Binary Tomography Reconstruction Algorithm Based on the Spectral Projected Gradient Optimization

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Abstract: In this paper we propose an optimization approach based on the Spectral Projected Gradient (SPG) method for solving the binary tomography reconstruction problem. Using a convex-concave regularization we treat the reconstruction problem as a convex and box constrained optimization problem which is suitable to solve by SPG method. Experimental results on the limited set of test problems show that the new method has competitive reconstruction performance in comparison with a well known non-deterministic simulated annealing approach. In addition, its capability to apply other regularization terms gives to this deterministic method a desirable properties.

Key words and phrases: Binary tomography, Spectral Projected Gradient method, Convex-concave regularization, Smooth regularization.

1 Introduction

Tomography is imaging by sections. It deals with recovering images from a number of projections. Since it is able to explore inside of object without touching it at all, tomography has a various application areas, for example in medicine, archaeology, geophysics and astrophysics. From the mathematical point of view, the object corresponds to a function and the problem posed is to reconstruct this function from its integrals or sums over subsets of its domain. In general, the tomographic reconstruction problem may be continuous or discrete. In *Discrete Tomography* (DT) the range of the function is a finite set. More details about DT and its applications you can find in [7, 8]. In addition to other, it has a wide range of application in medical imaging, for example within Computer Tomography (CT), Positron Emission Tomography (PET) and Electron Tomography (ET). A special case of DT, which is called *Binary Tomography* (BT), deals with the problem of the reconstruction of a binary image.

In many applications like in medical imaging, DT reconstruction problem leads to solving a large-scale and ill-posed optimization problem. Efficiency of this optimization has significant influence on the real applicability of the DT method. Therefore, this issue is actual and subject of several recently published papers, see [15, 13, 14, 16, 17]. In this paper we propose a new optimization approach BT image reconstruction based on the SPG optimization method in conbination with *convex-concave* regularization. SPG is introduced by Birgin, Martínez and Raydan (2000) in [2] and it is further analyzed and developed in [1, 4, 5]. The main motivation for application of SPG lies in the fact that SPG is a very efficient method for solving large-scale and convex-constrained problems, especially when the projection onto the feasible set is easy to compute, see [11]. Numerous numerical tests in [2] and [3] shown the superiority of SPG method in compare with other ones. Our problem is obviously a large-scale (regarding to the image resolution) and we reformulate it as a convex and box-constrained optimization problem where the projection is trivial to compute. Therefore the application of the SPG method became a suitable choice.

2 Reconstruction problem

A main problem in connection with DT refers to the image reconstruction. We consider a BT image reconstruction problem where the imaging process is represented by the following linear system of equations

$$Ax = b, \quad A \in \mathbb{R}^{m \times n}, \;\; x \in \{0,1\}^n, \; b \in \mathbb{R}^m. \tag{1}$$

The matrix A is a so called projection matrix, whose each row corresponds to one projection ray, the corresponding components of vector b contain the detected projection values, while binary-vector x represents the unknown image to be reconstructed. The row entries a_i of A represent the length of the intersection of pixels of the discretized volume and the corresponding projection ray, see Figure 1. Components of the vector x are binary variables indicating the membership of the corresponding pixel to the object: for $x_i = 1$ pixel belongs to the object, while for $x_i = 0$ does not. In a general case the system (1) is under-determined ($m < n$) and has no unique solution. Therefore the minimization of the squared projection error,

$$\min_{x \in \{0,1\}^n} \|Ax - b\|^2$$

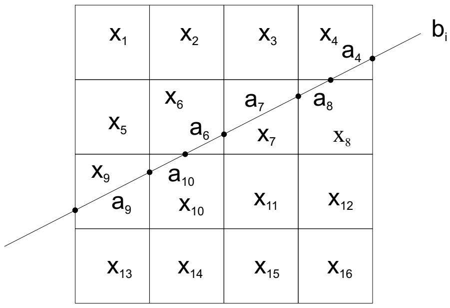


Figure 1: The discretization model. The corresponding reconstruction problem is represented in a form of a linear system of equations, see (1).

can not lead to the satisfactory result. To avoid this problem an appropriate prior regularization is need. We consider a well know smoothness prior defined by

$$\sum_{i} \sum_{j \in N(i)} (x_i - x_j)^2, \tag{2}$$

where $N(i)$ represents a set of indices of image neighbour pixels right and below from x_i. This prior is quadratic and convex and its role is to enforce the spatial coherency of the solution. In this paper we focus on the binary tomography problem given by

$$\min_{x \in \{0,1\}^n} \Phi_\alpha(x), \tag{3}$$

where the objective function is defined by

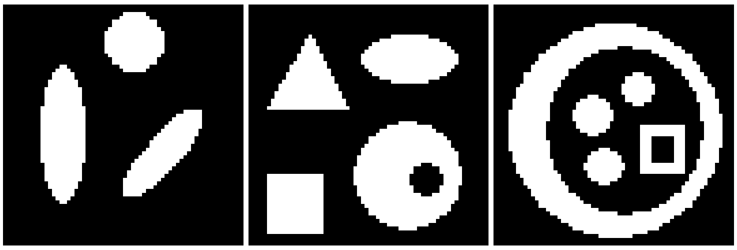
$$\Phi_\alpha(x) = \frac{1}{2} \left(\|Ax - b\|^2 + \alpha \sum_{i} \sum_{j \in N(i)} (x_i - x_j)^2 \right), \tag{4}$$

parameter $\alpha > 0$ is the balancing parameter between projection error and smoothing term. First term in (4) measures the accordance of a solution with a projection data while a rule of the last term is to enforce the coherency of the solution.

2.1 SPG Optimization Algorithm

In this section we give a short description of the SPG optimization algorithm. It is a deterministic, iterative algorithm, introduced by Birgin, Martínez and Raydan (2000) in [2] for solving a convex-constrained optimization problem

$$\min_{x \in \Omega} f(x),$$



PH1 PH2 PH3

Figure 2: Phantom images used in our experiments. All images have the same resolution 64×64.

where the feasible region Ω is a closed convex set in R^n. The requirements for application of SPG algorithm are: *i*) f is defined and has continuous partial derivatives on an open set that contains Ω; *ii*) the projection P_Ω of an arbitrary point $x \in R^n$ onto a set Ω is defined. The algorithm uses the following parameters: integer $m \geq 1$; $0 < \alpha_{min} < \alpha_{max}$, $\gamma \in (0,1)$, $0 < \sigma_1 < \sigma_2 < 1$ and initially $\alpha_0 \in [\alpha_{min}, \alpha_{max}]$ (see [3] for details). Starting from an arbitrary configuration $x^0 \in \Omega$, the below computation is iterated until convergence.

SPG iterative step [3]**.**

Given x^k and α_k, the values x^{k+1} and α_{k+1} are computed as follows:

$d^k = P_\Omega(x^k - \alpha_k \nabla f(x^k)) - x^k$;
$f_{max} = \max\{f(x^{k-j}) \mid 0 \leq j \leq \min\{k, m-1\}\}$;
$x^{k+1} = x^k + d^k$; $\delta = \langle \nabla f(x^k), d^k \rangle$; $\lambda_k = 1$;
while $f(x^{k+1}) > (f_{max} + \gamma \lambda_k \delta)$
  $\lambda_{temp} = -\frac{1}{2}\lambda_k^2 / (f(x^{k+1}) - f(x^k) - \lambda_k \delta)$;
  if $(\lambda_{temp} \geq \sigma_1 \wedge \lambda_{temp} \leq \sigma_2 \lambda)$ **then** $\lambda_k = \lambda_{temp}$ **else** $\lambda_k = \lambda_k / 2$;
  $x^{k+1} = x^k + \lambda_k d^k$;
end while;
$s^k = x^{k+1} - x^k$; $y^k = \nabla f(x^{k+1}) - \nabla f(x^k)$; $\beta_k = \langle s^k, y^k \rangle$;
if $\beta_k \leq 0$ **then** $\alpha_{k+1} = \alpha_{max}$ **else** $\alpha_{k+1} = \min\{\alpha_{max}, \max\{\alpha_{min}, \langle s^k, s^k \rangle \beta_k\}\}$

The SPG algorithm is particularly suited for the situations when the projection calculation is inexpensive, as in box-constrained problems, and its performance is shown to be very good in large-scale problems (see [3]).

3 The proposed method

We transform the binary tomography problem (3) to the convex-constrained problem defined by

$$\min_{x\in[0,1]^n} \Phi_\alpha(x) + \mu \cdot x^T(e-x)\,, \quad \mu > 0 \tag{5}$$

where $e = [1,1,1,..,1]^n$. In (5) we relax the feasible set of the optimization to the convex set, $[0,1]^n$ and add a concave regularization term $x^T(e-x)$ with aim to enforce binary solution. Parameter μ regulates the influence of this term. Due to the convex smoothness regularization (2) and the concave binary enforcing regularization the problem (5) belongs to the class of *convex-concave* regularized methods [17, 13]. Soundness of the problem (5) is ensured by the following theorem which establishes an equivalence between (3) and (5).

Theorem 1 *[6, 9] Let E be Lipschitzian on an open set $A \supset [0,1]^n$ and twice continuously differentiable on $[0,1]^n$. Then there exist a $\mu_* \in R$ such that for all $\mu > \mu_*$:*

(i) the integer (binary) programming problem

$$\min_{x\in\{0,1\}^n} E(x)$$

is equivalent with the concave minimization problem

$$\min_{x\in[0,1]^n} E(x) + \tfrac{1}{2}\mu\langle x, e-x\rangle,$$

(ii) the function $E(x) + \frac{1}{2}\mu\langle x, e-x\rangle$ is concave on $[0,1]^n$.

Requirements for application of the SPG algorithm for solving the problem (5) are satisfied. Indeed, it is obvious that the objective function is differentiable and the projection onto a feasible set P_r is given by

$$[P_r(x)]_i = \begin{cases} 0, & x_i \le 0 \\ 1, & x_i \ge 1 \\ x_i, & \text{elsewhere} \end{cases}\,, \qquad \text{where } i = 1,\dots,n\,. \tag{6}$$

where $x \in \mathbb{R}^n$. P_r is a projection with respect to the Euclidean distance and its calculation is inexpensive. Therefore the SPG algorithm is suitable choice for solving (5) for every fixed $\mu > 0$.

Our strategy is to solve a sequence of optimization problems (5), with gradually increasing μ, which will lead to a solution of the binary solution. More precisely, we suggest the following optimization algorithm.

SPG Algorithm for binary tomography.

Parameters: $\epsilon_{in} > 0$; $\epsilon_{out} > 0$; $\delta > 1$; μ_0; $maxit$.
$x^0 = [0.5, 0.5, \dots, 0.5]^T$; $\mu = \mu_0$; $k = 0$;
do

do

x^{k+1} from x^k by SPG iterative step; $k = k + 1$;

until $\| P_\Omega(x^k - \nabla\Phi(x^k)) - x^k \| > \epsilon_{in}$ and $k < maxit$

$\mu = \delta * \mu$;

until $\max_i\{\min\{x_i^k, 1 - x_i^k\}\} > \epsilon_{out}$.

The initial configuration is the image with all pixel values equally to 0.5. In each iteration in the outer loop we solve an optimization problem (5) for a fixed binary factor $\mu > 0$ by using the SPG method. By iteratively increasing the value of μ in the outer loop the binary solutions are enforced. The termination criterion for the outer loop, ϵ_{out}, regulates the tolerance for the finally accepted (almost) binary solution.

It is easy to show that the function Φ_α is quadratic and convex, see for example [13]. However, by increasing the μ factor during the optimization process the influence of the concave regularization term becames larger which leads to the non-convex objective function. Therefore, we cannot guaranty that this approach always end up in a global minimum. However, experimental results in section 4 show its very good performance.

An important characteristic of this method is its flexibility regarding the inclusion of other or additional constraints and regularization terms, for example the Gibbs prior, see [17]. The only requirement is the differentiability of the constraint.

4 Experimental results

In this section we compare the performance behavior of the SPG method with the well know SA method. SA algorithm in [15] is compered with a powerful DC based reconstruction algorithm for binary tomography introduced by Schüle et al. in [13]. The main conclusion of this Benchmark evaluation was that "there is no huge difference between the qualities of the reconstructed images of the two methods". This fact gives more validity for our evaluation.

We performed experiments on the binary test images (phantoms) presented in Figure 2. Reconstruction problems are composed by taking parallel projections from different directions. We take 64 parallel projections for each direction. Regarding to direction we distinguish reconstructions with 2,3,5 and 6 projections. For 2,3 and 5 projections directions are uniformly chosen within $[0^0, 90^0]$ and for 6 projections within $[0^0, 150^0]$.

The quality of reconstruction (solution) is measured by the following two error measure functions

$$E_1(x) = \|Ax - b\|,$$

$$E_2(x) = \|x - x^*\|_1,$$

where x is the reconstructed image. Function E_1 measures the accordance with the projection data, while E_2 gives the number of failed pixels in compare with the original image x^*.

Simulated Annealing (SA) is a stochastic optimization algorithm based on the simulation of physical process of slow cooling of the material in a heat bath. Based on the ideas from a paper published by Metropolis et al.(1953) [12] the SA algorithm is introduced by Kirkpatrick et. al.(1983) [10]. In our experiments we use the following SA algorithm adapted for BT problem by Weber et al. in [15].

SA Algorithm.

Parameters:
$\alpha > 0$; $T_{start} > 0$; $T_{min} > 0$; $T_{factor} \in (0, 1)$; $S > 0$.
Initial variable setting:
$x = [0, 0, \dots, 0]^T$; $T = T_{start}$; $S_{nr} = 0$; $E_{old} = \Phi_\alpha(x^0)$.
while $(T \geq T_{min}) \wedge (S_{nr} < S)$
  for $i = 1$ to sizeof(x),
    choose a random position j in the vector x;
    $\widetilde{x} = x$;
    $\widetilde{x}[j] = 1 - x[j]$;
    $E_{new} = \Phi_\alpha(\widetilde{x})$;
    $z = rand()$;
    $\Delta E = E_{new} - E_{old}$;
    if $\Delta E < 0 \vee exp(-\Delta E/T) > z$, **then**
     $x = \widetilde{x}$ {accept changes}
     $E_{old} = E_{new}$; $S_{nr} = 0$;
    end if
  end for
  $T = T * T_{factor}$; $S_{nr} = S_{nr} + N$;
end while.

For SA based reconstructions we use the following parameter settings: $\alpha = 5$, $T_{start} = 4$, $T_{min} = 10^{-14}$, $T_{factor} = 0.97$, $S = 10 *$ sizeof(x). The SPG algorithm is implemented in Matlab. Its parameter settings is the following: $\alpha = 3$, $\delta = 1.2$, $E_{in} = 0.1$, $E_{out} = 0.001$, $maxit = 100$.

Proj.	Alg.	PH1	PH2	PH3
2	SA	7.874/806	6.782/950	7.615/1177
	SPG	6.324/849	5.831/499	11.225/1432
3	SA	2.554/3	9.587/580	11.959/858
	SPG	2.554/3	3.182/ 3	11.769/1198
5	SA	0/0	4.030/3	17.015/589
	SPG	0/0	3.070/2	12.321/538
6	SA	0/0	3.444/2	0/0
	SPG	0/0	2.205/1	0/0

Table 1: The measured error values $E_1(x)/E_2(x)$ of the reconstructed images.

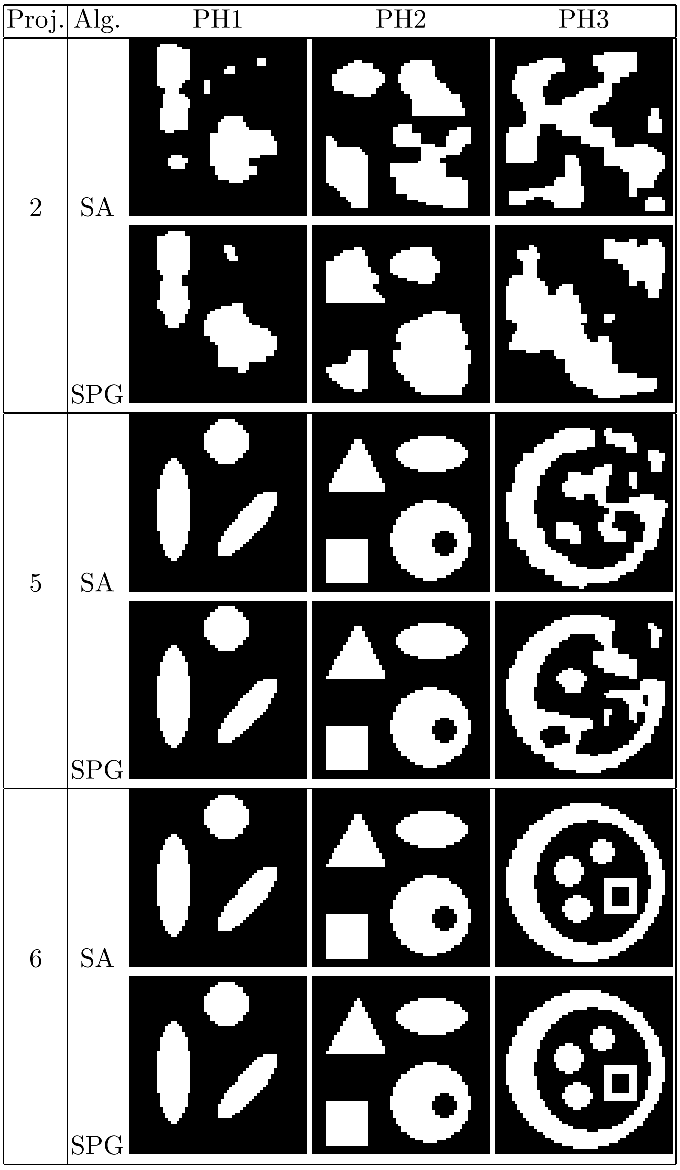


Figure 3: The phantom images reconstructed from 2, 5 and 6 projections without noise.

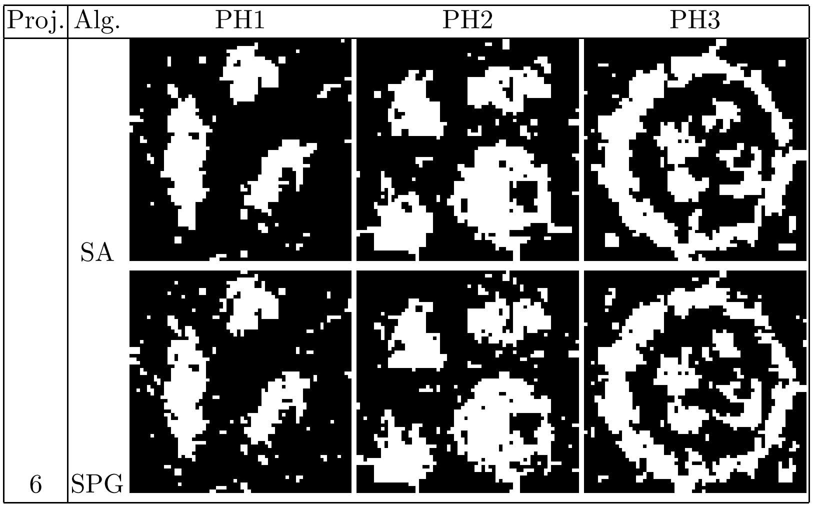


Figure 4: Phantom images reconstructed from 5 and 6 projections from noisy projection data.

Quality of reconstructed images we can follow in Table 1 and also in Figure 3. The results are similar or exactly the same, expect reconstructions of PH2 for 3 projections where SPG has better solution and PH3 for 2 projections where SA has better solution. Figure 4 represents reconstructions obtained from corrupted data (projection vector) with a Gaussian white noise with standard deviation 5. The results are very similar.

5 Conclusion

We successfully developed an optimization method based on the SPG algorithm for binary tomography reconstruction problem. Performance evaluation based on the comparison with SA method shows its competence regarding to the quality of reconstructions.

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References

[1] E. Birgin and J. Martínez. A box-constrained optimization algorithm with negative curvature directions and spectral projected gradients. *Computing*, 15:49–60, 2001.

[2] E. Birgin, J. Martínez, and M. Raydan. Nonmonotone spectral projected gradient methods on convex sets. *SIAM J. on Optimization*, 10:1196–1211, 2000.

[3] E. Birgin, J. Martínez, and M. Raydan. Algorithm: 813: Spg - software for convex-constrained optimization. *ACM Transactions on Mathematical Software*, 27:340–349, 2001.

[4] E. Birgin, J. Martínez, and M. Raydan. Inexact spectral projected gradient methods on convex sets. *IMA Journal of Numerical Analysis*, 23:539–559, 2003.

[5] E. Birgin, J. Martínez, and M. Raydan. Spectral projected gradient methods. *Encyclopedia of Optimization*, pages 3652–3659, 2009.

[6] F. Giannessi and F. Niccolucci. Connections between nonlinear and integer programming problems. *Symposia Mathematica*, pages 161–176, 1976.

[7] G. T. Herman and A. Kuba. *Discrete Tomography: Foundations, Algorithms and Applications.* Birkhäuser, 1999.

[8] G. T. Herman and A. Kuba. *Advances in Discrete Tomography and Its Applications.* Birkhäuser, 2006.

[9] R. Horst and H. Tuy. *Global Optimization: Determinitic Approaches, third ed.* Springer-Verlag, Berlin, 1996.

[10] S. Kirkpatrick, C. Gelatt, and M. Vecchi. Optimization by simulated annealing. *Science*, 220(4598):671–681, 1983.

[11] T. Lukić, N. Sladoje, and J. Lindblad. Deterministic Defuzzification Based on Spectral Projected Gradient Optimization. In *Proc. of 30th DAGM Symposium (DAGM 2008)*, volume 5096 of *LNCS*, pages 476–485, Munich, Germany, 2008. Springer-Verlag.

[12] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller. Equation of state calculations by fast computing machines. *Journal of Chemical Physics*, 21(6):1087–1092, 1953.

[13] T. Schüle, C. Schnörr, S. Weber, and J. Hornegger. Discrete tomography by convex-concave regularization and D.C. Programming. *Discrete Appl. Math.*, 151:229–243, 2005.

[14] T. Schüle, S. Weber, and C. Schnörr. Adaptive reconstruction of discrete-valued objects from few projections. In *Proceedings of the Workshop on Discrete Tomography and its Applications*, volume 20 of *Electronic Notes in Discrete Mathematics*, pages 365–384. Elsevier, 2005.

[15] S. Weber, A. Nagy, T. Schüle, C. Schnörr, and A. Kuba. A benchmark evaluation of large-scale optimization approaches to binary tomography. In *Proc. of 13th International Conference on Discrete Geometry for Computer Imagery (DGCI2006)*, volume 4245 of *LNCS*, pages 146–156, Szeged, Hungary, 2006. Springer-Verlag.

[16] S. Weber, C. Schnörr, T. Schüle, and J. Hornegger. Binary tomography by iterating linear programs from noisy projections. In *Proceedings of 10th International Workshop on Combinatorial Image Analysis*, volume 3322 of *LNCS*, pages 38–51. Springer-Verlag, 2005.

[17] S. Weber, T. Schüle, and C. Schnörr. Prior learning and convex-concave regularization of binary tomography. In *Proceedings of the Workshop on Discrete Tomography and its Applications*, volume 20 of *Electronic Notes in Discrete Mathematics*, pages 313–327. Elsevier, 2005.