Fuzzy Rule Interpolation by the Least Squares Method

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Abstract: Fuzzy rule interpolation-based reasoning methods are the most common choices for cases when the applied rule base is not dense. This paper presents a new technique called LESFRI, which is based on the method of least squares. Its central idea is the conservation of the shape type specific to a fuzzy partition. The method has low computational complexity.

Keywords: fuzzy rule interpolation, LESFRI, sparse fuzzy rule base

1 Introduction

Rule-based fuzzy systems can be divided into two groups depending on whether they require or not the dense character of a rule base. The classical fuzzy reasoning methods like Zadeh's, Mamdani's, Larsen's or Sugeno's belong to the first group. Having an observation that do not intersect or at least overlap partially any of the rule antecedents their response is either a 'do nothing' or the last cached valid result.

In several practical applications it is required from the system to give a reasonable output even in such cases. The members of the second group of fuzzy inference techniques were developed to fulfil the above discussed demand by the help of approximate approaches. Generally they are based on fuzzy rule interpolation and/or extrapolation. For the sake of simplicity further on they are referred as Fuzzy Rule Interpolation (FRI) methods. One can distinguish two separate approaches in the way of they are calculating the conclusion.

The members of the first subgroup determine the conclusion directly from the observation taking into consideration the nearest two or more rules that surround the observation. Relevant members of this subgroup are the α-cut-based interpolation (KH) proposed by Kóczy and Hirota in [13], which was the first developed technique, the modified α-cut-based interpolation (MACI) [17] introduced by Tikk and Baranyi, the fuzzy interpolation based on vague environment (FIVE) [11] developed by Kovács and Kóczy, its extended version suggested by Kovács in [12], the improved fuzzy interpolation technique for multi-dimensional input spaces (IMUL) [18] proposed by Wong, Gedeon and Tikk, the interpolative reasoning based on graduality (IRG) [4] introduced by Bouchon-Meunier, Marsala and Rifqi, the method based on the conservation of the relative fuzziness (CRF) [14] proposed by Hirota, Kóczy and Gedeon.

The members of the second subgroup follow a two-step approach whose basic concept was formalized by the Generalized Methodology of fuzzy rule interpolation (GM) introduced by Baranyi et al. in [3]. They determine first a new rule in the position of the observation and calculate the conclusion by its help. Relevant members of this subgroup are the technique group suggested by Baranyi, Kóczy and Gedeon in [3], the Interpolation with Generalized Representative Values (IGRV) [5] developed by Huang and Shen, the technique proposed by Jenei in [6], and the method FRIPOC introduced by Johanyák and Kovács in [7]. The solvability of fuzzy relation equations as the solvability of interpolating and approximating fuzzy functions with respect to a given set of fuzzy rules was studied by Perfilieva in [15].

The common ground of all the methods belonging to the two subgroups is that they suppose the existence of regularity between the linguistic terms of a fuzzy partition regardless of it is part of an antecedent or consequent universe of discourse. In a similar way they assume regularity in the case of the rules of the knowledge base, too.

In several practical cases all linguistic terms of a fuzzy partition belong to the same shape type and they have the same height. Having such a system it seems to be a natural requirement that the calculated conclusion and also the antecedent and consequent sets of the interpolated rule adhere to this regularity as well.

However, several FRI methods do not fulfil this requirement. Therefore we have developed a new inference technique especially for these purposes. It is called LESFRI.

2 The Method LESFRI

The method LESFRI (LEast Squares-based Fuzzy Rule Interpolation) belongs to the second subgroup of FRI techniques. In order to simplify the further calculations first the range of all partitions is normalized to the interval [0,1]. Due to the fact that LESFRI is an α-cut-based method the characteristic height of all partitions should be the same. If this condition is not met a normalization of the affected partitions is needed as an additional preparatory work.

LESFRI essentially follows the concepts laid down in [3]. It assumes that a better approximation of the real relation between the antecedent and consequent universes can be attained by determining first an auxiliary rule in the position of the observation and than calculating the conclusion by firing this rule.

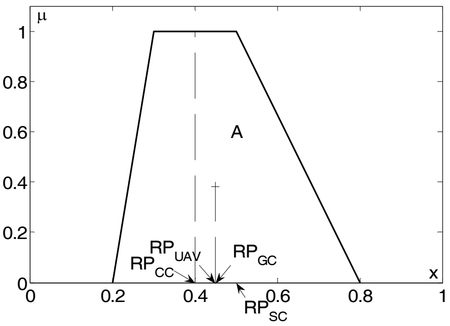

Figure 1
Reference point types

The above mentioned coincidence between the position of the antecedent and observation sets in each dimension can be interpreted thanks to the reference point-based evaluation of the location of the linguistic terms. The selection of the type of this point introduces a free parameter into the method. However, it is worthy of note that the same type of reference point have to be used for each set in each dimension in the frames of a fuzzy system. Usual choices are the centre of the core (RP_{CC}) [3][4][7], the centre of the support (RP_{SC}) [4], the centre of gravity (RP_{GC}) [5] and the unweighted or weighted average of the abscissas of the characteristic (break) points of the shape (RP_{UAV}, RP_{WAV}) [5]. Figure 1 presents the listed reference point types in case of a trapezoid shaped fuzzy set.

The centre of the core offers the most advantages among them. It contains information about the middle one from the most relevant – having the maximal membership value – elements of the set and facilitates the division of the shape of a linguistic term into two flanks that can be calculated separately.

The distance between the fuzzy sets is measured as the horizontal distance between their reference points. Furthermore the use of a representative point simplifies the evaluation of the ordering of the sets.

The first step of the method LESFRI, the interpolation of the new rule is achieved in three stages.

1 The antecedent sets are calculated in each dimension by a set interpolation technique called FEAT-LS.

2 The position of the consequent sets is calculated in each consequent dimension using a crisp interpolation technique, which is an extended version of the Shepard interpolation.

3 The shape of the consequent sets is determined by using the same set interpolation technique as in stage 1.

The conclusion is produced in the second step by the help of the new rule. The difficulty of the task is created by the fact that in the general case the antecedent part of the new rule does not fit perfectly the observation in each input dimension. Therefore a special single rule reasoning technique is needed for the calculations.

3 FEAT-LS

Fuzzy Set Interpolation (FSI) aims the determination of a new linguistic term in a given point of a fuzzy partition called interpolation point. This means that the new fuzzy set is generated in such way that its reference point coincides with the interpolation point. The problem is called interpolation when the given point is situated between the reference points of the first and last linguistic terms. Otherwise it is called an extrapolation task.

In case of two-step FRI methods an FSI technique is used for the calculation of the antecedent and consequent sets of the new rule. Thus the interpolation point is either the reference point of the observation or the reference point of the consequence in the current dimension. An FSI technique works only with one partition. Therefore the calculations in the different dimensions in both the antecedent and consequent cases can be done separately.

In several applications all sets of a partition belong to the same shape type and the characteristic (break) points are also situated at the same α-level. In such cases it seems to be a natural condition on the new linguistic term to suit this regularity. Furthermore if one or more shape-pieces delimited by the characteristic points are linear it could be expected that the corresponding pieces of the new set to be linear as well. The latter expectation is also formulated in requirement 7 of the General conditions on rule interpolation methods introduced in [8].

The Fuzzy sEt interpolAtion Technique based on the method of Least Squares (FEAT-LS) was developed especially for these purposes. It is applicable for both of the problems interpolation and extrapolation.

Most of the FSI methods (e.g. SCM [2]) seek for those two linguistic terms that surround the interpolation point supposing regularity between the sets of the partition. Keeping on this way often a better set approximation can be obtained by taking into consideration all the linguistic terms of the partition. Although at first sight it could lead to increased computational complexity, but usually it is simpler to draw into calculations the whole bulk of the sets. It is because a fuzzy partition contains a small (generally smaller or equal to 11 [9][1]) number of linguistic terms. Besides this approach ensures extrapolation capability of the method as well.

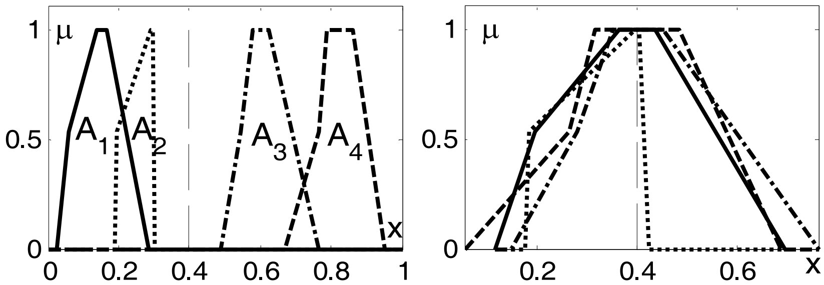

Figure 2
The original partition and the shifted linguistic terms

As a first step all the sets of the partition are shifted horizontally in order to reach the coincidence between their reference points and the interpolation point. The left part of Figure 2 presents an example for a partition containing four linguistic terms and an interpolation point at $x^i=0.4$. The effect of the shifting is presented on the right part of the figure.

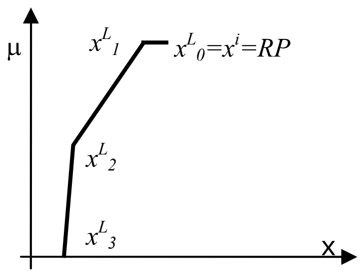

Figure 3
The left flank with numbered characteristic points

The shifting is not permanent. It is only used during the determination of the new set. Next the shape of the new linguistic term is calculated from the overlapped set shapes in a set form that belongs to the characteristic shape type of the partition (e.g. singleton, triangle, trapezoid, polygonal, etc). The calculations are done separately for the left and right flanks of the new set. In the followings only the case of the left flank is presented, the right flank is determined similarly.

During the calculations one has to determine only the abscissas of the characteristic points, the ordinate values are the same for all the sets of the partition. Proceeding on counter clockwise direction the first point is the reference point (see Fig. 3), which is determined by the interpolation point ($x_0^L = x^i$). For each remaining characteristic point we build the weighted sum of the squares of the differences between the abscissa of the new point and the abscissas of the corresponding points of the shifted linguistic terms

$$Q_j^L = \sum_{l=1}^{n} w_l \cdot \left(x_{lj}^L - x_j^L\right)^2, \tag{1}$$

where Q_j^L is the sum corresponding to the j^{th} point of the left flank of the interpolated set, w_l is the weighting factor of the l^{th} linguistic term of the partition, x_{lj}^L is the abscissa of the j^{th} point of the left flank of the l^{th} set and x_j^L is the abscissa of the j^{th} point of the left flank of the interpolated set.

The weighting expresses that the sets situated originally in closer neighborhood of the interpolation point should exercise a higher influence than those situated originally in farther regions of the partiton. The simplest weighting factor is the reciprocal value of the distance, but there are some others in the literature that were developed for more or less analogue cases. The formula (2) has been chosen as the most suitable for the method FEAT-LS

$$w_l = \frac{1}{d\left(A_l, A^i\right)^p}, \tag{2}$$

where $d\left(A_l, A^i\right)$ is the original distance between l^{th} set and the interpolated one, p is a constant. The selection of the type of the weighting factor and the selection of its parameters gives tuning possibility to the system.

There are two conditions on x_j^L: it should yield a minimal Q_j^L value and it should avoid the arising of an abnormal set shape. The latter one can be expressed by the formula

$$x_j^L \le x_{j-1}^L \qquad j = \overline{1,k}, \tag{3}$$

where k is the ordinal number of the last characteristic point of the left flank. Supposing that the first condition is fulfilled by all linguistic terms of the partition it can be proved that it is also fulfilled by the points calculated by the method of least squares. Thus results the formula

$$x_j^L = \begin{cases} \dfrac{\sum_{l=1}^{n} w_l \cdot x_{lj}^L}{\sum_{l=1}^{n} w_l} & d(A_l, A^i) \neq 0 \\ x_{lj}^L & d(A_l, A^i) = 0. \end{cases} \tag{4}$$

4 The Position of the Consequent Sets

The second stage of the first step of the method LESFRI is the determination of the position of the fuzzy sets belonging to the consequent part of the new rule. It is calculated independently in each output dimension. Further on for better lucidity the index identifying the current output dimension is not indicated in the formulas.

Each rule of the rule base can be represented as a point on a hyper-surface using as co-ordinates the reference points of the antecedent linguistic terms and the reference point of the rule consequent in the current output dimension. Figure 4 presents the case of a system having two antecedent dimensions. The point outlined by a bullet on the surface (Fig. 4) symbolises the interpolated point on the hyper-surface corresponding to the reference points of the interpolated antecedent sets ($RP(A^i_1)$ and $RP(A^i_2)$) and the reference point of the rule consequent ($RP(B^i)$).

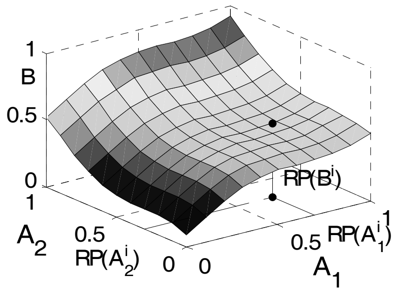

Figure 4
Hyper-surface representing the rules

The task of the current stage can be defined as a problem of finding a point on the hyper-surface. Due to the sparse character of the rule base an n_a dimensional interpolation has to be done for irregularly spaced data, where n_a is the number of the antecedent dimensions. It can be expressed in general by the formula

$$RP(B^i) = f\left(RP(A_1^i), RP(A_2^i), \ldots, RP(A_j^i), \ldots, RP(A_{n_a}^i), \Re\right) \tag{5}$$

where $RP(B^i)$ is the reference point of the interpolated consequent set, $RP(A^i_j)$ is the reference point of the interpolated antecedent set in the j^{th} input dimension and $\Re$ is the rule base. The function should pass through the known points of the hyper-surface. We suggest the use of an interpolation function that is an extension and adaptation of the Shepard interpolator [16] for the case of arbitrary number of antecedent dimensions.

The antecedent part of each rule can be thought of as a point in the antecedent hyper-space. Its co-ordinates are given by the reference points of the sets belonging to it. The point corresponding to the antecedent of the interpolated rule is at the same time also the representing point of the observation. Further on the Euclidean distance between these points is used as the measure of the closeness of the antecedents and by this means also the closeness of the rules. The proposed interpolation function determines the reference point of the conclusion as a weighted average of the reference points of the consequent sets of the known rules

$$RP\left(B^i\right)=\frac{\sum_{l=1}^{N} RP\left(B_l\right)\cdot s_l}{\sum_{l=1}^{N} s_l}, \tag{6}$$

where N is the number of the rules, l denotes the current rule, s_l is the weight attached to the l^{th} rule.

The rules whose antecedent part is in the closer neighbourhood of this point should exercise higher influence than those situated farther. Therefore the weighting factor is a distance function. Shepard in [16] proposed several variants of the weighting factors for its interpolation function. The first of them, which applies the reciprocal value of the square of the distance, was chosen to be applied considering it as the one having the lowest computational complexity. Its adapted version is the formula

$$s_l=\frac{1}{d\left(RA^i,RA_l\right)^2}=\frac{1}{\sum_{j=1}^{n_a}\left(RP\left(A^i_j\right)-RP\left(A_{lj}\right)\right)^2}, \tag{6}$$

where RA^i is the antecedent of the interpolated rule, RA_l is the antecedent of the l^{th} rule, $RP(A^i_j)$ is the reference point of the interpolated antecedent in the j^{th} dimension (identical with the reference point of the observation in the j^{th} dimension), $RP(A_{lj})$ is the reference point of the antecedent set of the l^{th} rule in the j^{th} dimension and n_a is the number of the antecedent dimensions.

5 SURE-LS

The Single RUle REasoning based on the method of least squares aims the determination of the conclusion using the observation and the new rule. In the general case differences could exist between the shape of the observation sets and the shape of the corresponding antecedent linguistic terms. Therefore the task is to modify the form of the rule consequents in each output dimension by taking into consideration the differences on the antecedent side.

SURE-LS is developed as a complement of FEAT-LS. Therefore during its application is assumed that in case of each dimension only one shape type is present and the characteristic points are situated at the same α-levels. Due to the possible diversity of the applied α-level collections first a new set of α-levels (Λ) is compiled that contains all the levels used in antecedent and consequent dimensions

$$\Lambda = \left(\bigcup_{j=1}^{n_a} \Lambda_j^a \right) \bigcup \left(\bigcup_{k=1}^{n_c} \Lambda_k^c \right), \tag{7}$$

where Λ_j^a is the set of α-levels in the j^{th} antecedent dimension, Λ_k^c is the set of α-levels in the k^{th} consequent dimension, n_a and n_c are numbers of the antecedent and consequent dimensions.

The rest of the calculations are done separately for the left and right flanks of the conclusion sets. Further on only the case of the left flank is presented. The right flank is calculated similarly. Next the difference ($d_{\alpha j}^{aL}$) is calculated between the antecedent and observation sets for each α-level and for each input dimension

$$d_{\alpha j}^{aL} = \inf\left\{A_{\alpha j}^{i}\right\} - \inf\left\{A_{\alpha j}^{*}\right\}, \tag{8}$$

where $A_{\alpha j}^{i}$ is the α-cut of the antecedent set of the interpolated rule in the j^{th} input dimension, $A_{\alpha j}^{*}$ is the α-cut of the observation set in the j^{th} input dimension. Hereupon the weighted average difference (ad_{α}^{aL}) is calculated for each α-level by the formula

$$ad_{\alpha}^{aL} = \frac{\sum_{j=1}^{n_a} we_j \cdot d_{\alpha j}^{aL}}{\sum_{j=1}^{n_a} we_j}, \tag{9}$$

where we_j is the weighting factor of the j^{th} antecedent dimension. This weighting factor offers a possibility to take into consideration each input state variable with different importance.

The following part of the calculations is done separately for each output dimension. Thus it can also be parallelized. Further on the case of the k^{th} consequent dimension is presented. For better lucidity the index identifying the current output dimension is not indicated in the formulas.

The basic idea of the method is the conservation of the weighted average difference measured on the antecedent side by considering the same difference between the consequent and conclusion sets as the ad_α^{aL} at the same α-level

$$d_\alpha^{cL} = \inf\{B_\alpha^i\} - \inf\{B_\alpha^*\} = ad_\alpha^{aL}\ , \tag{10}$$

where B_α^i is the consequent set of the interpolated rule and B_α^* is the conclusion. There are two important conditions to be met by the result: (1) the shape should adhere to the characteristic shape type of the current consequent partition and (2) abnormal set shapes should be avoided. In order to met the second condition the points defining the conclusion have to be situated on the left side of the reference point. Thus the formula describing the collection of the points is the following

$$\inf\{B_\alpha^*\} = \min\left(\inf\{B_\alpha^i\} - ad_\alpha^{aL}, RP\left(B^*\right)\right). \tag{11}$$

The geometric form resulting from binding by lines the points obtained in (11) generally does not fulfil the shape criterion. Threfore the method of least squares is applied to find the flank that best fits the calculated points.

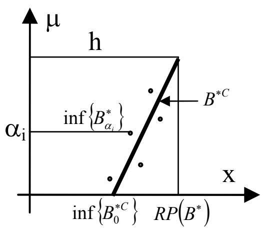

Figure 4
The left flank of a triange shaped conclusion

Hereupon the rest of the calculations depends on the characteristic shape of the current output partition. For example in the singleton case the conclusion will be a singleton placed in the reference point defined in the previous section. In the triangle case (Fig. 4) one vertex is given by the reference point and the characteristic height of the partition. The other one corresponding to the left end of the base of the triangle is calculated by the formula

$$\inf\left\{B_0^{*C}\right\}=RP\left(B^*\right)-h\frac{\sum_{i=1}^{n_\Lambda}\left(h-\alpha_i\right)\left(RP\left(B^*\right)-\inf\left\{B_{\alpha_i}^*\right\}\right)}{\sum_{i=1}^{n_\Lambda}\left(h-\alpha_i\right)^2}, \tag{12}$$

where B^{*C} is the corrected conclusion set, h is the characteristic height of the partition and n_Λ is the number of α-levels.

Conclusions

FRI-based inference techniques ensure the adequate conclusions even in case of sparse rule bases. The method LESFRI introduced in this paper follows the two-step concept. First it interpolates an auxiliary rule in the position of the observation than determines the conclusion by the help of the new rule. Its main advantages are (1) its capability to produce new linguistic terms that fit into the regularity of the original partitions and (2) its low computational complexity. Being an α-cut-based method its application is restricted to the case when the height of all sets is the same. In this paper only the case of piece-wise linear membership functions was presented. The case of smooth set shapes is subject of farther research work.

The method is implemented in Matlab and can be downloaded from [19]. This website is dedicated to a fuzzy rule interpolation Matlab toolbox development project (introduced in [10]) aiming the implementation of various FRI techniques.

Acknowledgement

This research was supported by GAMF Faculty Kecskemét College grant no: 1N076/2006.

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