Homotopy Solution of GPS - N Point Navigation Problem

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Abstract: Linear homotopy solution is given for GPS N-point navigation problem. The overdetermined polynomial system has been solved without initial guessed value using natural start system resulting real solutions.

Keywords: linear homotopy, GPS, navigation problem

1 Introduction. Theoretical Concepts

1.1 Homotopy Solution of Polynomial Systems

The continuous deformation of an object to an other object is called as *homotopy*. Let us consider two multivariate polynomials $p(x)$ and $q(x)$. Define the linear convex function H in variables x and λ, called homotopy function as,

$$H(x,\lambda) = (1-\lambda)\,p(x) + \lambda q(x) \qquad (1)$$

In geometric terms, the homotopy H provides us a continuous deformation from $p(x)$ – which is obtained for $\lambda = 0$ by $H(x, 0)$ – to $q(x)$ – which is obtained for $\lambda = 1$ by $H(x, 1)$. We call it linear homotopy because H is a linear function of the variable λ. If the solution of $p(x)$ is x_p, namely $p(x_p) = 0$, then $H(x_p, 0) = 0$. After this continuous deformation, x_q defined by $H(x_q, 1) = 0$ will be the solution of the system $q(x)$, namely $q(x_q) = 0$. The polynomial system $p(x)$ is called start system and the polynomial system $q(x)$ is called target systems. During the deformation $H(x, \lambda) = 0$ for every $\lambda \in [0,1]$ and the function $x(\lambda)$ is called the path of the homotopy. This homotopy path can be computed as an initial value problem [2]:

$$H_x \frac{\partial x(\lambda)}{\partial \lambda} + H_\lambda = 0 \quad (2)$$

with

$$x(0) = x_p \quad (3)$$

Computing the path, the solution of the target system is $x(1) = x_q$.

The start system can be constructed intuitively, reducing the original system (target system) to a more simple system (start system), which roots can be easily computed. In order to get all of the roots of the target system, the start system should have so many roots as many the target system has. The start system can be constructed in different ways, however there are two typical types of the start sytems, which are usually employed.

How can we find the proper start system, which will provide all of the solutions of the target system? This problem can be solved if the nonlinear system is specially a system of polynomial equations.

Let us consider the case when we are looking for the homotopy solution of $f(x) = 0$, where $f(x)$ is a polynomial system, $f(x): \mathrm{R}^n \to \mathrm{R}^n$. To get all of the solutions, one should find out a proper polynomial system, as start system, $g(x) = 0$, where $g(x): \mathrm{R}^n \to \mathrm{R}^n$ with known and easily computable solutions.

An appropriate start system can be generated in the following way, [1].

Let $f_i(x_1,...,x_n)$, $i = 1, ..., n$ be a system of n polynomials. We are interested in the common zeros of the system, namely $f = (f_1(x),...,f_n(x)) = 0$.

Let d_j denote the degree of the jth polynomial – that is the degree of the highest order monomial in the equation. Then such a starting system is,

$$g_j(x) = e^{i\varphi_j}\left(x_j^{d_j} - (e^{i\theta_j})^{d_j}\right) = 0 \quad , \quad j = 1, \ldots, n \quad (4)$$

where θ_j, φ_j are random real numbers in the interval $[0, 2\pi]$. The equation above has the obvious particular solution $x_j = e^{i\theta_j}$ and the complete set of the starting soltuions for $j = 1, \ldots, n$ is given by:

$$e^{\left(i\theta_j+\frac{2\pi ik}{d_j}\right)}, \quad k = 0, 1, \ldots, d_j\text{-}1 \tag{5}$$

Bézout's theorem [2], states that the number of isolated roots of such a system is bounded by the total degree of the system, $d_1 d_2 \ldots d_n$.

Let us consider the following system:

$$f_1(x, y) = x^2 + y^2 - 1 \tag{6}$$

$$f_2(x, y) = x^3 + y^3 - 1 \tag{7}$$

The degrees of the polynomials are $d_1 = 2$ and $d_2 = 3$. Indeed, this system has the following six roots ($d_1 d_2 = 2 * 3 = 6$), as it is expected (Table 1).

Now we compute a start system and its solutions considering (4) and (5). A start system is:

$$g_1(x, y) = (0.673116 + 0.739537i)\left((-0.933825 + 0.35773i) + x^2\right) \tag{8}$$

$$g_2(x, y) = (-0.821746 - 0.569853i)\left((-0.957532 - 0.288325i) + y^3\right) \tag{9}$$

and its solutions are:

$$\begin{aligned}
\{x_1, y_1\} &= \{0.983317 - 0.1819i, -0.413328 - 0.910582i\} \\
\{x_2, y_2\} &= \{0.983317 - 0.1819i, 0.995251 + 0.0973382i\} \\
\{x_3, y_3\} &= \{0.983317 - 0.1819i, -0.581923 + 0.813244i\} \\
\{x_4, y_4\} &= \{-0.983317 + 0.1819i, -0.413328 - 0.910582i\} \\
\{x_5, y_5\} &= \{-0.983317 + 0.1819i, 0.995251 + 0.0973382i\} \\
\{x_6, y_6\} &= \{-0.983317 + 0.1819i, -0.581923 + 0.813244i\}
\end{aligned} \tag{10}$$

The homotopy function is:

$$H(x, y, \lambda) = (1-\lambda)\begin{pmatrix} g_1(x, y) \\ g_2(x, y) \end{pmatrix} + \lambda \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} \tag{11}$$

The corresponding differential equation system, see (2):

$$\frac{d}{d\lambda}\begin{pmatrix} x(\lambda) \\ y(\lambda) \end{pmatrix} = -H_{x,y}{}^{-1} H_{\lambda} \tag{12}$$

where

Table 1
Roots of the system eqs. (6) and (7)

x	y
0	1
0	1
1	0
1	0
$\frac{-5i+\sqrt{2}}{4i+\sqrt{2}}$	$-1+\frac{i}{\sqrt{2}}$
$\frac{5i+\sqrt{2}}{-4i+\sqrt{2}}$	$-1-\frac{i}{\sqrt{2}}$

$$H_{x,y}=\begin{pmatrix} \lambda\frac{\partial f_1}{\partial x}+(1-\lambda)\frac{\partial g_1}{\partial x} & \lambda\frac{\partial f_1}{\partial y}+(1-\lambda)\frac{\partial g_1}{\partial y} \\ \lambda\frac{\partial f_2}{\partial x}+(1-\lambda)\frac{\partial g_2}{\partial x} & \lambda\frac{\partial f_2}{\partial y}+(1-\lambda)\frac{\partial g_2}{\partial y} \end{pmatrix} \tag{13}$$

and

$$H_{\lambda}=\begin{pmatrix} f_1(x,y)-g_1(x,y) \\ f_2(x,y)-g_2(x,y) \end{pmatrix} \tag{14}$$

So we have to solve this system with six initial values. These initial values – the solutions of the start system – will provide the start points of the six homotopy paths. The end points of these paths are the six solutions we are looking for. Figure 1(a)-1(f) show the homotopy paths belonging to the six initial values.

The solutions – the end points of the paths – are in Table 2.

Table 2
Homotopy solutions of the system eqs. (6) and (7)

x	y
0	1
0	1
1	0
1	0
-1-0.707107i	-1+0.707107i
-1+0.707107i	-1-0.707107i

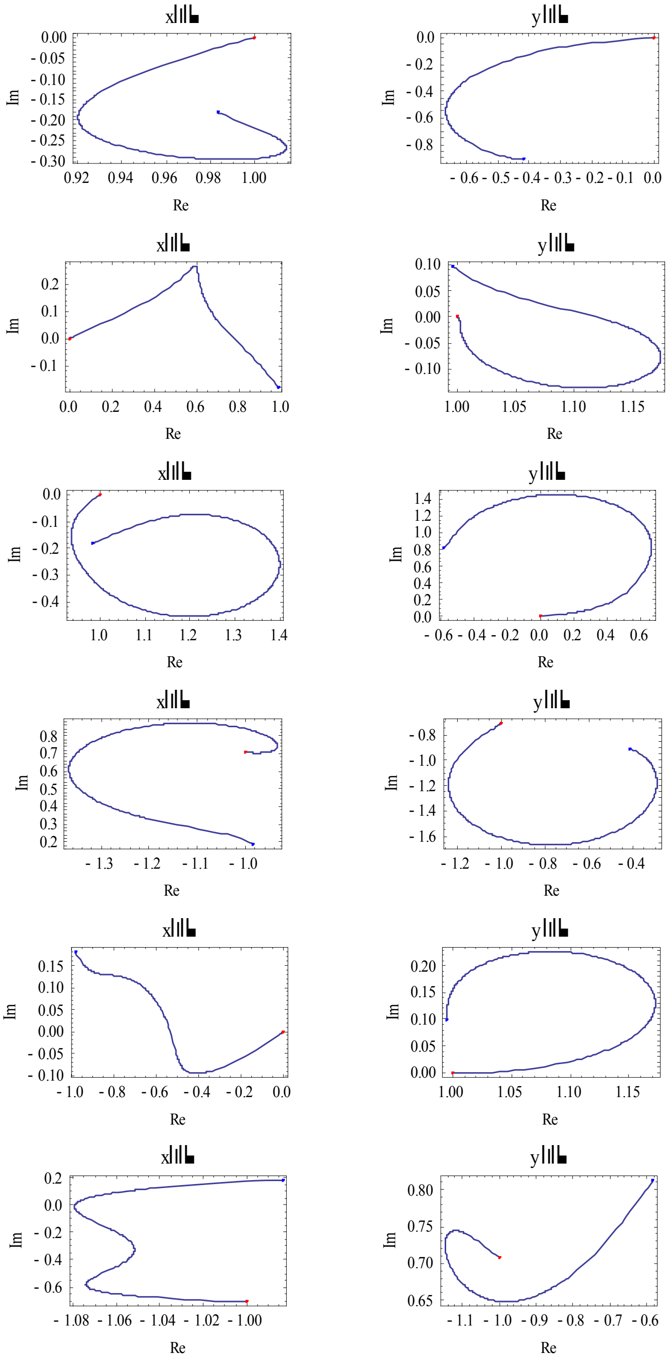

Figure 1(a)-(f)

Homotopy paths starting from the six different initial values

1.2 GPS – N Point Problem

Throughout history, position determination has been one of the most important tasks of mountaineers, pilots, sailor, civil engineers etc. In modern times, Global Positioning System (GPS) employing Global Navigation Satellite Systems (GNSS) provide an ultimate method to accomplish this task. If one has a hand held GPS receiver, the receiver measures the travel time of the signal transmitted from the satellites. Then this distance can be computed by multiplying the measured time by the speed of light in vacuum. The distance of the receiver from the *i*-th satellite, e_i is related to the unknown position of the receiver, $\{x_1, x_2, x_3\}$ [3]:

$$e_i = \sqrt{(x_1 - a_i)^2 + (x_2 - b_i)^2 + (x_3 - c_i)^2} + x_4 \tag{15}$$

where $\{a_i, b_i, c_i\}$, i=0,1,2,3 are the coordinates of the i^{th} satellite.

The distance is influenced also by the satellite and receiver' clock biases. The satellite clock biases can be modelled while the receiver' clock biases have to be considered as an unknown variable, x_4.

This means, we have four unknowns, consequently we need four satellite signals as minimum observation. The general form of the equation for the *i*-th satellite is given by:

$$f_i = (x_1 - a_i)^2 + (x_2 - b_i)^2 + (x_3 - c_i)^2 - (x_4 - d_i)^2 = 0 \tag{16}$$

In the case where more than four GPS satellites are in view as it is usually in practice, there are more independent equations than unknown variables. Considering least square method, we should minimize the following objective:

$$W(x_1, x_2, x_3, x_4) = \sum_{i=1}^{n} f_i^2 \tag{17}$$

This minimization problem can be transformed into a set of nonlinear equations:

$$g_1 = \frac{\partial W}{\partial x_1} = \sum_{i=1}^{n} (x_1 - a_i) f_i = 0 \tag{18}$$

$$g_2 = \frac{\partial W}{\partial x_2} = \sum_{i=1}^{n} (x_2 - a_i) f_i = 0 \tag{19}$$

$$g_3 = \frac{\partial W}{\partial x_3} = \sum_{i=1}^{n} (x_3 - a_i) f_i = 0 \tag{20}$$

$$g_4 = \frac{\partial W}{\partial x_4} = \sum_{i=1}^{n} (x_4 - a_i) f_i = 0 \tag{21}$$

For illustration in Table 3, we have data of six satellites with numerical values taken from [3].

Considering these data, the four equations (18)- (21) can be written as,

$$\begin{aligned}F_1 &= -9.05561\times10^{21} + 3.47408\times10^{15} x_1 - 1.58773\times10^{8} x_1^2 + 6x_1^3 - \\ &- 1.1956\times10^{15} x_2 + 1.84408\times10^{8} x_1 x_2 - 5.29243\times10^{7} x_2^2 + 6x_1 x_2^2 + \\ &+ 1.00201\times10^{15} x_3 - 1.38257\times10^{8} x_1 x_3 - 5.29243\times10^{7} x_3^2 + 6x_1 x_3^2 - \\ &- 2.44129\times10^{15} x_4 + 2.65299\times10^{8} x_1 x_4 + 5.29243\times10^{7} x_4^2 - 6x_1 x_4^2\end{aligned} \quad (22)$$

$$\begin{aligned}F_2 &= 1.99211\times10^{22} - 1.1956\times10^{15} x_1 + 9.22039\times10^{7} x_1^2 + \\ &+ 4.59673\times10^{15} x_2 - 1.05849\times10^{8} x_1 x_2 + 6x_1^2 x_2 + 2.76612\times10^{8} x_2^2 + \\ &+ 6x_2^3 - 1.67902\times10^{15} x_3 - 1.38257\times10^{8} x_2 x_3 + 9.22039\times10^{7} x_3^2 + \\ &+ 6x_2 x_3^2 + 4.06192\times10^{15} x_4 + 2.65299\times10^{8} x_2 x_4 - 9.22039\times10^{7} x_4^2 - \\ &- 6x_2 x_4^2\end{aligned} \quad (23)$$

$$\begin{aligned}F_3 &= -1.79393\times10^{22} + 1.00201\times10^{15} x_1 - 6.91284\times10^{7} x_1^2 - \\ &- 1.67902\times10^{15} x_2 - 6.91284\times10^{7} x_2^2 + 4.21573\times10^{15} x_3 - \\ &- 1.05849\times10^{8} x_1 x_3 + 6x_1^2 x_3 + 1.84408\times10^{8} x_2 x_3 + 6x_2^2 x_3 - \\ &- 2.07385\times10^{8} x_3^2 + 6x_3^3 - 2.9162\times10^{15} x_4 + 2.65299\times10^{8} x_3 x_4 + \\ &+ 6.91284\times10^{7} x_4^2 - 6x_3 x_4^2\end{aligned} \quad (24)$$

Table 3

Data for the GPS N-point problem

i	a_i	b_i	c_i	d_i
1	14 177 553.47	-18 814 768.09	12 243 866.38	21 119 278.32
2	15 097 199.81	-4 636 088.67	21 326 706.55	22 527 064.18
3	23 460342.33	-9 433 518.58	8 174 941.25	23 674 159.88
4	-8 206 488.95	-18 217 989.14	17 605 231.99	20 951 647.38
5	1 399 988.07	-17 563 734.90	19 705 591.18	20 155 401.42
6	6 995 655.48	-23 534 808.26	-9 927 906.48	24 222 110.91

$$\begin{aligned}F_4 &= -2.76891\times10^{22} + 2.44129\times10^{15} x_1 - 1.3265\times10^{8} x_1^2 - \\ &- 4.06192\times10^{15} x_2 - 1.3265\times10^{8} x_2^2 + 2.9162\times10^{15} x_3 - \\ &- 1.3265\times10^{8} x_3^2 - 4.6128\times10^{15} x_4 - 1.05849\times10^{8} x_1 x_4 + 6x_1^2 x_4 + \\ &+ 1.84408\times10^{8} x_2 x_4 + 6x_2^2 x_4 - 1.38257\times10^{8} x_3 x_4 + 6x_3^2 x_4 + \\ &+ 3.97949\times10^{8} x_4^2 - 6x_4^3\end{aligned} \quad (25)$$

2 Solution via Homotopy Method

According to the *Bezout's* theorem the upper bound of the number of the isolated roots of this system is 3^4 = 81. Therefore instead of generating a start system automatically, we can consider the univariate terms from each equations as natural start system. This seems to be more reasonable, because we are interested in only real solutions.

$$G_1 = -9.05561\times 10^{21} + 3.47408\times 10^{15} x_1 - 1.58773\times 10^{8} x_1{}^2 + 6x_1{}^3 \quad (26)$$

$$G_2 = 1.99211\times 10^{22} + 4.59673\times 10^{15} x_2 + 2.77612\times 10^{8} x_2{}^2 + 6x_2{}^3 \quad (27)$$

$$G_3 = -1.79393\times 10^{22} + 4.21573\times 10^{15} x_3 - 2.07385\times 10^{8} x_3{}^2 + 6x_3{}^3 \quad (28)$$

$$G_1 = -2.76891\times 10^{22} - 4.6128\times 10^{15} x_4 - 3.97949\times 10^{8} x_4{}^2 - 6x_4{}^3 \quad (29)$$

The real solutions of this system give the initial values (Table 4).

In this way we could reduce the 81 initial values to 3 initial values. Then to avoid singularity of the homotopy function, let

$$\gamma = i\,\{1,1,1,1\} \quad (30)$$

Table 4
The real solutions of the system eq. (26)-(29)

x_1	x_2	x_3	x_4
2.96291×10^6	-6.54641×10^6	5.51126×10^6	-4.30226×10^6
2.96291×10^6	-6.54641×10^6	5.51126×10^6	2.21079×10^7
2.96291×10^6	-6.54641×10^6	5.51126×10^6	4.85192×10^7

The homotopy function is the linear combination of the target system, eq. (22)-(25) and the start system, eq. (26)-(29):

$$H(x_1, x_1, x_1, x_1, \lambda) = \gamma(1-\lambda)\begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{pmatrix} + \lambda \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} \quad (31)$$

and the corresponding system of the differential equations:

$$\frac{d}{d\lambda}\begin{pmatrix} x_1(\lambda) \\ x_2(\lambda) \\ x_3(\lambda) \\ x_4(\lambda) \end{pmatrix} = -H_x{}^{-1} H_\lambda \quad (32)$$

where the Jacobian

$$\left(H_x\right)_{i,j} = \gamma(1-\lambda)\left(\frac{\partial G_i}{\partial x_j}\right)_{i,j} + \lambda\left(\frac{\partial F_i}{\partial x_j}\right)_{i,j} \tag{33}$$

and

$$\left(H_\lambda\right)_i = F_i(x_1, x_2, x_3) - \gamma\, G_i(x_1, x_2, x_3) \tag{34}$$

Now employing path tracing by integration with computing inverse, for the 3 initial values we get 3 solutions, but only one of them is the physically acceptable solution [3], [4]. Table 5 shows the values of this solution.

The trajectories belonging to this solution can be seen in Figures 2(a)-2(d).

Table 5
Solution of the GPS N-point problem

x_1	*596929*
x_2	-4.84785×10^6
x_3	4.08822×10^6
x_4	13.4526

Conclusions

The homotopy computation takes 0.203 sec in *Mathematica*. Neither reduced Groebner basis nor Kapur-Sexana-Yang Dixon resultant can solve the problem in practical time (less than 500 sec). Employing Gauss-Jacobi combinatoric solution we need to solve 3-point problem $\binom{6}{4} = 15$ times, which takes in *Mathematica* 0.718 sec, [5]. It means that homotopy method is three times faster in that case than the combinatoric solution based on the symbolic solution of the 3-point problem. Having more than six observations, this ratio can be even higher.

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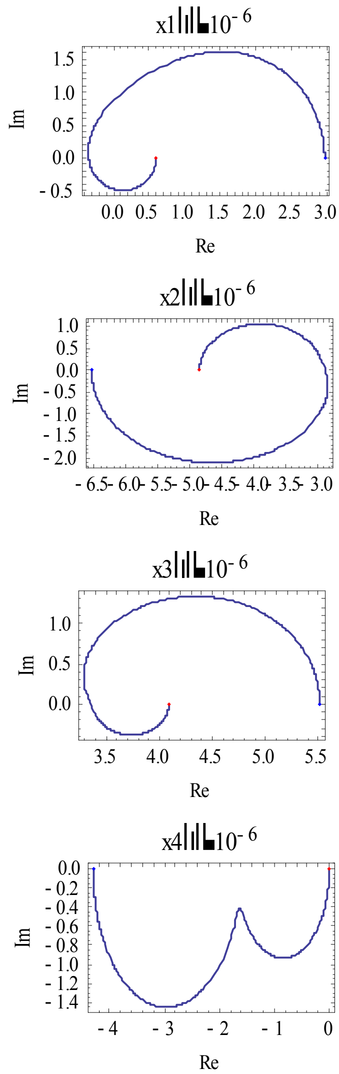

Figure 2(a)-(d)
Homotopy solution paths of the navigation problem