Nonlinear Adjustment with Parameter Estimation via Computer Algebra

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Abstract: Computer algebra has been used to solve nonlinear adjustment accompanied by simultaneous parameter estimation. The corresponding constrained minimization can be achieved by Groebner bases with lexicographic monomial order on the domain of rational numbers, employing Groebner Walk, when Buchberger algorithm failed. This method providing infinite precision is about minimum ten times faster than the traditional iterative numerical techniques with finite precision. In addition, computer algebra solution doesn't need initial approximation as the iterative methods do. The suggested technique is illustrated by a numerical example of solving the problem of 2D Helmert transformation. For the numeric and symbolic computations the Mathematica 6.01 system was employed.

Keywords: data adjustment, parameter estimation, computer algebra, Groebner basis, Groebner Walk, Helmert transformation

1 Introduction. Theoretical Concepts

1.1 Adjustment with Parameter Estimation

Least square has been by far predominant technique of data adjustment in photogrammetry, geodesy, surveying, robotics and many other fields. This technique can be used for estimating parameters of a model, employing observation data (measurements), which is called parameter estimaton, or for correcting certain observation data, called data adjustment. However, sometimes we need to do both simultaneously.

In general, these problems are nonlinear and the traditional solutions are mainly based on numerical iterative linearization. Unfortunately, there is no concrete and unique way of choosing initial approximation for linearization. Sometimes experience is relied on, and in other situations some computational shortcuts may be employed.

The proposed computer algebra method does not need such initial approximation and provides solution with infinite precision.

The problem of simultaneous parameter estimation with adjustment can be solved generally as follows.

Let us suppose we have $i = 1,..., n$ model equations:

$$f_i(y_1, y_2, \ldots y_S : \pi_1, \pi_1, \ldots, \pi_X) = 0 \tag{1}$$

where y_i are observable variables and π_i are unknown parameters.

The observable variables are measured, y_{m_i}. Now we should like to estimate the values of the unknown parameters and at the same time to adjust the measured values. Let consider Δ_i as the adjustment for the i-th measured value, with weight w_i, then one has to minimize the following objective function:

$$F_S = \sum_{i=1}^{S} w_i \Delta_i^2 \tag{2}$$

under the constraines represented by the model equations,

$$g_i = f_i(y_{m_1} + \Delta_1, y_{m_2} + \Delta_2, \ldots y_{m_S} + \Delta_S : \pi_1, \pi_1, \ldots, \pi_X) = 0 \tag{3}$$

Employing Lagrange multipliers, λ_i, the constrained minimization problem can be transformed into an unconstrained one, namely:

$$F(\Delta_i, \pi_i, \lambda_i) = F_S(\Delta_i) + \sum_{j=1}^{X} \lambda_j g_j(\Delta_i, \pi_i) \tag{4}$$

In case f_i's are linear functions, the solution in matrix form is well known. However, is f_i's are multivariate polynomials, the necessary condition for the existence of the optimum:

$$\frac{\partial F}{\partial \Delta_i} = 0,\ \frac{\partial F}{\partial \pi_i} = 0,\ \frac{\partial F}{\partial \lambda_i} = 0 \tag{5}$$

leads to a multivariate polynomial equation system, which can be solved using computer algebra.

A Groebner basis for a system of polynomials is an equivalence system, [1]. The set of polynomials in Groebner basis have the same collection of roots as the original polynomials. Groebner bases are very useful for solving the system of polynomial equations and for elimination of variables. The algorithm for computing Groebner bases is known as Buchberger's algorithm, [2]. Calculating a Groebner basis in this way is typically very time-consuming for large polynomial systems and even we have no guarantee to get monomial in the resulted basis, which is the starting point for the futher elimination of the variables.

Frequently, the Groebner Walk, an alternative approach for the calculation of a Groebner basis working by partitioning the computation of the basis into several smaller computations, can be more efficient, [3].

In our case, Groebner Walk proved to be sucessful, when Buchberger algorithm was failed.

1.2 Helmert Transformation

Transformation from one system of coordinates to another is a very useful operation that is used frequently in photogrammetry, [4], geodesy and surveying, [5], as well as in robotics, [6], [7], [8]. Considering two dimensional space, the transformation from one cartesian coordinate system, (x, y) to another, (X, Y) with rotation and scale (Figure 1):

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x\, s\, cos(\Omega) & -y\, s\, sin(\Omega) \\ x\, s\, sin(\Omega) & y\, s\, cos(\Omega) \end{pmatrix} \tag{6}$$

or

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} cos(\Omega) & -sin(\Omega) \\ sin(\Omega) & cos(\Omega) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{7}$$

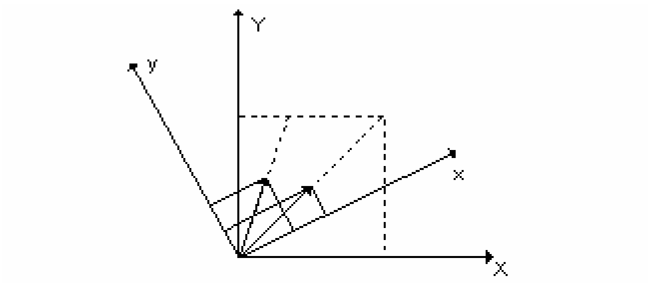

Figure 1
Two-parameter transformation with rotation and scale

Let us suppose, that we have measurements, three corresponding data pairs $(x_i, y_i) \rightarrow (X_i, Y_i)$, $i = a, b, c$, see Table 1, [9].

Table 1
Measured coordinates in both systems

i	x_i	y_i	X_i	Y_i
a	0	1	$-\frac{21}{10}$	$\frac{11}{10}$
b	1	0	1	2
c	1	1	$-\frac{9}{10}$	$\frac{28}{10}$

We require the least square estimates of the transformation parameters, α and β and simultaneouesly the adjustment of the first coordinates, namely x_a, x_b, x_c, X_a, X_b and X_c.

2 Solution via Computer Algebra

Now let us employ the general procedure described above. Applying the two equations of the transformation for each of the three point - pairs with the adjusted values:

$$
\begin{aligned}
f_a &= \alpha(x_a + \Delta_{x_a}) - \beta\, y_a - (X_a + \Delta_{X_a}) = 0 \\
g_a &= \beta(x_a + \Delta_{x_a}) + \alpha\, y_a - Y_a = 0 \\
f_b &= \alpha(x_b + \Delta_{x_b}) - \beta\, y_b - (X_b + \Delta_{X_b}) = 0 \\
g_b &= \beta(x_b + \Delta_{x_b}) + \alpha\, y_b - Y_b = 0 \\
f_c &= \alpha(x_c + \Delta_{x_c}) - \beta\, y_c - (X_c + \Delta_{X_c}) = 0 \\
g_c &= \beta(x_c + \Delta_{x_c}) + \alpha\, y_c - Y_c = 0
\end{aligned}
\tag{8}
$$

In these 6 equations there are 8 unknowns, the adjustments ($\Delta_{x_a}, \Delta_{x_b}, \Delta_{x_c}, \Delta_{X_a}, \Delta_{X_b}, \Delta_{X_c}$) and the two parameters (α, β) to be estimated. The constrained minimization problem can be formulated as:

$$
\begin{aligned}
F = {} & \Delta_{x_a}^2 + \Delta_{x_b}^2 + \Delta_{x_c}^2 + \Delta_{X_a}^2 + \Delta_{X_b}^2 + \Delta_{X_c}^2 + \\
& + \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\} \times \{f_a, f_a, f_b, g_b, f_c, g_c\}
\end{aligned}
\tag{9}
$$

Now, we have 14 unknowns

$$d = (\Delta_{x_a}, \Delta_{x_b}, \Delta_{x_c}, \Delta_{X_a}, \Delta_{X_b}, \Delta_{X_c}, \alpha, \beta, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \tag{10}$$

The necessary condition for the existence of the optimum provides the following polynomial equations,

$$\begin{aligned}
&2\Delta_{x_a} + \alpha\lambda_1 + \beta\lambda_2 = 0\\
&2\Delta_{x_b} + \alpha\lambda_3 + \beta\lambda_4 = 0\\
&2\Delta_{x_c} + \alpha\lambda_5 + \beta\lambda_6 = 0\\
&2\Delta_{X_a} - \lambda_1 = 0\\
&2\Delta_{X_b} - \lambda_3 = 0\\
&2\Delta_{X_c} - \lambda_5 = 0
\end{aligned} \tag{11}$$

$$\begin{aligned}
&x_a\lambda_1 + \Delta_{x_a}\lambda_1 + y_a\lambda_2 + x_b\lambda_3 + \Delta_{x_b}\lambda_3 + y_b\lambda_4 + x_c\lambda_5 + \Delta_{x_c}\lambda_5 + y_c\lambda_6 = 0\\
&- y_a\lambda_1 + x_a\lambda_2 + \Delta_{x_a}\lambda_2 - y_b\lambda_3 + x_b\lambda_4 + \Delta_{x_b}\lambda_4 - y_c\lambda_5 + x_c\lambda_6 + \Delta_{x_c}\lambda_6 = 0\\
&\alpha x_a - X_a - \beta y_a + \alpha\Delta_{x_a} - \Delta_{X_a} = 0\\
&\beta x_a + \alpha y_a - Y_a + \beta\Delta_{x_a} = 0\\
&\alpha x_b - X_b - \beta y_b + \alpha\Delta_{x_b} - \Delta_{X_b} = 0\\
&\beta x_b + \alpha y_b - Y_b + \beta\Delta_{x_b} = 0\\
&\alpha x_c - X_c - \beta y_c + \alpha\Delta_{x_c} - \Delta_{X_c} = 0\\
&\beta x_c + \alpha y_c - Y_c + \beta\Delta_{x_c} = 0
\end{aligned}$$

It has turned out, that Buchberger algorithm can not find the lexicographics, but only the reverse lexicographics Groebner bases. This bases consists of 24 polynomials, but none of them is a monomial.

However, employing Groebner Walk, this basis could be reduced to 14 polynomials and already provides a monomial, a polynomial of order six with the variable λ_6 (see equation (12)).

$$\begin{aligned}
&-763094995516248252876710834 0 +\\
&+ 429598392278896826677930481 33\lambda_6 -\\
&- 489181086373271123938589713 61\lambda_6^2 +\\
&+ 104610954860700279913881577 80\lambda_6^3 +\\
&+ 104018749323715741160794050 00\lambda_6^4 -\\
&- 382929968026648328876789062 5\lambda_6^5 +\\
&+ 349089071788949996689453125\lambda_6^6
\end{aligned} \tag{12}$$

The positive real solution has only physical meaning, $\lambda_6 = 0.238268$.

The numerical computation of the further variables is easy because the exponents of the variables in the polynomials of the basis are very simple, see Table 2.

Table 2
The maximum power of the variables in the polynomials of the Groebner basis resulted by Groebner Walk

Δ_{x_a}	Δ_{x_b}	Δ_{x_c}	Δ_{X_a}	Δ_{X_b}	Δ_{X_c}	α	β	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0	0	0	0	0	0	0	0	0	0	0	0	0	**6**
0	0	0	0	0	0	0	0	0	0	0	0	**1**	**5**
0	0	0	0	0	0	0	0	0	0	0	**1**	0	**5**
0	0	0	0	0	0	0	0	0	0	**1**	0	0	**5**
0	0	0	0	0	0	0	0	0	**1**	0	0	0	**5**
0	0	0	0	0	0	0	0	**1**	0	0	0	0	**5**
0	0	0	0	0	0	0	**1**	0	0	0	0	0	**5**
0	0	0	0	0	0	**1**	0	0	0	0	0	0	**5**
0	0	0	0	0	**1**	0	0	0	0	0	0	0	**5**
0	0	0	0	**1**	0	0	0	0	0	0	0	0	**5**
0	0	0	**1**	0	0	0	0	0	0	0	0	0	**5**
0	0	**1**	0	0	0	0	0	0	0	0	0	0	**5**
0	**1**	0	0	0	0	0	0	0	0	0	0	0	**5**
1	0	0	0	0	0	0	0	0	0	0	0	0	**5**

The result of the numerical computation is in Table 3.

Table 3
Result of the computation

Δ_{x_a}	0.0218895
Δ_{x_b}	0.0215372
Δ_{x_c}	-0.109804
Δ_{X_a}	0.165307
Δ_{X_b}	0.0799118
Δ_{X_c}	-0.116768
α	1.05714
β	1.95783

The computations were carried out on hp xw4100 workstation using Intell Pentium 4 processor with 3 GHz and 1GB RAM.

Conclusions

The computation time of the Groebner basis with infinite precision, using Groebner Walk took 0.203 seconds and that of the solution of the system of polynomials of the basis is neglegible. However, to solve the nonlinear minimization problem numerically took 2.884 seconds with 16 significant digits. It means that the solution using computer algebra method - Groebner basis provided by Groebner Walk is ten times faster than the traditional numerical solution. In addition, increasing precision of the solution needs more and more time using traditional methods, while the computer algebra solution is invariant for that. But perhaps the main goal of the computer algebra solution is, that it does not need an initial approximation for the linearization

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