Uninorm-based Residuated Lattice

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Abstract: Lattice-ordered monoids are important backgrounds and algebraic foundations of residuum in general. The t-norm based lattices are investigated widely in fuzzy models, but in recent time while researching new approximate reasoning methods in soft computing based models and fuzzy models, the investigations are focused on new types of operators, like uninorms. It is necessary to find and define correct residual operators, based on these operators, namely it can be the generalization of the already introduced residuum for t-norms.

Keywords: fuzzy operators, residuum, uninorm

1 Introduction

In complex systems with insufficient information about objects and there properties, fuzzy models seek to give acceptable results close to human way of thinking. One of the important components in a fuzzy system is the approximate reasoning process, which is usually based on residuum and implication. Lattice-ordered monoids are important backgrounds and algebraic foundations of residuum in general. The t-norm based lattices are investigated widely in fuzzy models, but in recent time researching new approximate reasoning methods in soft computing based models and fuzzy models, the investigations are focused on new types of operators, like uninorms. It is necessary to find and define correct residual operators, based on those operators, namely it can be the generalization of the introduced residuum for t-norms. Taking known algebraic structures as the basic for this theory, many other properties of this structure can be generalize.

In this paper the residuated l-monoid for left continuous uninorm is introduced as the first step to the generalization.

2 Preliminares

In many-valued logics since modeling ‘if … then …’ rule with fuzzy predicates is based on fuzzy implications, it is essential to study their mathematical properties. In fuzzy logic, the basic theory of connectives *and, or, not* is well developed and their functional models (t-norms, t-conorms and strong negations) are widely accepted [1]. However, there is no such clear, and, in some sense, unique way of defining fuzzy implications [2].

Implications

Definition 1

An implication is a function $I:[0,1]^2 \rightarrow [0,1]$ with following properties>

(I1) if $x \leq z$ then $I(x,y) \geq I(z,y)$ for $\forall y \in [0,1]$,

(I2) if $y \leq t$ then $I(x,y) \leq I(x,t)$ for $\forall x \in [0,1]$,

(I3) $I(0,x) = 1$ for $\forall x \in [0,1]$,

(I4) $I(x,1) = 1$ for $\forall x \in [0,1]$,

(I5) $I(1,0) = 0$.

The natural interpretations of those implication axioms are described in [3]. Some further axioms for implication are required in different applications (see [4] and its reference list).

(I6) $I(1,x) = x$ for $\forall x \in [0,1]$,

(I7) $I(x, I(y,z)) = I(y, I(x,z))$ for $\forall x,y,z \in [0,1]$,

(I8) $x \leq y$ if and only if $I(x,y) = 1$ for $\forall x,y \in [0,1]$,

(I9) $I(x,0) = N(x)$ is a strong negation,

(I10) $I(x,y) \geq y$ for $\forall x,y \in [0,1]$,

(I11) $I(x,x) = 1$ for $\forall x \in [0,1]$,

(I12) $I(1x,y) = I(N(y), N(x))$ with a strong negation N, and for $\forall x,y \in [0,1]$,

(I13) I is a continuous function.

There exist some interdependence among these axioms [5], and in applications usually there is no need for all of these axioms at the same time.

Implications by T-norms, T-conorms and Negations

Definition 2

An *S*-implication associated with a t-conorm *S* and a strong negation *N* is defined by:

(*i*) $I_{S,N}(x,y) = S(N(x,y))$

An *R*-implication associated with a t-norm *T* is defined by

(*ii*) $I_T(x,y) = \sup\{z | T(x,z) \le y\}$

A *QL*-implication associated with a t-norm *T*, a t-conorm *S* and a strong negation *N* is defined by

(*iii*) $I_{T,S,N}(x,y) = S(N(x),T(x,y))$.

$I_{S,N}$ and I_T satisfy the conditions (I1)-(I5), thus they are implications. The idea behind (*i*) is the elementary logical rule $p \to q \Leftrightarrow \neg p \vee q$. *QL*-implication, in general, violates the property (I1).

In [5] we can find a set of conditions (from (I1) to (I13)), which are satisfied by the implications from Definition 2, i.e., a function $I:[0,1]^2 \to [0,1]$ is an *R*-implication based on a left-continuous t-norm if and only if *I* satisfies conditions (I2), (I7), (I8) and $I(x,.)$ is right-continuous for any fixed $x \in [0,1]$.

Although $I(x,y) = T(x,y)$ (*T* is a t-norm), operators do not verify the properties of the implications they are used as a implication model in many applications in fuzzy logic, for example as Mamdani 'implication'.

3 Lattice-ordered Monoids and Left Continuous Uninorms and T-norms

Let *L* be a non-empty set. Lattice is a partially (totally) ordered set which for any two elements $x, y \in L$ also contains their *join* $x \vee y$ (i.e., the least upper bound of the set $\{x,y\}$), and their *meet* $x \wedge y$ (i.e., the greatest lower bound of the set $\{x,y\}$), denoted by $(L,\preceq)$. Secondly, $(L,*)$ is a semi-group with the neutral element. Following [6] and [7] let the following be introduced:

Definition 3

[11] Let $(L,\preceq)$ be a lattice and $(L,*)$ a semi-group with the neutral element.

(i) The triple $(L,*,\preceq)$ is called a *lattice-ordered monoid* (or an *l-monoid*) if for all $x,y,z\in L$ we have

(*LM1*) $x*(y\vee z)=(x*y)\vee(x*z)$ and

(*LM2*) $(x\vee y)*z=(x*z)\vee(y*z)$.

(ii) An $(L,*,\preceq)$ *l*- monoid is said to be *commutative* if the semi-group $(L,*)$ is commutative.

(iii) A commutative $(L,*,\preceq)$ *l*- monoid is said to be *commutative, residuated l-monoid* if there exists a further binary operation $\rightarrow_*$ on *L*, i.e., a function $\rightarrow_*: L^2\rightarrow L$ (*the* $*$ *residuum*), such that for all $x,y,z \in L$ we have

(*Res*) $x*y\preceq z$ if and only if $x\preceq(y\rightarrow_* z)$.

(iv) An *l*-monoid $(L,*,\preceq)$ is called an *integral* if there is a greatest element in the lattice $(L,\preceq)$ (often called the universal upper bound) which coincides with the neutral element of the semi-group $(L,*)$.

Obviously, each *l*-monoid $(L,*,\preceq)$ is a partially ordered semi-group, and in the case of commutativity the axioms (*LM1*) and (*LM2*) are equivalent.

In the following investigations the focus will be on the lattice $([0,1],\leq)$, we will usually work with a complete lattice, i.e., for each subset *A* of *L* its join $\vee A$ and its $\wedge A$ exist and are contained in *L*. In this case, *L* always has a greatest element, also called the *universal upper bound.*

Example 1

If we define $*:[0,1]^2\rightarrow[0,1]$ by

$$x*y=\max_{0.5}^{\min}=\begin{cases}\min(x,y) & \mathit{if} \quad x+y\leq 1\\ \max(x,y) & \mathit{otherwise}\end{cases}$$

then $([0,1],*,\leq)$ is a commutative, residuated *l*-monoid, and the *-residuum is given by

$$x\rightarrow_* y=\begin{cases}\max(1-x,y) & \mathit{if} \quad x\leq y\\ \min(1-x,y) & \mathit{otherwise}\end{cases}.$$

It is not an integral, since the neutral element is 0.5.

The operation * is an *uninorm*, and special type of distance based operators, maximum distance minimum operator, with the parameter and unit element 0.5, [8], [9], [10].

The following result is on important characterization of left-continuous uninorms.

Theorem 1

For each function $U : [0,1]^2 \to [0,1]$ the following are equivalent:

(*i*) $([0,1], U, \leq)$ is a commutative, residuated *l*-monoid, with a neutral element

(*ii*) *U* is a left continuous uninorm.

In this case the *U*-residuum $\to_U$ is given by

(*ResU*) $$x \to_U y = \sup\{z \in [0,1] U(x,z) \leq y\}.$$

Proof. It is easy to see, that $([0,1], U, \leq)$ is a commutative, residuated *l*-monoid with a neutral element if and only if *U* is a uninorm.

Therefore, in order to prove that (*i*)⇒(*ii*), assume that $([0,1], U, \leq)$ is residuated, fix and arbitrary sequence $(x_n)_{n \in N}$ in $[0,1]$ and put $x_0 = \sup_{n \in N} x_n$.

Let $y_0 \in [0,1]$, and $z_0 = \sup_{n \in N} U(x_n, y_0)$.

Obviously $z_0 \leq U(x_0, y_0)$, and (*ResU*) implies $(y_0 \to_U z_0) \geq x_n$ for all $n \in N$, subsequently, $(y_0 \to_U z_0) \geq x_0$.

Applying again (*ResU*) in the opposite direction, we obtain $U(x_0, y_0) \leq z_0$. Because of the monotonicity of uninorm U, (U3), and based on *Proposition 1.22.* from [11], we have $U(x_0, y_0) = z_0$, i.e.,

$$\sup_{n \in N} U(x_n, y_0) = U\left(\sup_{n \in N}(x_n), y_0\right)$$

Conversely, if the uninorm *U* is left-continuous, define the operation $\to_U$ by (*ResU*). Then it is clear that for all $x, y, z \in [0,1]$, $x \leq (y \to_U z)$ whenever $U(x, y) \leq z$. The left-continuity of U then implies $U(y \to_U z, y) \leq z$, which together with the monotonicity (U3), ensures that $\to_U$ is indeed the *U*-residuum. ◊

The work [9] presents general theoretical results related to residual implicators of uninorms, based on residual implicators of t-norms and t-conorms.

Residual operator R_U, considering the uninorm U , can be represented in the following form:

$$R_U(x,y) = \sup\{z | z \in [0,1] \wedge U(x,z) \le y\}.$$

Uninorms with the neutral elements $e = 0$ and $e = 1$ are t-norms and t-conorms, respectively, and related residual operators are widely discussed [5], [12]. In [9] we also find suitable definitions for uninorms with neutral elements $e \in]0,1[$.

If we consider a uninorm U with the neutral element $e \in]0,1[$, then the binary operator R_U is an implicator if and only if $(\forall z \in]e,1[)(U(0,z) = 0)$. Furthermore R_U is an implicator if U is a disjunctive right-continuous idempotent uninorm with unary operator g satisfying $(\forall z \in [0,1])(g(z) = 0 \Leftrightarrow z = 1)$.

The residual implicator R_U of uninorm U can be denoted by Imp_U.

Corollary 1

Consider a uninorm U, then R_U is an implicator in the following cases:

(i) U is a conjunctive uninorm,

(ii) U is a disjunctive representable uninorm,

(iii) U is a disjunctive right-continuous idempotent uninorm with unary operator g satisfying

$(\forall z \in [0,1])(g(z) = 0 \Leftrightarrow z = 1)$.

Theorem 1. implies in a special case Proposition 2.47. from [11]:

Corollary 2

For each function $T : [0,1]^2 \to [0,1]$ the following are equivalent:

(*i*) $([0,1], T, \le)$ is a commutative, residuated integral *l*-monoid,

(*ii*) T is a left continuous t-norm.

In this case the T-residuum $\to_T$ is given by

(*ResT*) $$x \to_T y = \sup\{z \in [0,1] T(x,z) \le y\}.$$

Because of its interpretation in [0,1]-valued logics, the T-residuum $\to_T$ is also called *residual implication* (or briefly *R-implication*). (see [13])

Example 2

For continuous basic t-norms T_M, T_P, T_L the corresponding residuum is given by

$$x \to_M y = \begin{cases} 1 & \text{if} \quad x \le y \\ y & \text{otherwise} \end{cases}, \qquad (\textit{Gödel implication}),$$

$$x \to_P y = \begin{cases} 1 & \text{if} \quad x \le y \\ \dfrac{y}{x} & \text{otherwise} \end{cases} \qquad (\textit{Goguen implication}),$$

$$x \to_L y = \min(1 - x + y, 1) \qquad (\textit{Lukasiewicz implication}),$$

where respectively $\to_M, \to_P, \to_L$ are used instead of $\to_{T_M}, \to_{T_P}, \to_{T_L}$.

Remark 1

It should be emphasized that the formula (*ResT*) can be computed for arbitary t-norms *T*, but that, as pointed out in the *Corollary 2*, the resulting operation $\to_T$ equals the *T*- residuum, i.e., satisfies axiom (*R*) only if the t-norm *T* is left-continuous [13].

In [14] residual implications and left-continuous t-norms are discussed, which are ordinal sums of semigroups.

The problem of *S*-implications and QL implications and the problem of $p \to q \Leftrightarrow \neg p \vee q$ equality in fuzzy logic theory based on uninorms are still an open problems.

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