Approximation and interpolation sum of exponential functions

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1 Learning pliant operator

Recall Dombi operator[2]

$$o(\underline{x}) = o(x_1, \dots, x_n) = \frac{1}{1 + \left(\sum \left(\frac{(1-x_i)}{x_i} \right)^{\alpha} \right)^{1/\alpha}} \tag{1}$$

$x_i \in [0, 1]$ If $\alpha > 0$ then $o(x_1, \dots, x_n)$ is a conjunctive operator, and
if $\alpha < 0$ then $o(x_1, \dots, x_n)$ is a disjunctive operator
and $\alpha \neq 0$
From (1)

$$\left(\frac{o(\underline{x}) - 1}{o(x)} \right)^{\alpha} = \sum_{i=i}^{n} \left(\frac{1 - x_i}{x_i} \right)^{\alpha} \tag{2}$$

Let us suppose, that α is known. (In most cases is $\alpha = \pm 1$)

In the pliant concept we use the Dombi operator with sigmoid function

$$\sigma_i(t) = \frac{1}{1 + e^{\lambda_i (t - d_i)}} \tag{3}$$

so $\sigma_i(t) = x_i$

Substituting (3) into (2)

$$y = \left(\frac{o(x)-1}{o(x)}\right)^{\alpha} = \sum e^{\lambda_i \alpha (t-d_i)} = \sum a_i e^{\alpha_i t}$$

where $a_i = e^{\alpha \lambda_i d_i}$ and $\alpha_i = -\alpha\lambda_i$

2 Interpolation sum of exponential functions

Our task is now to approximate or interpolate the following function:

$$\varphi(t) = a_1 e^{\alpha_1 t} + a_2 e^{\alpha_2 t} + \cdots + a_n e^{\alpha_n t} \tag{4}$$

Let us given $(t_1, y_1), (t_2, y_2), \ldots, (t_{2n}, y_{2n})$ (*and let us suppose that the abscissas are equidistant i. e.* $t_{i+1} - t_i = h, i = 1, 2, \ldots, 2n-1$). First we deal with the interpolation case. Because in (1) we have $2n$ variables $(a_1, a_2, \ldots, a_n, \alpha_1, \alpha_2, \ldots, \alpha_n)$ we have to give $2n$ equations:

$$\begin{aligned} y_1 &= a_1 e^{\alpha_1 t_1} + a_2 e^{\alpha_2 t_1} + \ldots + a_n e^{\alpha_n t_1} \\ y_2 &= a_1 e^{\alpha_1 t_2} + a_2 e^{\alpha_2 t_2} + \ldots + a_n e^{\alpha_n t_2} \\ &\vdots \\ y_{2n} &= a_1 e^{\alpha_1 t_{2n}} + a_2 e^{\alpha_2 t_{2n}} + \ldots + a_n e^{\alpha_n t_{2n}}. \end{aligned} \tag{5}$$

Let us introduce new variables:

$$p_k = a_k e^{\alpha_k t_k}\ ,\ z_k = e^{\alpha_k h},\ k = 1 \ldots n. \tag{6}$$

Using this notation one term is in (4):

$$a_k e^{\alpha_k t_j} = a_k e^{\alpha_k (t_1 + (j-1)h)} = a_k e^{\alpha_k t_1} (e^{\alpha_k h})^{j-1} = p_k z^{j-1}, (j = 1, \ldots, 2n), (k = 1, \ldots, n).$$

Now (5) has the following form and we get the equations:

$$\begin{aligned} y_1 &= p_1 + p_2 + p_3 + \ldots + p_n \\ y_2 &= p_1 z_1 + p_2 z_2 + p_3 z_3 + \ldots + p_n z_n \\ y_3 &= p_1 z_1^2 + p_2 z_2^2 + p_3 z_3^2 + \ldots + p_n z_n^2 \\ y_4 &= p_1 z_1^3 + p_2 z_2^3 + p_3 z_3^3 + \ldots + p_n z_n^3 \\ &\vdots \\ y_{2n} &= p_1 z_1^{2n-1} + p_2 z_2^{2n-1} + p_3 z_3^{2n-1} + \ldots + p_n z_n^{2n-1}. \end{aligned} \tag{7}$$

where $p_1, p_2, p_3, \ldots, p_n$ and $z_1, z_2, z_3, \ldots, z_n$ are $2n$ unknown quantities.

3 Ramanujan solution

Ramanujan find an interesting solution of (7)[1]. In the following we make a detailed construction of the proof. It is easy to verify that

$$\begin{aligned} \frac{p_1}{1-\theta z_1} &= p_1 + p_1 z_1 \theta + p_1 z_1^2 \theta^2 + \dots \\ \frac{p_2}{1-\theta z_2} &= p_2 + p_2 z_2 \theta + p_2 z_2^2 \theta^2 + \dots \\ &\vdots \\ \frac{p_n}{1-\theta z_n} &= p_n + p_n z_n \theta + p_n z_n^2 \theta^2 + \dots \end{aligned} \tag{8}$$

The sum of (8) is:

$$\phi(\theta) = \sum \frac{p_i}{1-\theta z_i} \quad = \quad \sum p_i + \theta \sum p_i z_i + \theta^2 \sum p_i z_i^2 + \dots$$

Using (8) we get:

$$\phi(\theta) = \sum_{i=1}^{n} \frac{p_i}{1-\theta z_i} = y_1 + \theta y_2 + \theta^2 y_3 \dots + \theta^n y_{n-1} + \dots + \theta^{2n-1} y_{2n} + \dots \tag{9}$$

The left hand side of (9) is a rational expression, and when it is simplified, then its numerator is an expression of the $(n-1)$-th degree in θ, and its denominator is an expression of the n-th degree in θ

$$\begin{aligned} \phi(\theta) = \sum_{i=1}^{n} \frac{p_i}{1-\theta z_i} = \frac{p_1 \frac{\prod(1-\theta z_i)}{1-\theta z_1} + p_2 \frac{\prod(1-\theta z_i)}{1-\theta z_2} + \dots + p_n \frac{\prod(1-\theta z_i)}{1-\theta z_n}}{\prod(1-\theta z_i)} = \frac{D(\theta)}{N(\theta)} = \\ = \frac{Y_1 + Y_2\theta + Y_3\theta^2 + \dots + Y_n\theta^{n-1}}{1 + Z_1\theta + Z_2\theta^2 + Z_3\theta^3 + \dots + Z_n\theta^n} = y_1 + y_2\theta + y_3\theta^2 + \dots + y_{2n}\theta^{2n-1}, \end{aligned} \tag{10}$$

and so

$$(1+Z_1\theta+Z_2\theta^2+\ldots+Z_n\theta^n)(y_1+y_2\theta+y_3\theta^2+\ldots+y_{2n}\theta^{2n-1}) = Y_1+Y_2\theta+Y_3\theta^2+\ldots+Y_n\theta^{n-1}$$

Equating the coefficiens of the powers of θ, we have

$$\begin{aligned} Y_1 &= y_1 \\ Y_2 &= y_2 + y_1 Z_1 \\ Y_3 &= y_3 + y_2 Z_1 + y_1 Z_2 \\ &\vdots \\ Y_n &= y_n + y_{n-1} Z_1 + y_{n-2} Z_2 + \dots + y_1 Z_{n-1} \end{aligned} \tag{11}$$

and

$$
\begin{aligned}
0 &= y_{n+1} + y_n Z_1 + \cdots + y_1 Z_n \\
0 &= y_{n+2} + y_{n+1} Z_1 + \cdots + y_2 Z_n \\
0 &= y_{n+3} + y_{n+2} Z_1 + \cdots + y_3 Z_n \\
&\vdots \\
0 &= y_{2n} + y_{2n-1} Z_1 + \cdots + y_n Z_n.
\end{aligned} \tag{12}
$$

From (12) $Z_1, Z_2, \ldots, Z_n$ can easily be found, and since $Y_1, Y_2, \ldots, Y_n$ in (11) depend upon these values they can also be found.

Now, our task is splitting

$$\phi(\theta) = \sum_{i=1}^{n} \frac{p_i}{1-\theta z_i} = \frac{Y_1 + Y_2\theta + Y_3\theta^2 + \cdots + Y_n\theta^{n-1}}{1 + Z_1\theta + Z_2\theta^2 + Z_3\theta^3 + \cdots + Z_n\theta^n} =$$

(where $Y_1, \ldots, Y_n, Z_1, \ldots, Z_n$ are known) into partial fractions, i.e.

$$\phi(\theta) = \frac{q_1}{1-r_1\theta} + \frac{q_2}{1-r_2\theta} + \frac{q_3}{1-r_3\theta} + \cdots + \frac{q_n}{1-r_n\theta},$$

and comparing with (1), we see that

$$p_1 = q_1, z_1 = r_1;$$

$$p_2 = q_2, z_2 = r_2;$$

$$p_3 = q_3, z_3 = r_3;$$

$$\vdots$$

$$p_n = q_n, z_n = r_n.$$

which is the desired solution.

The question, how we can get this partial fraction form.

The denominator of (10) is:

$$D(\theta) = 1 + Z_1\theta + Z_2\theta^2 + \cdots + Z_n\theta^n = \prod_{i=1}^{n}(1 - r_i\theta)$$

Let us substitude: $\theta = \frac{1}{\xi}$

$$1 + Z_1\frac{1}{\xi} + Z_2\frac{1}{\xi^2} + \cdots + Z_n\frac{1}{\xi^n} = (1 - r_1\frac{1}{\xi})(1 - r_2\frac{1}{\xi})\ldots(1 - r_n\frac{1}{\xi})) =$$

$$= \frac{1}{\xi^n}\prod_{i=1}^{n}(\xi - r_i)$$

Multipling both side by ξ^n

$$D^*(\xi) = \xi^n + Z_1\xi^{n-1} + \cdots + Z_n = \prod_{i=1}^{n}(\xi - r_i)$$

From $D(\theta)$ we build $D^*(\xi)$ polinom, and the roots of D^* are the r_i.

The nominator of (10) is:

$$\begin{gathered} N(\theta) = \sum_{i=1}^{n} Y_i\theta^{i-1} = \sum_{i=1}^{n} p_i \frac{\prod_{j=1}^{n}(1-\theta r_j)}{1-\theta r_i} = \\ p_1(1-r_2\theta)(1-r_3\theta)\ldots(1-r_n\theta)+ \\ +p_2(1-r_1\theta)(1-r_3\theta)\ldots(1-r_n\theta)+ \\ +p_3(1-r_1\theta)(1-r_2\theta)\ldots(1-r_n\theta)+ \\ \vdots \\ +p_n(1-r_2\theta)(1-r_3\theta)\ldots(1-r_{n-1}\theta) \\ = Y_1 + Y_2\theta + Y_3\theta^2 + \ldots + Y_n\theta^{n-1} \end{gathered}$$

From this we get the following linear equation systems:

$$\begin{aligned} Y_1 &= \sum p_i \\ Y_2 &= \sum p_i \sum_{i\neq j,} r_j \\ Y_3 &= \sum p_i \sum_{i\neq j, i\neq k, j\neq k} r_j r_k \\ &\vdots \\ Y_n &= \sum p_i \prod_{i\neq j} r_j \end{aligned} \tag{13}$$

From (14) we get p_i.

p_i can be determined by n equation of (7)
if p_i is given

$$a_i = \frac{p_i}{e^{\alpha_i t_i}}.$$

4 Ramanujan's example

As an example we may solve the equations:

$$x + y + z + v + u = 2,$$

$$px + qy + rz + su + tv = 3,$$
$$p^2x + q^2y + r^2z + s^2u + t^2v = 16,$$
$$p^3x + q^3y + r^3z + s^3u + t^3v = 31,$$
$$p^4x + q^4y + r^4z + s^4u + t^4v = 103,$$
$$p^6x + q^5y + r^5z + s^5u + t^5v = 235,$$
$$p^6x + q^6y + r^6z + s^6u + t^6v = 674,$$
$$p^7x + q^7y + r^7z + s^7u + t^7v = 1669,$$
$$p^8x + q^8y + r^8z + s^8u + t^8v = 4526,$$
$$p^9x + q^9y + r^9z + s^9u + t^9v = 11595,$$

where $x, y, z, u, v, p, q, r, s, t$ are the unknowns. Proceeding as before, we have

$$\frac{x}{1-\theta p} + \frac{y}{1-\theta q} + \frac{z}{1-\theta r} + \frac{u}{1-\theta s} + \frac{v}{1-\theta t}$$

$$= 2 + 3\theta + 16\theta^2 + 31\theta^3 + 103\theta^4 + 235\theta^5 + 674\theta^6 + 1669\theta^7 + 4526\theta^8 + 11595\theta^9 + ...$$

By the method of indeterminate coefficients, this can be shown to be equal to

$$\frac{2 + \theta + 3\theta^2 + 2\theta^3 + \theta^4}{1 - \theta - 5\theta^2 + \theta^3 + 3\theta^4 - \theta^5}.$$

Splitting this into partial fractions, we get the values of the unknowns, as follows:

$$x = -\frac{3}{5}, p = -1,$$

$$y = \frac{18+\sqrt{5}}{10}, q = \frac{3+\sqrt{5}}{2},$$

$$z = \frac{18-\sqrt{5}}{10}, r = \frac{3-\sqrt{5}}{2},$$

$$u = -\frac{8+\sqrt{5}}{2\sqrt{5}}, s = \frac{\sqrt{5}-1}{2},$$

$$v = \frac{8-\sqrt{5}}{2\sqrt{5}}, t = \frac{\sqrt{5}+1}{2}.$$

5 New solution

We start with (7):

$$\begin{aligned} y_1 &= p_1 + p_2 + p_3 + \ldots + p_n \\ y_2 &= p_1 z_1 + p_2 z_2 + p_3 z_3 + \ldots + p_n z_n \\ y_3 &= p_1 z_1^2 + p_2 z_2^2 + p_3 z_3^2 + \ldots + p_n z_n^2 \\ y_4 &= p_1 z_1^3 + p_2 z_2^3 + p_3 z_3^3 + \ldots + p_n z_n^3 \\ &\vdots \\ y_{2n} &= p_1 z_1^{2n-1} + p_2 z_2^{2n-1} + p_3 z_3^{2n-1} + \ldots + p_n z_n^{2n-1}. \end{aligned} \tag{14}$$

Let us build the polinom

$$z^n + s_1 z^{n-1} + s_2 z^{n-2} + \ldots + s_n = 0 \tag{15}$$

The roots of (15) are $z_1, \ldots, z_n$. The following wellknown identities are valid.

$$\begin{aligned} s_1 &= -\sum z_i \\ s_2 &= \sum_{i \neq j} z_i z_j \\ &\vdots \\ s_n &= (-1)^n \prod z_i \end{aligned} \tag{16}$$

Based on (14) we show that:

$$\begin{aligned} y_1 s_n + y_2 s_{n-1} + \ldots + y_n s_1 + y_{n+1} &= 0 \\ y_2 s_n + y_3 s_{n-1} + \ldots + y_{n+1} s_1 + y_{n+2} &= 0 \\ &\vdots \\ y_n s_n + y_{n+1} s_{n-1} + \ldots + y_{2n-1} s_1 + y_{2n} &= 0 \end{aligned} \tag{17}$$

The k^{th} row is in (17):

$$y_k s_n + y_{k+1} s_{n-1} + y_{k+2} s_{n-2} + \ldots + y_{n+k-1} s_1 + y_{n+k} = 0$$

Using (14):

$$s_n \sum p_i z_i^{k-1} + s_{n-1} \sum p_i z_i^k + \ldots + s_1 \sum p_i z_i^{n+k} + \sum p_i z_i^{n+k-1} = 0$$

Let us find the coefficient of p_j:

$$p_j (s_n z_j^{k-1} + s_{n-1} z_j^k + \ldots + z_j^{n+k-1}) = 0$$

because (15) is valid.

From (17) if $y_1, \dots, y_{2n}$ are given we can get s_i. If s_i is given, then the roots of (15) are the solution for z_i.

$$\alpha_i = \frac{lnz_i}{h} \tag{18}$$

If z_i are given, then from (14) we get p_i.

6 Approximation sum of exponential functions

Now we have more than $2n$ points:

$$(t_1, y_1), (t_2, y_2), ..., (t_m, y_m)$$

and

$$\varphi(t) = a_1 e^{\alpha_1 t} + a_2 e^{\alpha_2 t} + \dots + a_m e^{\alpha_m t}$$

Suppose that the abscissas are equidistant i.e. $t_{i+1} - t_i = h, i = 1, 2, \dots, 2n-1$, and $m > 2n$.

We build the polinom (15) and using the same construction:

$$\begin{aligned} y_1 s_n + y_2 s_{n-1} + \dots + y_n s_1 + y_{n+1} &= 0 \\ y_2 s_n + y_3 s_{n-1} + \dots + y_{n+1} s_1 + y_{n+2} &= 0 \\ &\vdots \\ y_n s_n + y_{n+1} s_{n-1} + \dots + y_{m-1} s_1 + y_m &= 0 \end{aligned} \tag{19}$$

we get it instead of (17).

The A matrix of (19) and the constant vector $\underline{b}$

$$A = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_2 & y_3 & \cdots & y_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m-n} & y_{m-n+1} & \cdots & y_{m-1} \end{bmatrix}$$

$$\underline{b} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The best approximation of (19) is based on the least square method.

$$(A^T A)\underline{s} + A^T \underline{b} = 0 \tag{20}$$

The solution of (20) gives the s̲ values: If s̲ is given we can determine (15), and the roots of (15) are the z_i values, and based on (16) α_i can be determined.
Now we have to find the a_i-values. We use also the least square method:

$$\sum_{i=1}^{n} (\varphi(t_i) - y_i)^2 = \sum_{j=1}^{m} (\sum_{i=1}^{n} a_i e^{\alpha_i t_j} - y_j)^2 \tag{21}$$

The matrix is:

$$F = \begin{bmatrix} e^{\alpha_1 t_1} & e^{\alpha_2 t_1} & \dots & e^{\alpha_n t_1} \\ e^{\alpha_1 t_2} & e^{\alpha_2 t_2} & \dots & e^{\alpha_n t_2} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\alpha_1 t_m} & e^{\alpha_2 t_m} & \dots & e^{\alpha_n t_m} \end{bmatrix}$$

And the $\underline{b} = (y_1, \dots, y_n)$

The best approximation of (21) is:

$$(F^T F)\underline{a} + F^T \underline{b} = 0$$

where a is unknown.

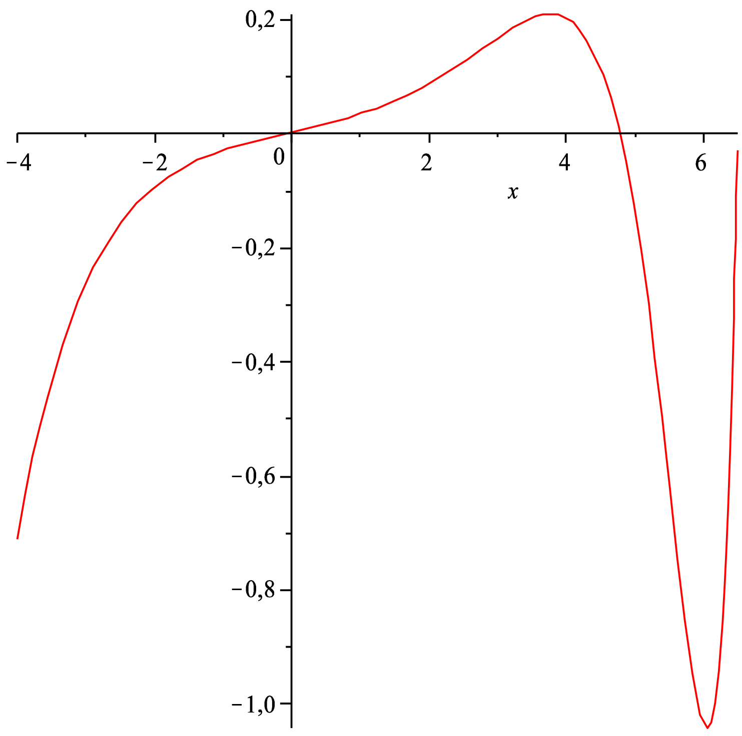

Figure 1: $\phi(x) = (20.09e^{x} - 13.03e^{-x} - 4.481e^{1.5x} + e^{1.7x} + 0.0608e^{1.4x})10^{-3}$

References

[1] S. Ramanujan. Note on a set of simultaneous equations. Journal of the Indian Mathematical Society, 94-94, 1912.

[2] J. Dombi A general class of fuzzy operators, the De Morgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. Fuzzy Sets and Systems Vol 8., 1982, pp 149-163.