A Theory for Measuring the Preference of Fuzzy Numbers

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Introduction

- In theory and practice of multi-criteria decision making, fuzzy numbers are widely used.
- Ranking of fuzzy numbers is an important topic in computer science, especially in fuzzy decision-making.
- Without doubt, the ranking of fuzzy numbers is a challenging problem, and there are plenty of methods available for tackling this problem (see, e.g. Jain (1976); Yager (1981); Chen (1985); Cheng (1998); Chu and Tsao (2002); Asady and Zendehnam (2007); Wang et al. (2006); Abbasbandy and Asady (2006); Wang and Luo (2009)).

Introduction

- In this study, we
  - introduce sigmoid function-based preference measures for intervals and fuzzy numbers.
  - present formulas for the numerical computation of the proposed preference measures.
  - show that the proposed preference measures for intervals and fuzzy numbers asymptotically correspond to the well-known probability-based preference measures for intervals and fuzzy numbers.
  - introduce two parametric crisp relations, which have common parameters, over a collection of fuzzy numbers.
  - prove that the limits of these relations can be used to rank fuzzy numbers. Here, we show that
    - the limit of one of these relations is a strict order relation, while the limit of the other one may be viewed as an indifference relation.
    - the latter can be used to capture situations where the order of two fuzzy numbers cannot be judged; and then, their order may be viewed as being indifferent.

Introduction

- It should be added that in a recent paper by Zumelzu et al. (2020), the authors pointed out that among more than two hundred partial order relations for fuzzy numbers studied, they found just a few that are total orders. They introduced and analyzed the notion of admissible orders for fuzzy numbers with respect to a partial order.

A preference measure for two intervals

Definition 1.

The fuzzy relation y is preferred over x (i.e. $x \prec y$) is given by the membership function $\mu_{\prec}^{(\lambda)} \colon \mathbb{R}^2 \to (0,1)$

$$\mu_{\prec}^{(\lambda)}(x,y) = \frac{1}{1 + \mathrm{e}^{-\lambda(y-x)}},$$

where $\lambda > 0$.

The following proposition concerns the main properties of the preference measure $\mu_{\prec}^{(\lambda)}$.

A preference measure for two intervals

Proposition 1.

The preference measure $\mu_{\prec}^{(\lambda)}$ *has the following properties:*

$$\mu_{\prec}^{(\lambda)}(y,x) < \frac{1}{2} \text{ if and only if } y < x$$

$$\mu_{\prec}^{(\lambda)}(y,x) = \frac{1}{2} \text{ if and only if } y = x$$

$$\mu_{\prec}^{(\lambda)}(y,x) > \frac{1}{2} \text{ if and only if } y > x$$

$$\lim_{(y-x)\to-\infty} \mu_{\prec}^{(\lambda)}(y,x) = 0$$

$$\lim_{(y-x)\to+\infty} \mu_{\prec}^{(\lambda)}(y,x) = 1$$

$$\mu_{\prec}^{(\lambda)}(x,y) + \mu_{\prec}^{(\lambda)}(y,x) = 1.$$

A preference measure for two intervals

Using the fuzzy preference relation $\mu_{\prec}^{(\lambda)}$, we introduce a preference relation for two intervals as follows. Let $\mathbf{I}$ be a collection of intervals on the real line and let $I_1, I_2 \in \mathbf{I}$, $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$. Henceforth, we shall assume that $a_1 < b_1$ and $a_2 < b_2$.

Definition 2.

The preference measure $M_{I,\prec}^{(\lambda)} \colon \mathbf{I} \times \mathbf{I} \to [0,1]$ is given by

$$M_{I,\prec}^{(\lambda)}(I_1, I_2) =$$

$$= \frac{1}{(b_1 - a_1)(b_2 - a_2)} \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} \mu_{\prec}^{(\lambda)}(x, y) \mathrm{d}x \right) \mathrm{d}y,$$

where $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$.

A preference measure for two intervals

The following proposition concerns the reciprocity property of the preference relation $M^{(\lambda)}_{I,\prec}$.

Proposition 2.

For any $I_1 = [a_1, b_1]$ *and* $I_2 = [a_2, b_2]$ *intervals on the real line and a uniquely determined* $\lambda > 0$ *parameter value,*

$$M^{(\lambda)}_{I,\prec}(I_1, I_2) + M^{(\lambda)}_{I,\prec}(I_2, I_1) = 1.$$

A preference measure for two intervals

Connection with the probability-based approach used to measure the preference of two intervals

The following probability-based approach used to measure the preference of two intervals is well-known.

Definition 3.

The probability-based preference intensity index $M^*_{I,\prec}\colon \mathbf{I} \times \mathbf{I} \to [0,1]$ is given by

$$M^*_{I,\prec}(I_1, I_2) = \frac{\mu(A)}{\mu(\Omega)},$$

where I_1, I_2 are two intervals in the collection $\mathbf{I}$,

$$\Omega = I_1 \times I_2,$$

$$A = \{(x,y)\colon (x,y) \in I_1 \times I_2, x < y\} \subseteq \Omega,$$

and $\mu(R)$ is the area of the two-dimensional region R for any $R \subseteq \Omega$.

A preference measure for two intervals

Connection with the probability-based approach used to measure the preference of two intervals

Here, the function value $M^*_{I,\prec}(I_1, I_2)$ represents the probability of $x < y$, where the values of x and y have been randomly chosen from the intervals I_1 and I_2, respectively, (see Huynh et al. (2008); Yue (2016); Sengupta and Pal (2009); Dombi and Jónás (2020)).

Proposition 3.

Let $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ be two intervals on the real line. Then,

$$\lim_{\lambda \to \infty} M^{(\lambda)}_{I,\prec}(I_1, I_2) = M^*_{I,\prec}(I_1, I_2).$$

A preference measure for two intervals

Computing the preference measure for two intervals

We can readily get that

$$M_{I,\prec}^{(\lambda)}(I_1, I_2) = 1 + \frac{\mathcal{I}(a_1, b_1, a_2, b_2, \lambda)}{(b_1 - a_1)(b_2 - a_2)},$$

where

$$\mathcal{I}(a_1, b_1, a_2, b_2, \lambda) = \int\limits_{a_2}^{b_2} \left(\frac{1}{\lambda} \ln \left(\frac{\mathrm{e}^{\lambda a_1} + \mathrm{e}^{\lambda y}}{\mathrm{e}^{\lambda b_1} + \mathrm{e}^{\lambda y}} \right) \right) \mathrm{d}y.$$

Note that the integral $\mathcal{I}(a_1, b_1, a_2, b_2, \lambda)$ has no closed form. We can approximate it quite well by using the trapezoidal rule. That is,

$$\mathcal{I}(a_1, b_1, a_2, b_2, \lambda) \approx$$

$$\approx \frac{\Delta y}{2\lambda} \sum_{i=1}^{n} \left(\ln \left(\frac{\mathrm{e}^{\lambda a_1} + \mathrm{e}^{\lambda(a_2 + (i-1)\Delta y)}}{\mathrm{e}^{\lambda b_1} + \mathrm{e}^{\lambda(a_2 + (i-1)\Delta y)}} \right) + \right.$$

$$\left. + \ln \left(\frac{\mathrm{e}^{\lambda a_1} + \mathrm{e}^{\lambda(a_2 + i\Delta y))}}{\mathrm{e}^{\lambda b_1} + \mathrm{e}^{\lambda(a_2 + i\Delta y)}} \right) \right),$$

where $\Delta y = \frac{b_2 - a_2}{n}$ and n is sufficiently large (e.g. $n = 1000$).

A preference measure for two fuzzy numbers

Definition 4.

The fuzzy number A is given by the membership function $\mu_A \colon \mathbb{R} \to [0,1]$,

$$\mu_A\left(x; \underline{x}_A^L, \overline{x}_A^L, \overline{x}_A^R, \underline{x}_A^R\right) = \begin{cases} 0, & \textit{if } x < \underline{x}_A^L \\ l_A(x), & \textit{if } \underline{x}_A^L \leq x < \overline{x}_A^L \\ 1, & \textit{if } \overline{x}_A^L \leq x < \overline{x}_A^R \\ r_A(x), & \textit{if } \overline{x}_A^R \leq x < \underline{x}_A^R \\ 0, & \textit{if } \underline{x}_A^R \geq x, \end{cases}$$

where $\underline{x}_A^L < \overline{x}_A^L \leq \overline{x}_A^R < \underline{x}_A^R$, and $l_A \colon [\underline{x}_A^L, \overline{x}_A^L) \to [0,1)$ and $r_A \colon [\overline{x}_A^R, \underline{x}_A^R) \to [0,1)$ are continuous, strictly increasing and decreasing functions with the inverse functions $l_A^{-1} \colon [0,1) \to [\underline{x}_A^L, \overline{x}_A^L)$ and $r_A^{-1} \colon [0,1) \to [\overline{x}_A^R, \underline{x}_A^R)$, respectively.

Note that in Definition 4, the functions l_A and r_A determine the left hand side and the right hand side of the membership function of fuzzy number A.

A preference measure for two fuzzy numbers

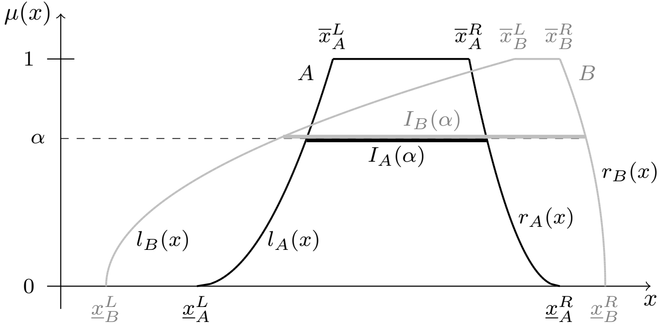

Figure 1: Two fuzzy numbers with their α-cut intervals

A preference measure for two fuzzy numbers

Definition 5.

Let $\mathbf{F}$ be a collection of fuzzy numbers. Let $A, B \in \mathbf{F}$, and let $\lambda > 0$. The preference measure $M^{(\lambda)}_{F,\prec} \colon \mathbf{F}^2 \to (0,1)$ of the preference $A \prec B$ (i.e. $M^{(\lambda)}_{F,\prec}(A,B)$) is given by

$$M^{(\lambda)}_{F,\prec}(A,B) = \int_0^1 M^{(\lambda)}_{I,\prec}(I_A(\alpha), I_B(\alpha))\mathrm{d}\alpha,$$

where $I_A(\alpha)$ and $I_B(\alpha)$ are the α-cut intervals of the membership functions of A and B, respectively, $\alpha \in [0,1]$, and the preference measure $M^{(\lambda)}_{I,\prec}$ for two intervals is given by Definition 2.

A preference measure for two fuzzy numbers

The following proposition is about the reciprocity property of the preference measure $M_{F,\prec}^{(\lambda)}$.

Proposition 4.

For any fuzzy numbers A, B and a uniquely determined $\lambda > 0$ parameter value

$$M_{F,\prec}^{(\lambda)}(A,B) + M_{F,\prec}^{(\lambda)}(B,A) = 1.$$

Using the trapezoidal rule, if n is sufficiently large (e.g. $n = 1000$), then $M_{F,\prec}^{(\lambda)}(A,B)$ can be approximated by

$$M_{F,\prec}^{(\lambda)}(A,B) \approx 1+$$
$$+\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{\mathcal{I}\left(a_A\left(\frac{i-1}{n}\right), b_A\left(\frac{i-1}{n}\right), a_B\left(\frac{i-1}{n}\right), b_B\left(\frac{i-1}{n}\right), \lambda\right)}{\left(b_A\left(\frac{i-1}{n}\right) - a_A\left(\frac{i-1}{n}\right)\right)\left(b_B\left(\frac{i-1}{n}\right) - a_B\left(\frac{i-1}{n}\right)\right)}+\right.$$
$$\left.+\frac{\mathcal{I}\left(a_A\left(\frac{i}{n}\right), b_A\left(\frac{i}{n}\right), a_B\left(\frac{i}{n}\right), b_B\left(\frac{i}{n}\right), \lambda\right)}{\left(b_A\left(\frac{i}{n}\right) - a_A\left(\frac{i}{n}\right)\right)\left(b_B\left(\frac{i}{n}\right) - a_B\left(\frac{i}{n}\right)\right)}\right).$$

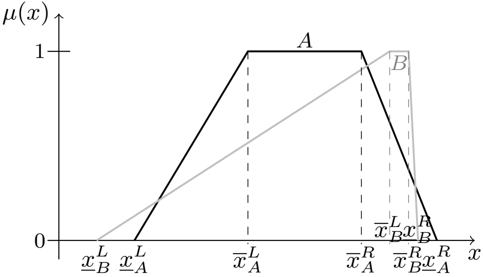

Figure 2: The membership functions of two trapezoidal fuzzy numbers.

$$
\begin{array}{llll}
\underline{x}_A^L = 3.00 & \overline{x}_A^L = 6.00 & \overline{x}_A^R = 9.00 & \underline{x}_A^R = 11.00 \\
\underline{x}_B^L = 2.00 & \overline{x}_B^L = 9.75 & \overline{x}_B^R = 10.25 & \underline{x}_B^R = 10.50.
\end{array}
$$

A preference measure for two fuzzy numbers

A demonstrative example

Using the the α-cut intervals and the approximation formulas presented with $n = 1000$, we computed the values of the preference measure $M_{F,\prec}^{(\lambda)}(A, B)$ for certain λ values.

Table 1: Values of $M_{F,\prec}^{(\lambda)}(A, B)$ for various values of λ

λ	$M_{F,\prec}^{(\lambda)}(A, B)$
1	0.6412
2	0.6715
10	0.6873
30	0.6879
50	0.6880
75	0.6880

Ranking fuzzy numbers

In Dombi and Jónás (2020), we used the so-called probability-based preference intensity index for two fuzzy numbers to derive a crisp strict order relation over fuzzy numbers. The probability-based preference intensity index for two fuzzy numbers is defined as follows.

Definition 6.

Let $\mathbf{F}$ be a collection of fuzzy numbers. Let $A, B \in \mathbf{F}$, and let $\lambda > 0$. The probability-based preference intensity index $M^*_{F,\prec}\colon \mathbf{F}^2 \to [0,1]$ of the preference $A \prec B$ (i.e. $M^*_{F,\prec}(A,B)$) is given by

$$M^*_{F,\prec}(A,B) = \int_0^1 M^*_{I,\prec}(I_A(\alpha), I_B(\alpha))\mathrm{d}\alpha,$$

where $I_A(\alpha)$ and $I_B(\alpha)$ are the α-cut intervals of the membership functions of A and B, respectively, $\alpha \in [0,1]$, and the probability-based preference intensity index $M^*_{I,\prec}$ for two intervals is given by Definition 3.

Ranking fuzzy numbers

The following proposition states an important connection between the preference measures $M_{F,\prec}^{(\lambda)}$ and $M_{F,\prec}^{*}$.

Proposition 5.

Let $\mathbf{F}$ *be a collection of fuzzy numbers and let* $A, B \in \mathbf{F}$. *Then,*

$$\lim_{\lambda \to \infty} M_{F,\prec}^{(\lambda)}(A,B) = M_{F,\prec}^{*}(A,B).$$

Ranking fuzzy numbers

Here, we introduce a parametric crisp relation over a collection of fuzzy numbers and show that the limit of this relation is a strict order relation.

Definition 7.

Let $\mathbf{F}$ be a collection of fuzzy numbers. The binary relation $\prec_F^{(\lambda,\delta)}$ over the collection $\mathbf{F}$ is given by

$$\prec_F^{(\lambda,\delta)} = \left\{ (A,B) \in \mathbf{F} \times \mathbf{F} \colon M_{F,\prec}^{(\lambda)}(A,B) \geq \frac{1}{2} + \delta \right\},$$

where $\delta \in (0, 1/2]$ and $\lambda > 0$.

Theorem 1.

Let $\mathbf{F}$ be a collection of fuzzy numbers. If $\lambda \to \infty$, then there exists a $\delta \in (0, 1/2]$ such that $\prec_F^{(\lambda,\delta)}$ is a strict order relation over $\mathbf{F}$.

Ranking fuzzy numbers

- In practice, for a given finite collection $\mathbf{F}$, the smallest value of $\delta \in (0, \frac{1}{2}]$, for which relation $\prec_F^{(\delta)}$ is transitive, can be numerically determined by using searching methods such as a binary search.
- Now, suppose that A and B are two different elements of $\mathbf{F}$ such that $\frac{1}{2} - \delta < M^*_{F,\prec}(A, B) < \frac{1}{2} + \delta$ holds. This means that neither $A \prec_F^{(\delta)} B$ nor $B \prec_F^{(\delta)} A$ nor $A = B$ holds; that is, $\prec_F^{(\delta)}$ is not a total order.
- In this case, the order of A and B may be viewed as being indifferent. With the purpose of expressing the fact that the order of two fuzzy numbers is really indifferent, we introduce the following indifference relation.

Ranking fuzzy numbers

Definition 8.

Let $\mathbf{F}$ be a collection of fuzzy numbers. The indifference relation relation $\lesseqqgtr_F^{(\lambda,\delta)}$ over the collection $\mathbf{F}$ is given by

$$\lesseqqgtr_F^{(\lambda,\delta)} = \left\{ (A,B) \in \mathbf{F} \times \mathbf{F} : \left| M_{F,\prec}^{(\lambda)}(A,B) - \frac{1}{2} \right| < \delta \right\},$$

where $\delta \in (0, 1/2]$ and $\lambda > 0$.

The following proposition is about the limit of the indifference relation $\lesseqqgtr_F^{(\lambda,\delta)}$.

Ranking fuzzy numbers

Proposition 6.

Let $\mathbf{F}$ *be a collection of fuzzy numbers. Then,*

$$\lim_{\lambda \to \infty} \lesseqqgtr_F^{(\lambda,\delta)} = \lesseqqgtr_F^{(\delta)},$$

where

$$\lesseqqgtr_F^{(\delta)} = \left\{ (A,B) \in \mathbf{F} \times \mathbf{F} : \left| M^*_{F,\prec}(A,B) - \frac{1}{2} \right| < \delta \right\},$$

$\delta \in (0, 1/2]$, *and* $M^*_{F,\prec}$ *is the probability-based preference intensity index for two fuzzy numbers given in Definition 6.*

Ranking fuzzy numbers

Theorem 2.

Let $\mathbf{F}$ *be a collection of fuzzy numbers and let the relations* $\prec_F^{(\lambda,\delta)}$ *and* $\lesseqqgtr_F^{(\lambda,\delta)}$ *over the set* $\mathbf{F}$ *be given by Definition 7 and Definition 8, respectively, where* $\delta \in (0, 1/2]$ *has a fixed value and* $\lambda > 0$ *has a fixed value as well. If* $\lambda \to \infty$*, then there exists a* $\delta \in (0, 1/2]$ *such that* $\prec_F^{(\lambda,\delta)}$ *is a strict order relation over* $\mathbf{F}$*, and for any* $A, B \in \mathbf{F}$*, either* $A \prec_F^{(\lambda,\delta)} B$*, or* $B \prec_F^{(\lambda,\delta)} A$ *or* $A \lesseqqgtr_F^{(\lambda,\delta)} B$ *holds.*

Remark.

An important practical consequence of Theorem 1 and Theorem 2 is that the limits ($\lambda \to \infty$*) of the relations* $\prec_F^{(\lambda,\delta)}$ *and* $\lesseqqgtr_F^{(\lambda,\delta)}$ *can be used to rank fuzzy numbers.*

- *The limit of* $\prec_F^{(\lambda,\delta)}$ *is a strict order relation and it can be used to rank comparable fuzzy numbers.*
- *The limit of the indifference relation* $\lesseqqgtr_F^{(\lambda,\delta)}$ *can be used to express the fact that the order of some fuzzy numbers is indifferent.*

Conclusions

- Here, we introduced sigmoid function-based preference measures for intervals and fuzzy numbers, and described the main properties of these preference measures.
- Next, we presented formulas for the numerical computation of the proposed preference measures.
- Also, we showed that the proposed preference measures for intervals and fuzzy numbers are, asymptotically, the well-known probability-based preference measures for intervals and fuzzy numbers.
- Using the new preference measure, we introduced two parametric crisp relations, which have common parameters, over a collection of fuzzy numbers.
- Then we proved that the limits of these relations can be used to rank fuzzy numbers. Here, we showed that
  - the limit of one of these relations is a strict order relation.
  - the limit of the other may be viewed as an indifference relation. This latter can be used to capture the situations where the order of two fuzzy numbers cannot be judged; and so, their order may be viewed as being indifferent.

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