Approximation and Complexity Trade-off by TP model transformation in Controller Design: a Case Study of the TORA system

Zoltán Petres and Péter Baranyi

System and Control Laboratory,
Computer and Automation Research Institute, Hungarian Academy of Sciences

E-mail: {petres,baranyi}@sztaki.hu

Abstract — The main objective of the paper is to study the approximation and complexity trade-off capabilities of the recently proposed Tensor Product Distributed Compensation (TPDC) based control design framework. The TPDC is the combination of the TP model transformation and the Parallel Distributed Compensation (PDC) framework. The TP model transformation includes an HOSVD based technique to solve the approximation and complexity trade-off. In this paper we generate TP models with different complexity and approximation properties, and then we derive controllers for them. We analyze how the trade-off effects the model behavior and control performance. All these properties are studied via the state feedback controller design of the Translational Oscillations with an Eccentric Rotational Proof Mass Actuator (TORA) System.

1 Introduction

The TP model form is a dynamic model representation whereupon Linear Matrix Inequality (LMI) based control design techniques [1–3] can immediately be executed. It describes a class of Linear Parameter Varying (LPV) models in a polytopic form that is the convex combination of linear time invariant (LTI) models, where the convex combination is defined by the weighting functions of each parameter separately. This model is called TP model, or polytopic model.

The TP model transformation is a recently proposed numerical method to transform LPV models into TP model form (polytopic form) [4, 5]. It is capable of transforming different LPV model representations (such as physical model given by analytic equations, fuzzy, neural network, genetic algorithm based models) into TP model form in a uniform way. In this sense it replaces the analytical derivations and affine decompositions (that could be a very complex or even an unsolvable task), and automatically results in the TP model form. Execution of the TP model transformation takes a few minutes by a regular Personal Computer. The TP model transformation minimizes the number of the LTI components of the resulting TP model. Furthermore, the TP model transformation is capable of resulting different convex hulls of the given LPV model.

One can find a number of LMIs under the PDC framework which can immediately be applied to the TP model, according to various control design specifications. Therefore, it is worth linking the TP model transformation and the PDC design framework [6]. That is called Tensor Product Distributed Compensation (TPDC) in the literature.

During the TPDC controller design procedure complexity issues can occur that can inhibit the derivation of the controller, or the complexity of the resulting controller is so high that it is impossible to handle in real world operation. The TPDC framework offers trade-off techniques that help us to control the model complexity and approximation accuracy challenge. In this paper we derive TP models with different complexity of the TORA system, and design controllers that assures asymptotic stability.

In order to study the trade-off capability of TP model transformation we present a case study of the TORA system. We generate TP models with different complexity, and we derive controllers to each models. We analyze how the behavior of these models and the controllers' performance change compared to the exact TP model by reducing more and more the complexity.

The rest of the paper is organized as follows: Section 2 introduce the Tensor Product Distributed Compensation based controller design framework. Section 3 at first describes the TORA system, and discuss the goals and the specifications of the controller. Then, the different TP models are given, and through simulations the designed controllers are analyzed and compared. Finally, Section 4 concludes the results.

2 Tensor Product Distributed Compensation (TPDC) based Controller Design Framework

Consider the following parameter-varying state-space model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t), \tag{1}$$

$$\mathbf{y}(t) = \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{D}(\mathbf{p}(t))\mathbf{u}(t),$$

with input $\mathbf{u}(t)$, output $\mathbf{y}(t)$ and state vector $\mathbf{x}(t)$. The system matrix

$$\mathbf{S}(\mathbf{p}(t)) = \begin{pmatrix} \mathbf{A}(\mathbf{p}(t)) & \mathbf{B}(\mathbf{p}(t)) \\ \mathbf{C}(\mathbf{p}(t)) & \mathbf{D}(\mathbf{p}(t)) \end{pmatrix} \in \mathbb{R}^{O \times I} \tag{2}$$

is a parameter-varying object, where $\mathbf{p}(t) \in \Omega$ is time varying N-dimensional parameter vector, and is an element of the closed hypercube $\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N] \subset \mathbb{R}^N$. The parameter $\mathbf{p}(t)$ can also include some elements of $\mathbf{x}(t)$.

The TP model transformation starts with the given LPV model (1) and results in the TP model representation

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} \underset{\varepsilon}{\approx} \mathcal{S} \bigotimes_{n=1}^{N} \mathbf{w}_n(p_n) \begin{pmatrix} x \\ u \end{pmatrix} \tag{3}$$

that can always be transformed to the polytopic form:

$$\begin{pmatrix}\dot{\mathbf{x}}(t)\\ \mathbf{y}(t)\end{pmatrix} \underset{\varepsilon}{\approx} \sum_{r=1}^{R} w_r(\mathbf{p}(t))\mathbf{S}_r \begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{pmatrix} \tag{4}$$

where

$$\varepsilon = \left(\left\| \mathbf{S}(\mathbf{p}(t)) - \sum_{r=1}^{R} w_r(\mathbf{p}(t))\mathbf{S}_r \right\|_{L_2} \right)^2 \leq \sum_k \sigma_k^2. \tag{5}$$

Here, ε symbolizes the bounded approximation error, σ_k are the singular values discarded during the trade-off of the TP model transformation [5, 7], and $w_r(\mathbf{p}(t)) \in [0, 1]$ are the coefficient functions. For further details about TP model transformation we refer to [4, 5, 8].

The TP model transformation ensures the convexity of the convex combination of the LTI systems as follows:

Definition 1 *The model (4) is convex if:*

$$\forall r \in [1, R], \mathbf{p}(t) \quad : \quad w_r(\mathbf{p}(t)) \in [0, 1] \tag{6}$$

$$\forall r \in [1, R], \mathbf{p}(t) \quad : \quad \sum_{i=1}^{I_n} w_r(\mathbf{p}(t)) = 1 \tag{7}$$

This simply means that $\mathbf{S}(\mathbf{p}(t))$ is within the convex hull of the LTI vertex systems $\mathbf{S}_r$ for any $\mathbf{p}(t) \in \Omega$.

$\mathbf{S}(\mathbf{p}(t))$ has a finite element TP model representation in many cases ($\varepsilon = 0$ in (4)). In this case we say that the TP model is exact. However, exact finite element TP model representation does not exist in general ($\varepsilon > 0$ in (4)), see Ref. [9]. In this case $\varepsilon \to 0$, when the number of the LTI systems involved in the TP model goes to ∞.

The TP model transformation helps the trade-off between the complexity of the model, namely the number of LTI vertex systems, and the modeling accuracy, denoted by ε in Equation (5) [4]. The TP model transformation offers options to generate different types of the weighting functions $w(\cdot)$ to a given specification.

One class of the LMI based control design methods, the Parallel Distributed Compensation (PDC) framework was introduced by Tanaka and Wang [6]. The PDC design framework determines one LTI feedback gain to each LTI vertex system of a given TP model. The inputs of the framework are the LTI vertex systems $\mathbf{S}_r$, and the results are the LTI vertex gains $\mathbf{F}_r$ of the controller. These gains $\mathbf{F}_r$ are obtained from a feasible solution of the LMI based stability theorems. After having the $\mathbf{F}_r$, the control value $\mathbf{u}(t)$ is determined by the help of the same TP model structure used in (4):

$$\mathbf{u}(t) = -\left(\sum_{r=1}^{R} w_r(\mathbf{p}(t))\mathbf{F}_r \right) \mathbf{x}(t). \tag{8}$$

The LMI theorems, to be solved under the PDC framework, are selected according to the stability criteria and the desired control performance. For instance, the speed of response, constraints on the state vector or on the control value can be considered via properly selected LMI based stability theorems.

3 Complexity and approximation trade-off in the control of the TORA system

This section is devoted to show through the case study of the TORA system the approximation trade-off capabilities of the TP model transformation. Besides the exact TP model, we also generate complexity relaxed TP models. For each TP model a corresponding controller is designed. We analyze how the complexity relaxation changes the behavior of the TP models, and influence the controller performances. At the end of the section we make a comprehensive summary and comparison.

The study is conducted through a state feedback control design for the Translational Oscillations with an Eccentric Rotational Proof Mass Actuator (TORA) System, which was originally studied as a simplified model of a dual-spin spacecraft with mass imbalance to investigate the resonance capture phenomenon [10, 11]. The same plant was later studied involving the rotational proof-mass actuator for feedback stabilization of translational motion [12, 13]. The TORA system is also considered as a fourth-order benchmark problem [14–16]. The *International Journal of Robust and Nonlinear Control* published a series of studies about the control issue of the TORA system in Volume 8 in 1998 [17].

3.1 Nomenclature

- M = mass of cart
- k = linear spring stiffness
- m = mass of the proof-mass actuator
- I = moment of inertia of the proof-mass actuator
- e = distance between the rotation point and the center of the proof mass
- N = control torque applied to the proof mass
- F = is the disturbance force on the cart
- q = translational position of the cart
- θ = angular position of the rotational proof mass

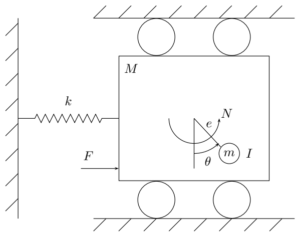

Figure 1: TORA system

3.2 Equations of motion

The TORA system is shown in Figure 1 with the notation defined above. The oscillation consists of a cart of mass M connected to a fixed wall by a linear spring of stiffness k. The cart is constrained to have one-dimensional travel in the horizontal plane. The rotating proof-mass actuator is attached to the cart. The control torque is applied to the proof mass. $\theta = 0°$ is perpendicular to the motion of the cart, while $\theta = 90°$ is aligned with the positive q direction. The equations of motion are given by [17]:

$$(M + m)\ddot{q} + kq = -me(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta) \tag{9}$$

$$(I + me^2)\ddot{\theta} = -me\ddot{q}\cos\theta + N \tag{10}$$

with the normalization [12]:

$$\xi \simeq \sqrt{\frac{M + m}{I + me^2}}q \quad \tau \simeq \sqrt{\frac{k}{M + m}}t \tag{11}$$

$$u \simeq \frac{M + m}{k(I + me^2)}N \tag{12}$$

the equations of motion become

$$\ddot{\xi} + \xi = \rho\left(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta\right) \tag{13}$$

$$\ddot{\theta} = -\rho\ddot{\xi}\cos\theta + u \tag{14}$$

where ξ is the normalized cart position, and u is the per unit control torque. τ is the normalized time whereupon the differentiation is understood. ρ is the coupling between the rotational and the translational motions:

$$\rho \simeq \frac{me}{\sqrt{(I + me^2)(M + m)}}. \tag{15}$$

The above equations can be given in the state-space model form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{g}(\mathbf{x}(t))u(t), \tag{16}$$

Table 1: Parameters of the TORA system

Description	Parameter	Value	Units
Cart mass	M	1.3608	kg
Arm mass	m	0.096	kg
Arm eccentricity	e	0.0592	m
Arm inertia	I	0.0002175	kg m^2
Spring stiffness	k	186.3	N/m
Coupling parameter	ρ	0.200	—

$$\mathbf{y}(t) = \mathbf{c}(\mathbf{x}(t)),$$

where

$$\mathbf{f}(\mathbf{x}(t)) = \begin{pmatrix} x_2 \\ \frac{-x_1+\rho x_4^2 \sin(x_3)}{1-\rho^2\cos^2(x_3)} \\ x_4 \\ \frac{\rho\cos(x_3)(x_1-\rho x_4^2\sin(x_3))}{1-\rho^2\cos^2(x_3)} \end{pmatrix}, \quad \mathbf{g}(\mathbf{x}(t)) = \begin{pmatrix} 0 \\ \frac{-\rho\cos(x_3)}{1-\rho^2\cos^2(x_3)} \\ 0 \\ \frac{1}{1-\rho^2\cos^2(x_3)} \end{pmatrix}, \tag{17}$$

$$\mathbf{c}(\mathbf{x}(t)) = \begin{pmatrix} 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix}.$$

and $\mathbf{x}(t) = \begin{pmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{pmatrix}^T = \begin{pmatrix} \xi & \dot{\xi} & \theta & \dot{\theta} \end{pmatrix}^T$. Let us write the above equation in the typical form of linear parameter-varying state-space model as

$$\dot{\mathbf{x}}(t) = \mathbf{S}(\mathbf{p}(t)) \begin{pmatrix} \mathbf{x}(t) \\ u(t) \end{pmatrix} \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \tag{18}$$

where system matrix $\mathbf{S}(\mathbf{p}(t))$ contains:

$$\mathbf{S}(\mathbf{p}(t)) = \begin{pmatrix} \mathbf{A}(\mathbf{p}(t)) & \mathbf{B}(\mathbf{p}(t)) \end{pmatrix}$$

and $\mathbf{p}(t) = \begin{pmatrix} x_3(t) & x_4(t) \end{pmatrix} \in \Omega$ is time-varying 2nd-order parameter vector, thus

$$\mathbf{A}(x_3(t), x_4(t)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{1-\rho^2\cos^2(x_3)} & 0 & 0 & \frac{\rho x_4\sin(x_3)}{1-\rho^2\cos^2(x_3)} \\ 0 & 0 & 0 & 1 \\ \frac{\rho\cos(x_3)}{1-\rho^2\cos^2(x_3)} & 0 & 0 & \frac{-x_4\rho^2\cos(x_3)\sin(x_3)}{1-\rho^2\cos^2(x_3)} \end{pmatrix} \tag{19}$$

$$\mathbf{B}(x_3(t)) = g(x(t)) \quad \mathbf{C} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The laboratory version of the TORA system is described in [15]. The nominal configuration of this version is given in Table 1.

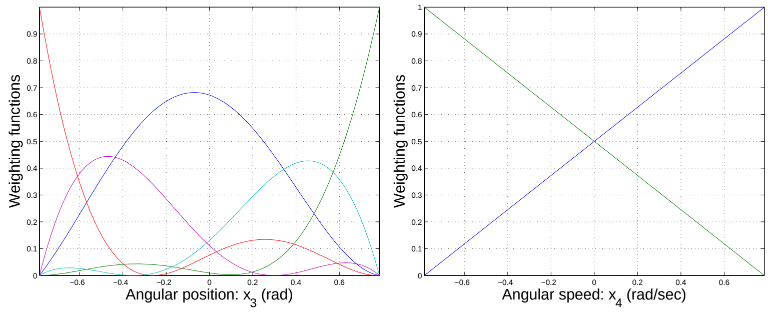

Figure 2: Weighting functions of exact TP model for $x_3(t)$ and $x_4(t)$

3.3 Different complexity relaxed TP models generated by the TP model transformation

We execute the TP model transformation on the LPV model (18) of the TORA. As a first step of the TP model transformation we have to define the transformation space Ω. If we see the simulations in the papers of special issue [17] and [18–20] we find that θ is always smaller than 0.85 rad, and according to the maximum allowed torque $u = 0.1$ N the system would not achieve larger $\dot{\theta}$ than 0.5 rad, but could have a little overshoot, see [17, page 392] and control specifications section of this paper. Therefore, we define the transformation space as $\Omega = [-a, a] \times [-a, a]$ ($x_3(t) \in [-a, a]$ and $x_4(t) \in [-a, a]$), where $a = \frac{45}{180}\pi$ rad (note that these intervals can be arbitrarily defined). The TP model transformation starts with the discretization over a rectangular grid. Let the density of the discretization grid be 137×137 on $(x_3(t) \in [-a, a]) \times (x_4(t) \in [-a, a])$. The result of the TP model transformation shows that TORA system can be exactly given in the HOSVD-based canonical polytopic model form with minimum $5 \times 2 = 10$ LTI vertex models.

TP MODEL 0 In this case we discard only zero singular values, then the TP transformation results in an exact representation of the TORA system:

$$\dot{\mathbf{x}}(t) = \mathbf{S}(\mathbf{p}(t)) \begin{pmatrix} \mathbf{x}(t) \\ u(t) \end{pmatrix} = \sum_{i=1}^{5} \sum_{j=1}^{2} w_{1,i}(x_3(t)) w_{2,j}(x_4(t)) \left(\mathbf{A}_{i,j}\mathbf{x}(t) + \mathbf{B}_{i,j}u(t) \right). \quad (20)$$

The close to NO type weighting functions that define the tight convex hull of the LPV model is depicted in Figure 2.

During the TP model transformation based controller design procedure complexity issues can occur that can inhibit the derivation of the controller, or the complexity of the resulting controller is so high that it is impossible to handle in real world operation. The TP model transformation based control design framework offers trade-off techniques that help us to control the model complexity and approximation accuracy challenge.

Besides the exact TP model, we also generate complexity relaxed TP models of the TORA system. In later section for each TP model a corresponding controller is designed. We analyze how the complexity relaxation changes the behavior of the TP models, and influence the controller performances.

In the followings we generate close to NO types weighting function, but in each TP model we reduce the number of weighting function, thus the complexity of the model.

As it has been already shown, the rank of the system matrix $\mathbf{S}(\mathbf{p}(t))$ in the dimension of $x_3(t)$ is 5, whilst in the dimension of $x_4(t)$ is 2. The nonzero singular values in each dimension is

$$
\begin{aligned}
\sigma_{1,1} &= 341.31 & \sigma_{2,1} &= 341.31\\
\sigma_{1,2} &= 5.5948 & \sigma_{2,2} &= 5.5948\\
\sigma_{1,3} &= 3.8334 & &\\
\sigma_{1,4} &= 0.085134 & &\\
\sigma_{1,5} &= 0.041615 & &
\end{aligned}
$$

TP MODEL 1 The complexity of the TP model can be reduced in the dimension of $x_3(t)$ by discarding some singular values. Note that in the dimension of $x_4(t)$ the reduction is not possible since convexity (Definition **??**) requires at least two weighting functions. Hence, let us keep the four largest singular values of dimension $x_3(t)$. It results an approximation of the TORA system that is composed of $4 \times 2 = 8$ LTI systems. Figure 3 shows the weighting functions of the tight convex hull of the reduced LPV model.

$$
\dot{\mathbf{x}}(t) = \mathbf{S}(\mathbf{p}(t)) \begin{pmatrix} \mathbf{x}(t) \\ u(t) \end{pmatrix} = \sum_{i=1}^{4} \sum_{j=1}^{2} w_{1,i}(x_3(t)) w_{2,j}(x_4(t)) \left(\mathbf{A}_{i,j}\mathbf{x}(t) + \mathbf{B}_{i,j}u(t) \right). \tag{21}
$$

Eq.$\tilde{(}$5) gives only an upper bound for the approximation error that is calculated by the discarded singular values. In case of TP MODEL 1 it is $\sigma_{1,5} = 0.041615$. In order to measure the actual modeling approximation error by numerical checking, the difference of the analytical model and TP models, in terms of L_2 norm, were calculated over 10 000 random sample points. The numerical approximation error of TP MODEL 2b is 0.0023.

TP MODEL 2 In the dimension of $x_3(t)$ further reduction is possible. By discarding the two smallest singular values, namely $\sigma_{1,4}$ and $\sigma_{1,5}$, we get a more complexity relaxed TP model of the TORA system. The resulting system equation is

$$
\dot{\mathbf{x}}(t) = \mathbf{S}(\mathbf{p}(t)) \begin{pmatrix} \mathbf{x}(t) \\ u(t) \end{pmatrix} = \sum_{i=1}^{3} \sum_{j=1}^{2} w_{1,i}(x_3(t)) w_{2,j}(x_4(t)) \left(\mathbf{A}_{i,j}\mathbf{x}(t) + \mathbf{B}_{i,j}u(t) \right). \tag{22}
$$

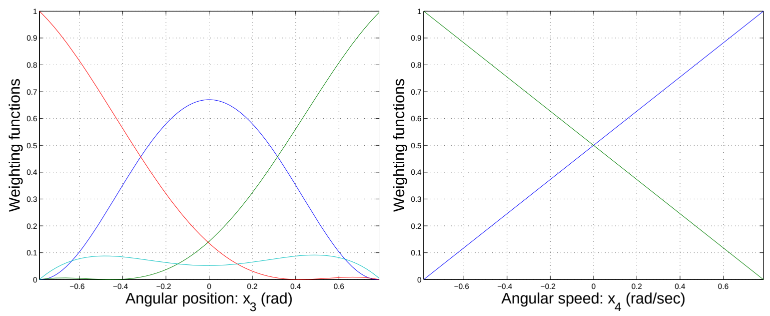

Figure 3: Close to NO type weighting functions of the reduced TP model of 8 LTI systems

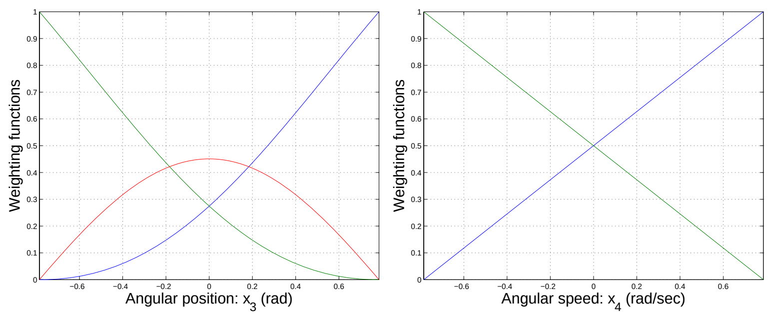

Figure 4: Close to NO type weighting functions of the reduced TP model of 6 LTI systems

In this case the upper bound of the approximation error is 0.126749, while the maximal numerical error is 0.0024. In Figure 4 the resulting weight functions are illustrated.

TP MODEL 3 By keeping only the two largest singular values in dimension $x_3(t)$, the most reduced TP model of the TORA system realizes. The system equation of this model is

$$\dot{\mathbf{x}}(t) = \mathbf{S}(\mathbf{p}(t)) \begin{pmatrix} \mathbf{x}(t) \\ u(t) \end{pmatrix} = \sum_{i=1}^{2} \sum_{j=1}^{2} w_{1,i}(x_3(t)) w_{2,j}(x_4(t)) \left(\mathbf{A}_{i,j}\mathbf{x}(t) + \mathbf{B}_{i,j}u(t) \right). \tag{23}$$

In this case the upper bound of the approximation error is 3.960149, while the maximal measured error is 0.21513. In Figure 5 the resulting weight functions are illustrated.

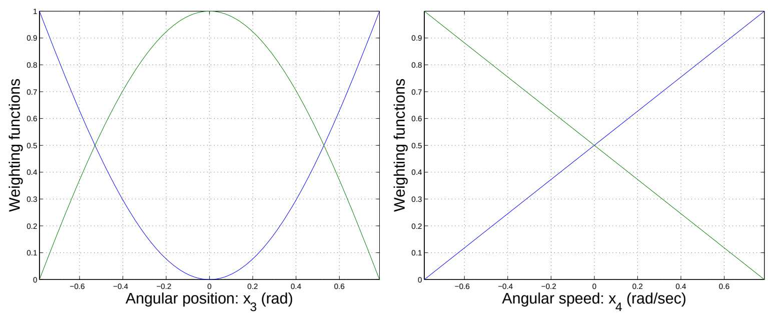

Figure 5: Close to NO type weighting functions of the reduced TP model of 4 LTI systems

Comparison of the resulting TP models The main conclusion of the comparison is the complexity of the model can be drastically reduced without causing unacceptable approximation error as the results, summarized in Table 2 on page 13, show. The second conclusion is that the estimated error bound from the singular values are much worse than the actual approximation error. As a matter of fact we have to be careful when selecting the non-exact approximation. In most cases, and in the case of TORA, the approximated model is suitable, but there are some systems, *e.g.* chaotic systems, when even slight changes can cause drastic differences. It is also worth noticing here that usually the error of typical identification techniques is in a larger range than the discarded singular values.

3.4 Derivation of controllers

For the present controller design, we apply the widely adapted design specifications detailed on [17, page 309] and in [18–20]. These specifications for the TORA system can be summarized as follows:

Design a controller that satisfies the following criteria:

- The closed-loop system exhibits good settling behavior for a class of initial conditions.
- The closed-loop system is stable (in this paper we aim at achieving asymptotic stability).
- The physical configuration of the system necessities the constraint $|q| \leq 0.025\,\mathrm{m}$.
- The control value is limited by $N \leq 0.100\,\mathrm{Nm}$, although somewhat higher torques can be tolerated for short periods.

Considering these specifications, we derive controllers by applying the TP model and the LMI theorems. Having the solution of the LMIs the feedback gains $\mathbf{F}_r$ are computed by Equation (26), and the control value is computed by Equation (8). In the present case it is:

$$u(t) = -\left(\sum_{r=1}^{R} w_r(x_3(t), x_4(t))\mathbf{F}_r\right) x(t),$$

where R is the number of LTI systems of the applied TP model.

We select the following LMI systems to guarantee the above given specifications. The derivations and the proofs of these theorems are fully detailed in [6].

Theorem 1 (Asymptotic stability) *TP model (4) with control value (8) is asymptotically stable if there exists* $\mathbf{X} > 0$ *and* $\mathbf{M}_r$ *satisfying equations*

$$-\mathbf{X}\mathbf{A}_r^T - \mathbf{A}_r\mathbf{X} + \mathbf{M}_r^T\mathbf{B}_r^T + \mathbf{B}_r\mathbf{M}_r > \mathbf{0} \tag{24}$$

for all r and

$$\begin{aligned} &-\mathbf{X}\mathbf{A}_r^T - \mathbf{A}_r\mathbf{X} - \mathbf{X}\mathbf{A}_s^T - \mathbf{A}_s\mathbf{X}+ \\ &+\mathbf{M}_s^T\mathbf{B}_r^T + \mathbf{B}_r\mathbf{M}_s + \mathbf{M}_r^T\mathbf{B}_s^T + \mathbf{B}_s\mathbf{M}_r \geq \mathbf{0}. \end{aligned} \tag{25}$$

for $r < s \leq R$*, except for the pairs* (r, s) *such that* $w_r(\mathbf{p}(t))w_s(\mathbf{p}(t)) = 0, \forall \mathbf{p}(t)$.

Theorem 2 (Constraint on the control value) *Assume that* $\|\mathbf{x}(0)\| \leq \phi$*, where* $\mathbf{x}(0)$ *is unknown, but the upper bound* ϕ *is known. The constraint* $\|\mathbf{u}(t)\|_2 \leq \mu$ *is enforced at all times* $t \geq 0$ *if the LMIs*

$$\begin{aligned} \phi^2\mathbf{I} &\leq \mathbf{X} \\ \begin{pmatrix} \mathbf{X} & \mathbf{M}_i^T \\ \mathbf{M}_i & \mu^2\mathbf{I} \end{pmatrix} &\geq \mathbf{0} \end{aligned}$$

hold.

Theorem 3 (Constraint on the output) *Assume that* $\|\mathbf{x}(0)\| \leq \phi$*, where* $\mathbf{x}(0)$ *is unknown, but the upper bound* ϕ *is known. The constraint* $\|\mathbf{y}(t)\|_2 \leq \lambda$ *is enforced at all times* $t \geq 0$ *if the LMIs*

$$\begin{aligned} \phi^2\mathbf{I} &\leq \mathbf{X} \\ \begin{pmatrix} \mathbf{X} & \mathbf{X}\mathbf{C}_i^T \\ \mathbf{C}_i\mathbf{X} & \lambda^2\mathbf{I} \end{pmatrix} &\geq \mathbf{0} \end{aligned}$$

hold.

We compose a joint LMI system of Theorem 1–3 to guarantee the stability issues and constraints defined in the above control specification. The feedback gains are determined form the solutions $\mathbf{X}$ and $\mathbf{M}_r$ as

$$\mathbf{F}_r = \mathbf{M}_r\mathbf{X}^{-1}. \tag{26}$$

The feasible solution of this joint LMI system can be easily computed by an LMI solver, *e.g.* the one included in the LMI package of MATLAB Robust Control Toolbox.

Controller 0 We consider this controller as the reference controller, and the response of the rest of the controllers are compared to this. The controller is designed for TP MODEL 0. This requires the solution of 67 LMI equations.

Controller 1 The feedback gains for the controller of TP MODEL 1 is obtained by the feasible solution of an LMI system containing 46 LMI equations.

Controller 2 The LMI system of TP MODEL 2c consists of 29 LMIs. The feedback gains of the controller, that is derived from the feasible solution of LMI system, is the following:

Controller 3 We applied the same LMI system to TP MODEL 3 that results a system of 16 LMI equations. The feedback gains of the controller is obtained from the feasible solution of the LMI system.

Simulation results and comparison of derived controllers In the simulation the system's initial configuration was $\mathbf{x}(0) = \begin{pmatrix} 0.023\,\mathrm{m} & 0 & 0 & 0 \end{pmatrix}$. The results of the four controllers are shown and are plotted together for better visualization in Figure 6. Figure 7 shows some parts of the simulation results magnified in order to highlight the differences. We can see on the figures, there are only slight differences between the responses of the different controllers, practically we can say that the results are identical despite of the applied reduction during the TP model transformation. The main reason behind this fact is that the strength of influence of the LTI models is proportional to the magnitude of the singular values. Therefore, the magnitude of differences between the designed controllers is also strongly related to the magnitude of the singular values. For illustration, we should analyze carefully the responses of the controller of the exact model (indicated with "CTRL 0" in the figures), and the controller Controller 2. The difference is so small, because the difference of the exact TP model, TP MODEL 0 and the TP MODEL 1 from which the controllers were derived is also really small. The contribution of the neglected LTIs to the TP model is proportional to the ratios of the singular values, and $\sigma_{1,5}$ has only an effect of 0.012%. If we analyze the response of Controller 2, we cannot see more significant difference, because the contribution of $\sigma_{1,4}$ is also around 0.025%, thus together with $\sigma_{1,5}$ it is still around 0.037% in total. A slightly significant change can be observer in the response of Controller 3. In case the sum of the discarded singular values has an effect of about 1.16%, thus a bigger results in the response of the controller.

Another important issue concerning these results is that in order to derive Controller 0, an LMI system of 67 LMIs has to be solved, whilst 16 LMIs can describe Controller 0, the controller of the most reduced TP model that is a 76% of reduction. The TORA system is a simple model, the number of LMIs is moderate, but by defining more constraints, applying more complex controller specifications, such as decay rate control, observer design, etc., and if the TP model of the system consists

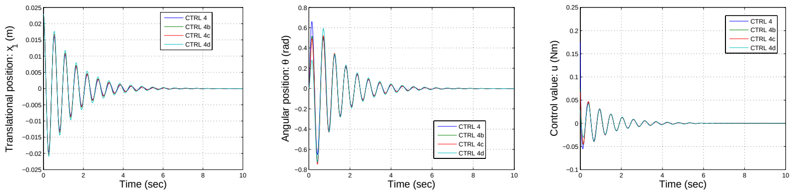

Figure 6: Asymptotic stability controller design of exact and reduced TP models

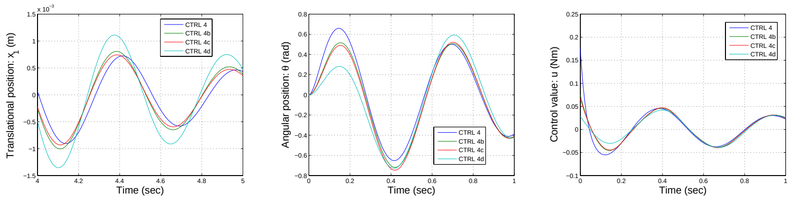

Figure 7: Magnification of Figure 6 for emphasizing the differences of controllers

of more LTI systems, the number of LMIs can easily explode to such a manner that is difficult to handle [21].

Table 2 shows a comprehensive chart on the approximation trade-off of the TORA system.

Again we have to be careful with the approximation trade-off. In case of exact TP models, the feasible solution of the LMI system are proofed to guarantee the stability and defined controller specifications for original model. In this case the solution is trackable through the LMIs and TP model transformation. On the other hand, if the TP model is only an approximation of the original model, then in mathematical sense we can only say that the stability and control specification are guaranteed only for the approximated model described by the TP model. Even in case of comprehensive series of simulation the controller shows stabilization capability, it is not trackable

Table 2: Summary of approximation trade-off of the TORA system

Number of singular values kept	Number of LTIs	Reduction ratio of model transformation	Upper-bound of estimated error	Measured maximal L_2 error	Number of LMIs of the controller	Reduction ration of the number of LMIs
5	10	0%	0	10^{-12}	67	0%
4	8	20%	0.00416	0.0023	46	31%
3	6	40%	0.12675	0.0024	29	56%
2	4	60%	3.96015	0.21513	16	76%

mathematically. For instance in case of a dynamic system with chaotic behavior, a small error can cause explosion in the system. However, in most cases the controllers derived from the approximated models are satisfactory as the modeling error of typical identification techniques in in larger range than the approximation error.

4 Conclusion

The paper presents a study how the TPDC controller design framework can handle the trade-off between approximation accuracy and model complexity through the case study of the TORA system. The proposed framework is shown to be an efficient tool for complexity reduction. The simulation proved that there is no significant difference in control response between the controllers derived from the reference model and the reduced models whilst major loss in model size and cut in computational necessity is achieved. As a matter of fact we should note that the stability issues of the original model are not guaranteed in mathematical sense by the controllers derived from the reduced TP models.

Acknowledgement

This research was supported by the Hungarian National Found OTKA No. T 049838, Dr. Petres is supported by Zoltán Magyary Postdoctoral Scholarship and Dr. Baranyi is supported by János Bolyai Postdoctoral Scholarship.

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