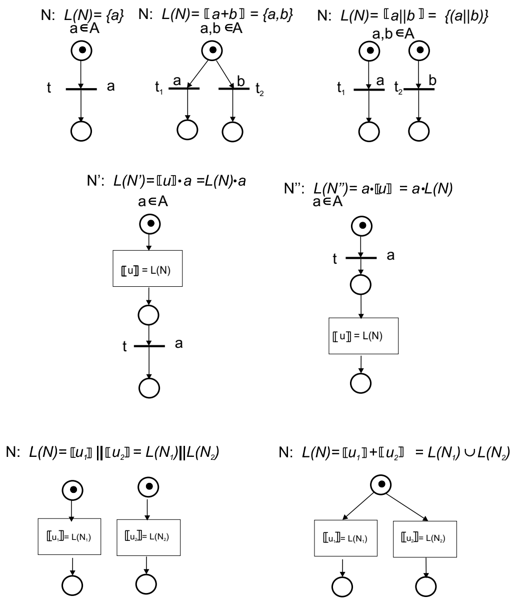

Figure 6: The PTN language

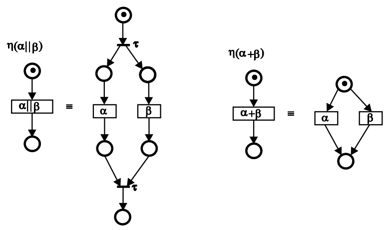

Figure 7: B-sharpened PN

$$u =_r v \Leftrightarrow [u] = [v] \tag{13}$$

In the case we use BPTN we will denote the relation as $=_{r_B}$.

From (13) and the definition of the reachability semantics it follows that the relation $=_r$ ($=_{r_B}$) is an equality relation.

2.1 Axioms of algebra of PTN processes

The aim of this section is to prove a set of equalities to hold for PN terms with respect to (wrt) equality relations defined, i.e. wrt $=_L$ and $=_r$.

Let us assume that we are given an alphabet $A, andlet$x, y, z $bem - stringsfrom$A$^{|*|}$, and let $a \in A$ be a symbol. Then obviously $x||y$, $x{+}y$, $x \cdot y$ are m-strings and the following equalities are going to be checked on their truth wrt equality relations defined: $=_L$ and $=_r$. In the following we are going to use the relation symbol $=$ to denote either $=_L$ or $=_r$.

First we introduce a set of equalities, that can be considered as candidates for to be chosen as axioms of prospective algebra of PN terms. So, here we go.

$$x||y \stackrel{?}{=} y||x \qquad \text{(C1)}$$

$$x + y \stackrel{?}{=} y + x \qquad \text{(C2)}$$

$$(x||y)||z \stackrel{?}{=} x||(y||z) \qquad \text{(A1)}$$

$$(x+y)+z \stackrel{?}{=} x+(y+z) \qquad \text{(A2)}$$

$$(x \cdot y) \cdot z \stackrel{?}{=} x \cdot (y \cdot z) \qquad \text{(A3)}$$

$$x \cdot (y||z) \stackrel{?}{=} (x \cdot y)||(x \cdot z) \qquad \text{(LD1)}$$

$$x+(y||z) \stackrel{?}{=} (x+y)||(x+z) \qquad \text{(LD2)}$$

$$x \cdot (y+z) \stackrel{?}{=} (x \cdot y)+(x \cdot z) \qquad \text{(LD3)}$$

$$x+(y \cdot z) \stackrel{?}{=} (x+y) \cdot (x+z) \qquad \text{(LD4)}$$

$$x||(y+z) \stackrel{?}{=} (x||y)+(x||z) \qquad \text{(LD5)}$$

$$x||(y \cdot z) \stackrel{?}{=} (x||y) \cdot (x||z) \qquad \text{(LD6)}$$

$$(x||y) \cdot z \stackrel{?}{=} (x \cdot z)||(y \cdot z) \qquad \text{(RD1)}$$

$$(x||y)+z \stackrel{?}{=} (x+z)||(y+z)||x \qquad \text{(RD2)}$$

$$(x+y) \cdot z \stackrel{?}{=} (x \cdot z)+(y \cdot z) \qquad \text{(RD4)}$$

$$(x+y)||z \stackrel{?}{=} (x||z)+(y||z) \qquad \text{(RD5)}$$

$$(x \cdot y)||z \stackrel{?}{=} (x||z) \cdot (y||z) \qquad \text{(RD6)}$$

Using the association introduced above, and the equivalence relation $=_L$, $=_r$, and $=_{r_B}$ we have been able to prove the following validity results, that is summarized in Table 1 .

So, now we may introduce

Table: Results of validation of distribution laws				
		$=_L$	$=_r$	$=_{r_B}$
LD1	$x \cdot (y\|\|z) \stackrel{?}{=} (x \cdot y)\|\|(x \cdot z)$	$-$	$+$	$+$
LD2	$x+(y\|\|z) \stackrel{?}{=} (x+y)\|\|(x+z)$	$-$	$-$	$-$
LD3	$x \cdot (y+z) \stackrel{?}{=} (x \cdot y)+(x \cdot z)$	$+$	$+$	$+$

LD4	$x + (y \cdot z) \stackrel{?}{=} (x + y) \cdot (x + z)$	$-$	$+$	$+$
LD5	$x\|\|(y + z) \stackrel{?}{=} (x\|\|y) + (x\|\|z)$	$+$	$-$	$+$
LD6	$x\|\|(y \cdot z) \stackrel{?}{=} (x\|\|y) \cdot (x\|\|z)$	$-$	$+$	$+$
RD1	$(x\|\|y) \cdot z \stackrel{?}{=} (x \cdot z)\|\|(y \cdot z)$	$-$	$-$	$-$
RD2	$(x\|\|y) + z \stackrel{?}{=} (x + z)\|\|(y + z)$	$-$	$-$	$-$
RD3	$(x + y) \cdot z \stackrel{?}{=} (x \cdot z) + (y \cdot z)$	$+$	$+$	$-$
RD4	$(x \cdot y) + z \stackrel{?}{=} (x + z) \cdot (y + z)$	$-$	$+$	$-$
RD5	$(x + y)\|\|z \stackrel{?}{=} (x\|\|z) + (y\|\|z)$	$+$	$-$	$+$
RD6	$(x \cdot y)\|\|z \stackrel{?}{=} (x\|\|z) \cdot (y\|\|z)$	$-$	$-$	$-$

Table 1: Results of validation of distribution laws

Definition 1 *Given an alphabet A we will denote by $PNTerm$ the set of terms defined as follows:*

- *λ - the empty symbol, and any $a \in A$, are PN terms (belonging to $PNTerm$);*
- *$\forall a, b \in A \cup \{\lambda\} : a \cdot b,\ a + b,\ a||b$ are PN terms ;*
- *let $\alpha,\ \beta \in PNTerm$ then $\alpha \cdot \beta,\ \alpha + \beta,\ \alpha||\beta$ are PN terms .*

The algebra of PN terms APT is given by a couple

$$APT =< PNTerm,\ \Omega >$$

where $PNTerm$ is the basis and Ω is the signature of APT consisting of operations $\cdot,\ +,\ ||$.
To relate PN terms we use the notions of equality $=_L,\ =_r$, that we have defined above.
In the algebra APT we are given the following axioms :
For any PN terms $x, y, z \in PNTerm$

$$x + y = y + x \qquad \text{(C1)}$$

$$x||y = y||x \qquad \text{(C2)}$$

provided $= \in \{=_L,\ =_r\}$

$$x \cdot (y + z) =_L x \cdot y + x \cdot z \qquad (\text{LD3}_L)$$

$$x||(y + z) =_L (x||y) + (x||z) \qquad (\text{LD5}_L)$$

$$x \cdot (y||z) =_r (x+y) \cdot (x+z) \qquad (\mathrm{LD1}_r)$$

$$x \cdot (y+z) =_r (x \cdot y) + (x \cdot z) \qquad (\mathrm{LD3}_r)$$

$$x + (y \cdot z) =_r (x+y) \cdot (x+z) \qquad (\mathrm{LD4}_r)$$

$$x||(y+z) =_r (x||y) + (x||z) \qquad (\mathrm{LD5}_r)$$

$$x||(y \cdot z) =_r (x||y) \cdot (x||z) \qquad (\mathrm{LD6}_r)$$

$$(x+y) \cdot z =_L (x \cdot z) + (y \cdot z) \qquad (\mathrm{RD3}_L)$$

$$(x+y)||z =_L (x||z) + (y||z) \qquad (\mathrm{RD5}_L)$$

$$(x+y) \cdot z =_r (x \cdot z) + (y \cdot z) \qquad (\mathrm{RD3}_r)$$

$$(x \cdot y) + z =_r (x+z) \cdot (y+z) \qquad (\mathrm{RD4}_r)$$

$$(x+y)||z =_{r_B} (x||z) + (y||z) \qquad (\mathrm{RD5}_{r_B})$$

□

3 PN term equations.

Let us start with a PT net that is depicted in Fig. 8a).

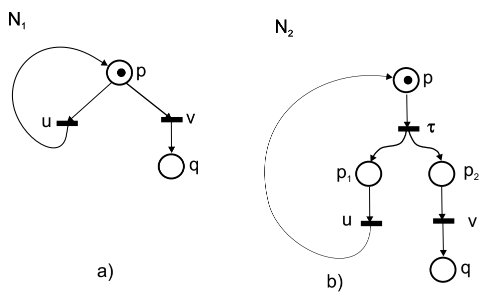

Figure 8: Petri Nets

For the PT net we may write

$$E(p) = u \cdot E(p) + v \cdot E(q) \tag{14}$$

By substituting $k-1$ times left alternative from RHS(14) we obtain from (14)

$$\forall k \geq 0 : E(p) = u^k \cdot v \cdot E(q) \tag{15}$$

or simply we may write

$$E(p) = u^* \cdot v \cdot E(q) \tag{16}$$

That is not difficult to see that prefix term $B(q) = u^* \cdot v$, that gives us

$$E(p) = B(q) \cdot E(q) \tag{17}$$

Actually for any PTN $N_0 = (P, T, pre, post, m_0)$ we may write

$$E(m_0) = B(m) \cdot E(m) \tag{18}$$

provided m is a reachable marking in PTN N_0.

Now let us turn our attention to the PTN which is depicted in Fig. 8b). For the PTN we may write the following equation

$$E(p) = uE(p)||vE(q) \tag{19}$$

We may substitute RHS(19) instead of $E(p)$ in RHS(19). As the result of the substitution we obtain

$$E(p) = u(uE(p)||vE(q))||vE(q) \tag{20}$$

Using property (LD1) we may rewrite (20)as

$$E(p) = u^2E(p)||uvE(q)||vE(q) \tag{21}$$

We can repeat the substitution and we obtain

$$\forall k \geq 0 : E(p) = u^{k+1}E(p)||u^kvE(q)||u^{k-1}vE(q)||...||vE(q) \tag{22}$$

By using properties of E-terms and the convention to represent postlanguage of nontrivial markings (see 1.1)we can write

$$\forall k \geq 0 : E(p) = u^{k+1}E(p)||(u^kv||u^{k-1}v||...||v)E((k+1)q) \tag{23}$$

We introduce here a shorthand notation for prefix language $B((k+1)q) = (u^kv||u^{k-1}v||...||v)$ (see (17)). Namely we will abreviate $(u^kv||u^{k-1}v||...||v)$ as $u^{|k|}v$. i.e.

$$u^{|k|}v = (u^kv||u^{k-1}v||...||v) \tag{24}$$

By that virtue we obtain

$$\forall k \geq 0 : E(p) = u^{k+1}E(p)||u^{|k|}vE((k+1)q) \tag{25}$$

How can we interpret equation (25) and the like? To answer the question, we make some rearrangement of (25) first. By using properties of E-terms and the convention to represent postlanguage of nontrivial markings (see the place in main file)we can write

$$\forall k \geq 0 : E(p) = (u^{k+1}||u^{|k|}v)E(p + (k+1)q) \tag{26}$$

We can see that $E(p+(k+1)q)$ stands for the post language of reachable marking $m = p+(k+1)q$, and so in the view of (**??**) we have

$$B(p + (k+1)q) = (u^{k+1}||u^kv||u^{k-1}v||...||v) \tag{27}$$

The equation (27) can be viewed as *'histories'* of tokens in the marking $m = p + (k+1)q$. There is some *synchronization* in (27), that reflects the fact, that tokens in the place q have been generated in some sequence due to repeated execution (firing) of the transition u. The situation is illustrated in Fig. 9.

The marking $m = p+(k+1)q$ represents the situation that there is one token in the place p and $k+1$ tokens in the place q.

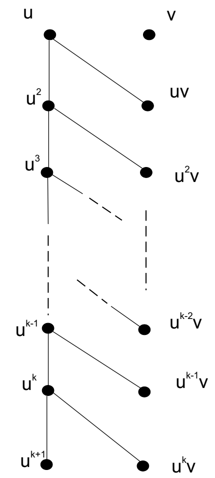

Figure 9: Tokens histories: the structure of $B(p + (k + 1)q) = (u^{k+1}||u^k v||u^{k-1}v||...||v)$

Conclusion

Special PN terms- *E-terms* and *B-terms*have been used to describe computational processes using non-interleaved approach. The description provides a new insight into PT nets' processes. The description is transparent enough showing explicitly simultaneously actual marking reached and the histories of transition firings involved. The inspiration to the present work has come from the work of several researchers: J. Baeten [1, 2], T.Basten [5], E.-R. Olderog [4], and J. Kollar [14]. Some tribute should be paid particularly to E.-R. Olderog [4], where relations among PTNs, terms and formulas are treated.

The results presented here form a basis for an Algebra of Petri Nets with operations allowing to de/compose Petri Nets in formally sound way. The properties of de/composed PNs can be readily expressed by E- and B-terms respectively. The methodology can be conceived that allows de/composition of PT nets on the sound formal basis. The methodology and results achieved in that respect will be presented elsewhere.

References

[1] BAETEN,J.C.M., WEIJLAND,W.P.: Process Algebra. Cambridge University Press 1990, ISBN 0 521 400430, 248pp.

[2] BAETEN,J.C.M.: Applications of Process Algebra. Cambridge University Press 1990, ISBN 0 521 400430, 248pp.

[3] EHRIG,H.,MAHR,B.: Fundamentals of Algebraic Specifications.(2 vol.)EATCS Monographs, 1985.

[4] OLDEROG,E.R.: Nets, Terms and Formulas.Cambridge University Press 1990, ISBN 0 521 400430, 248pp.

[5] BASTEN, T., VOORHOEVE,M.: An Algebraic Semantics of Hierarchical Petri Nets.Computing Science Report, Eindhoven University of Technology, pp. 32, 1995.

[6] HOPCROFT,J.E.,ULLMAN,J.D.:Formal Languages and Their Relations to Automata. Addison Wesley, Reading, Mass. 1969.

[7] HUDÁK, Š.: Extensions to Petri Nets. Habilitation Thesis (in Slovak), TU Košice, 1980, 107pp.

[8] VALK R.: Generalizations of Petri Nets.MFCS 1981, LNCS vol.118,(J.Gruska, M.Chytil Eds.)140 - 155pp.

[9] HUDÁK, Š.:De/compositional reachability analysis. Journal of Electrical Engineering, CS ISSN 0013-578X, vol.45,(1994),No.11,424-431.

[10] BEST,E.: Semantics of Sequential and Parallel Programs. Prentice Hall Europe, ISBN 0-13-460643-4, 1996, 352pp.

[11] TSEYTLIN, G.E.: Introduction into Algorithmics.The International Solomon University Kiew 1997, Printed by "Sfera" Kiew 1998, ISBN 960-7627-14-8,310pp.(in Russian)

[12] ŠIMOŇÁK, S.: Formal methods integration by using Perti Nets and Process Algebra transformations (in Slovak), FEI TU Košice, 2003, 112pp.

[13] HUDÁK, Š, ŠIMOŇÁK, S.: Algebra of PN Terms, Informatika 2003, Bratislava, Dom techniky ZSVTS Bratislava s.r.o., 2003, pp. 250-255, ISBN – 80-233-0491-7

[14] KOLLÁR, J., NOVITZKÁ, V.: Static Weaving at Dynamic Join Points, Acta Electrotechnica et Informatica, 4, 1, 2004, pp. 16-23, ISSN 1335-8243