An Optimality Criterion for Fuzzy Output Interfaces

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Abstract- **This paper is concerned with the issue of optimality for fuzzy output interfaces. A new criterion of optimality is introduced that leads to the definition of an optimality degree. The proposed optimality degree is particularly useful to evaluate defuzzification algorithms that are the basis of output interfaces and that seldom satisfy the optimality condition. The proposed criterion is applied to several Mamdani Fuzzy Inference Systems that differ in the output interfaces. Specifically, the optimality degree is used to compare different defuzzification strategies and to highlight quality variations attained by different output interfaces.**

I. INTRODUCTION

Fuzzy inference systems are linguistic modeling structures composed of well-defined functional blocks including an input interface, a processing module and an output interface. The input interface performs the conversion of information coming from the environment in an internal format acceptable by the processing module. Symmetrically, the output interface transforms information coming from the processing module into a representation to be used in the environment [1].

The performance of a fuzzy system heavily depends on the input/output interfaces, hence a careful design of such functional blocks is of crucial importance. The design of I/O interfaces should take into account the main objectives of the fuzzy model, i.e. those concerning a tradeoff between interpretability and accuracy of the modeling [2], [3]. In particular, an interface should conserve the information entering into (or exiting from) the processing module, as stated by the so-called "Information Equivalence Criterion". Under such condition, the interface is regarded as "optimal interface" [4], [5].

Various optimality conditions have been considered in the literature, particularly in the case of triangular fuzzy sets [5] [6], and recently for a wider class of fuzzy sets, named "bi-monotonic" [7]. In all cases, the same optimality condition has been considered for input and output interfaces. However, while input interfaces are usually implemented by a matching mechanism that makes use of possibility function to derive membership values from numerical values, output interfaces are defined by an aggregation operation that can easily hamper the optimality condition. Actually, it is very hard for output interfaces to be optimal in the sense of the Information Equivalence Criterion.

To this aim, in this paper we propose a new criterion of optimality for output interfaces that leads to an "optimality degree", which gives a measure of the quality of an output interface. Hence, different output interfaces can be compared and their quality ranked according to their optimality degrees. The proposed optimality degree is experimentally applied and discussed for several Fuzzy Inference Systems of Mamdani type that differ in the defuzzification method.

II. FUZZY I/O INTERFACES

A Fuzzy Inference System (FIS) is composed of three main functional blocks:

- Input interface
- Processing module
- Output interface

The Input interface (I-interface) provides a fuzzy discretization of external inputs by means of a family of K_x reference fuzzy sets. For n_x-dimensional inputs of domain $\mathbf{X}$, the (multidimensional) I-interface is defined as:

$$\mathfrak{I}:\mathbf{X}\subset\mathbb{R}^{n_x}\to[0,1]^{K_x} \tag{1}$$

such that:

$$\forall x\in\mathbf{X}\subseteq\mathbb{R}^{n_x}:\mathfrak{I}(x)=\left[A_i(x)\right]_{i=1,2,\ldots K_x} \tag{2}$$

where $A_i(x)$ is the membership degree of the i-th reference fuzzy set. Usually, a multidimensional I-interface is composed of several one-dimensional I-interfaces, each of them is defined over a single input dimension, as follows:

$$\forall x\in\mathbf{X}\subseteq\mathbb{R}^{n_x}:\mathfrak{I}(x)=\left[\mathfrak{I}^{(j)}(x_j)\right]_{j=1,2,\ldots n_x} \tag{3}$$

being $x=\left[x_1,x_2,\ldots,x_{n_x}\right]$, where each $\mathfrak{I}^{(j)}$ is a one-dimensional I-interface in the form:

$$\mathfrak{I}^{(j)}:X^{(j)}\subseteq\mathbb{R}\to[0,1]^{K_x^{(j)}},\quad j=1,2,\ldots n_x \tag{4}$$

The second functional block of a FIS is the Processing module, which usually embodies the knowledge base in form of fuzzy relations and transforms a structure of input membership degrees into a structure of output membership degrees. The processing module is a function defined as:

$$\mathfrak{P}:[0,1]^{K_x}\to[0,1]^{K_y} \tag{5}$$

Typically, the processing module computes output membership values by evaluating the truth values of the antecedents of each rule constituting the knowledge base.

The Output interface (O-interface) transforms the structure of membership degrees provided by the processing module into a vector of numerical values. From a functional point of view, it can be defined as:

$$\mathfrak{O}:[0,1]^{K_y} \to \mathbb{R}^{n_y} \tag{6}$$

In the following, we assume for simplicity that the O-interface provides only a scalar value, i.e. $n_y = 1$. The extension to multidimensional outputs is straightforward by considering n_y independent O-interfaces.

Generally, an O-interface is characterized by a family of referential fuzzy sets and calculates the final numerical output by means of a defuzzification algorithm, which operates by aggregating membership degrees to derive the final numerical value. Depending on the FIS type, the O-interface can be simple or complex. In Takagi-Sugeno FIS with R rules of the form:

$$\text{IF } x \text{ is } \mathbf{A}^{(r)} \text{ THEN } y = w^{(r)} \tag{7}$$

the O-interface is defined as follows:

$$\mathfrak{O}_{TS}\left(\pi_y\right) = \frac{\sum_{r=1}^{K_y} \pi_y^{(r)} \cdot w^{(r)}}{\sum_{r=1}^{K_y} \pi_y^{(r)}} \tag{8}$$

being $\pi_y = \left[\pi_y^{(1)}, \pi_y^{(2)}, \ldots, \pi_y^{(K_y)}\right]$ and $w^{(r)} \in \mathbb{R}$ the output of the r-th rule corresponding to a singleton set.

In a Mamdani FIS type, the O-interface is far more complex. In a FIS with R rules of the form:

$$\text{IF } x \text{ is } \mathbf{A}^{(r)} \text{ THEN } y \text{ is } B^{(r)} \tag{9}$$

the processing module returns a vector of R membership degrees corresponding to the values $\pi^{(r)} = \mathbf{A}^{(r)}(x)$. The evaluation of $\pi^{(r)}$ can be attained by conjunction of one-dimensional fuzzy sets. For each rule, a new fuzzy set is derived as follows:

$$\bar{B}^{(r)}(y) = B^{(r)}(y) \otimes \pi^{(r)} \tag{10}$$

being $\otimes$ a t-norm like the minimum or the product operator. Then, all such fuzzy sets are aggregated to form a unique fuzzy set as follows:

$$B = \bigcup_{r=1}^{R} \bar{B}^{(r)} \tag{11}$$

being the union defined by the *sup* operator. Finally the aggregate fuzzy set B is reduced to a single numerical value according to a defuzzification method, such as:

Centroid: $$\tilde{y} = \frac{\int_{\mathbb{R}} B(y) \cdot y \cdot dy}{\int_{\mathbb{R}} B(y) \cdot dy} \tag{12}$$

Bisector: $$\int_{-\infty}^{\tilde{y}} B(y)\,dy = \int_{\tilde{y}}^{+\infty} B(y)\,dy \tag{13}$$

$$M := \arg\max_{y \in \mathfrak{R}} B(y);$$

Mean of Maxima (MOM): $$\tilde{y} = \frac{\sum_{\psi \in M} \psi}{|M|} \tag{14}$$

Inf of Maxima (LOM): $\tilde{y} = \inf M$ (15)

Sup of Maxima (SOM): $\tilde{y} = \sup M$ (16)

Hence, in Mamdani FIS the O-interface is basically defined by a defuzzification formula.

III. OPTIMALITY OF I/O INTERFACES

In this section the concept of optimality for I/O interfaces is introduced.

A fuzzy interface is defined as *optimal* if it is invertible, i.e. if an error-free conversion exists that transforms numerical values in membership degrees and vice versa.

Optimality for I-interfaces can be formally defined as:

$$\operatorname{opt}(\mathfrak{I}, \mathbf{X}) \Leftrightarrow \exists \mathfrak{I}^{-1} \text{ s.t. } \forall x \in \mathbf{X} \subseteq \mathbb{R}^{n_x} : \mathfrak{I}^{-1}\left(\mathfrak{I}(x)\right) = x \tag{17}$$

It should be remarked that this definition is more general than other definitions given in literature. For example, in [8] and [9] the authors provide a definition of optimality for I/O interfaces, being the O-interface the inverse function of the I-interface and usually defined by means of a defuzzification formula. This is different from the above definition (17) where the inverse function of the I-interface is simply required to exist, but it is not given. This implies that such inverse function may not coincide with any known defuzzification method.

The optimality condition for I-interfaces can be easily guaranteed if proper choices of referential fuzzy sets are made. In particular, in [7] we have characterized the family of fuzzy sets which guarantees optimal I-interfaces.

Similarly to I-interfaces, the optimality condition for an O-interface is satisfied if the interface is invertible, that is:

$$\operatorname{opt}\left(\mathfrak{O}, [0,1]^{K_y}\right) \Leftrightarrow \exists \mathfrak{O}^{-1} \text{ s.t. } \forall \pi_y \in [0,1]^{K_y} : \mathfrak{O}^{-1}\left(\mathfrak{O}(\pi_y)\right) = \pi_y \tag{18}$$

Since the processing module provides only a small subsets of elements in $[0,1]^{K_y}$, the optimality condition of an O-interface can be conveniently redefined as:

$$\operatorname{opt}'(\mathfrak{O}, \mathbf{X}) \Leftrightarrow \exists \mathfrak{O}^{-1} \text{ s.t. } \forall x \in \mathbf{X} : \pi_y = \mathfrak{P}\left(\mathfrak{I}(x)\right) \to \mathfrak{O}^{-1}\left(\mathfrak{O}(\pi_y)\right) = \pi_y \tag{19}$$

The last definition restricts optimality condition only for those values of π_y that can be effectively returned by the processing module. Even with such a restricted condition, however, optimality of O-interfaces is hard to achieve. This is essentially due to the aggregation operation of the defuzzification procedure, which is adopted to reduce a highly dimensional structure of membership degrees into low dimensional numerical values (one dimension only in case of single output). As a consequence, an optimality measure is desirable to better evaluate and compare the quality of different O-interfaces, as well as to assess the quality of newly devised defuzzification methods.

IV. A NEW OPTIMALITY CRITERION FOR O-INTERFACES

As optimality of I-interfaces can be easily checked and optimal I-interfaces can be effortlessly designed, we restrict our study only to optimality of fuzzy O-Interfaces. In this section, we introduce a new criterion for optimal O-interfaces which is an extension of the standard optimality condition defined in (18). The new condition provides an optimality degree of a fuzzy interface ranging from 0 (excluded) to 1. When the optimality degree is 1, the membership values provided by the Processing module can be perfectly reconstructed by the inferred numerical value, according to (19). In a rule-based system as in (9), this means that the inferred value y has the following property:

$$\forall r : y \in \left\{ \omega \middle| B^{(r)}(\omega) = \pi^{(r)} \right\} \tag{20}$$

Condition (20) states that, for each rule, the inferred numerical value exactly corresponds to one of the numerical values that belong to the consequent fuzzy set with the membership degree provided by the Processing Module. In this ideal situation, inference is perfectly transparent and exactly reflects the semantics of each rule of the knowledge base.

In real situations, however, ideal condition rarely takes place since the inferred value does not belong to the sets (20). Intuitively, the more distant is the inferred value from those contained in such sets, the less transparent is the inference process since it badly reflects the semantics of the rules. The optimality degree is designed to take values as small as more distant is the inferred value w.r.t. values belonging to sets in (20). Hence, the optimality degree provides a measure of the quality (transparency) of the inference process.

The new optimality criterion is defined by considering, for a given O-interface, a related I-interface. More specifically, given an O-interface:

$$\mathfrak{D} : [0,1]^{K_y} \to \mathbb{R}^{n_y} \tag{21}$$

an I-interface is considered as follows:

$$\bar{\mathfrak{D}} : \mathbb{R}^{n_y} \to [0,1]^{K_y} \tag{22}$$

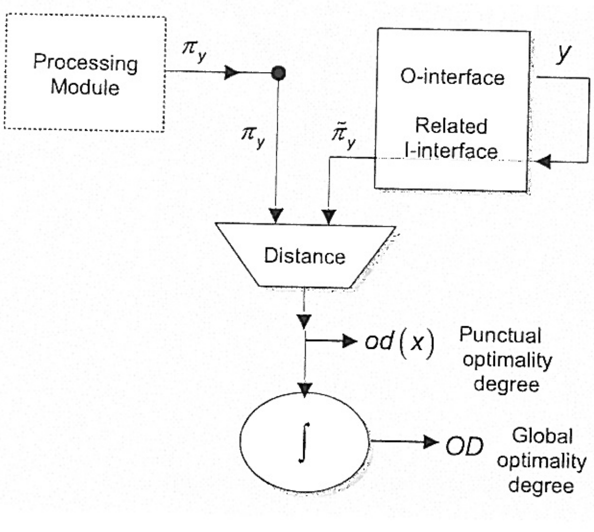

Fig. 1. Computation of the optimal degree

The choice of $\bar{\mathfrak{D}}$ should be guided by the mechanism implementing $\mathfrak{D}$. For a FIS of Mamdani-type, we define $\bar{\mathfrak{D}}$ as follows:

$$\forall y \in \mathbb{R}^{n_y} : \bar{\mathfrak{D}}(y) = \left[B^{(r)}(y) \right]_{r=1,2,\ldots,K_y} \tag{23}$$

Once the I-interface $\bar{\mathfrak{D}}$ has been derived, its optimality (in the classical sense) can be easily checked. If $\bar{\mathfrak{D}}$ is optimal, then $\mathfrak{D}$ is referred as *reversely optimal*. The optimality degree of a reversely optimal O-interface is computed as follows (see Fig. 1). When an input x is presented to the FIS, a structure of membership degrees π_y is computed by the processing module. The vector π_y is further elaborated by the O-interface $\mathfrak{D}$, which provides a numerical value y. Now the I-interface $\bar{\mathfrak{D}}$ is considered, and the structure of membership values $\tilde{\pi}_y = \bar{\mathfrak{D}}(y)$ is computed for the numerical value y. If $\pi_y = \tilde{\pi}_y$, then the O-interface is optimal. Conversely, the more π_y is different form $\tilde{\pi}_y$, the lower is the optimality of the O-interface. Given a distance measure $d : [0,1]^{K_y} \times [0,1]^{K_y} \to \mathbb{R}$ (e.g. the Euclidean distance), the optimality degree is defined as a fuzzy equality relation:

$$od(x) = \exp\left(-d\left(\pi_y, \tilde{\pi}_y\right)\right) \tag{24}$$

By integrating such pointwise optimal degree on the whole input domain, a global optimality degree can be derived as follows:

$$OD = \frac{\int_X od(x)\,dx}{\int_X dx} \tag{25}$$

provided that $\int_X dx$ exists and is finite.

The optimality degree of an O-interface can be affected by several factors:

- an inadequate choice of the output fuzzy sets;
- a poor defuzzification procedure implementing the O-interface;
- an inexact design of the processing module and/or the I-interface.

Hence, the optimality degree can be also regarded as an evaluation index of the quality of a FIS.

V. AN ILLUSTRATIVE EXAMPLE

The proposed optimality degree is experimentally applied and discussed for several FIS of Mamdani type that differ in the defuzzification method. A very simple synthetic Mamdani FIS is considered with one input and one output. Two fuzzy sets are defined both on the input and the output domain, as depicted in fig. 2. The fuzzy rules are:

IF x is MF1 THEN y is MF2
IF x is MF2 THEN y is MF1

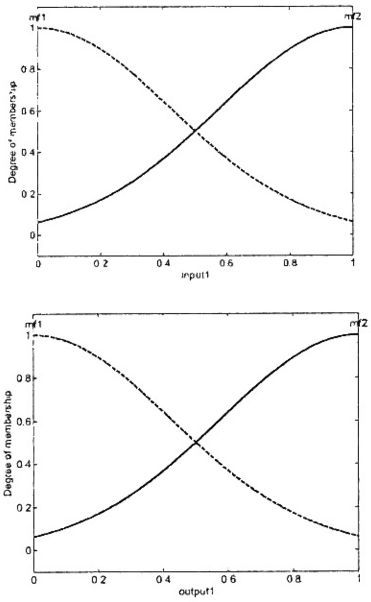

Fig. 2. Input and output fuzzy sets for the synthetic FIS

The min operator is used for conjunction and approximate inference is carried out through the *sup-min* composition. All the defuzzification methods defined in (12)-(16) are used.

In Fig. 4 the pointwise optimality degree is plotted for each defuzzification procedure, together with the global optimal degree. It can be seen that none of the considered defuzzification methods achieves maximal optimal degree (od=1) on the entire input domain. In particular, the centroid and the bisector methods show a good behaviour only in the middle of the input domain, while the remaining methods behave symmetrically.

For the considered synthetic FIS, the better defuzzification (in terms of transparency) method seems to be the bisector method, that provides the highest global optimality degree and achieves high values of the pointwise optimality degree on a wide range of the input domain.

Such simulation does not aim to rank the quality of each defuzzification method in an absolute sense, but to discuss the transparency issue of a specific inference process with varying defuzzification methods in a quantitative rather than qualitative manner. More specifically, the simulation highlights that the use of a standard defuzzification method does not guarantee a high optimality degree for all inputs. This suggests that new defuzzification procedures could be conveniently defined so as to maximize the optimality degree on the whole input domain.

VI. CONCLUSIONS

In this paper a measure of optimality for fuzzy interfaces has been proposed. The measure, called optimality degree, is especially suited for output interfaces in fuzzy inference systems. The pointwise definition of the

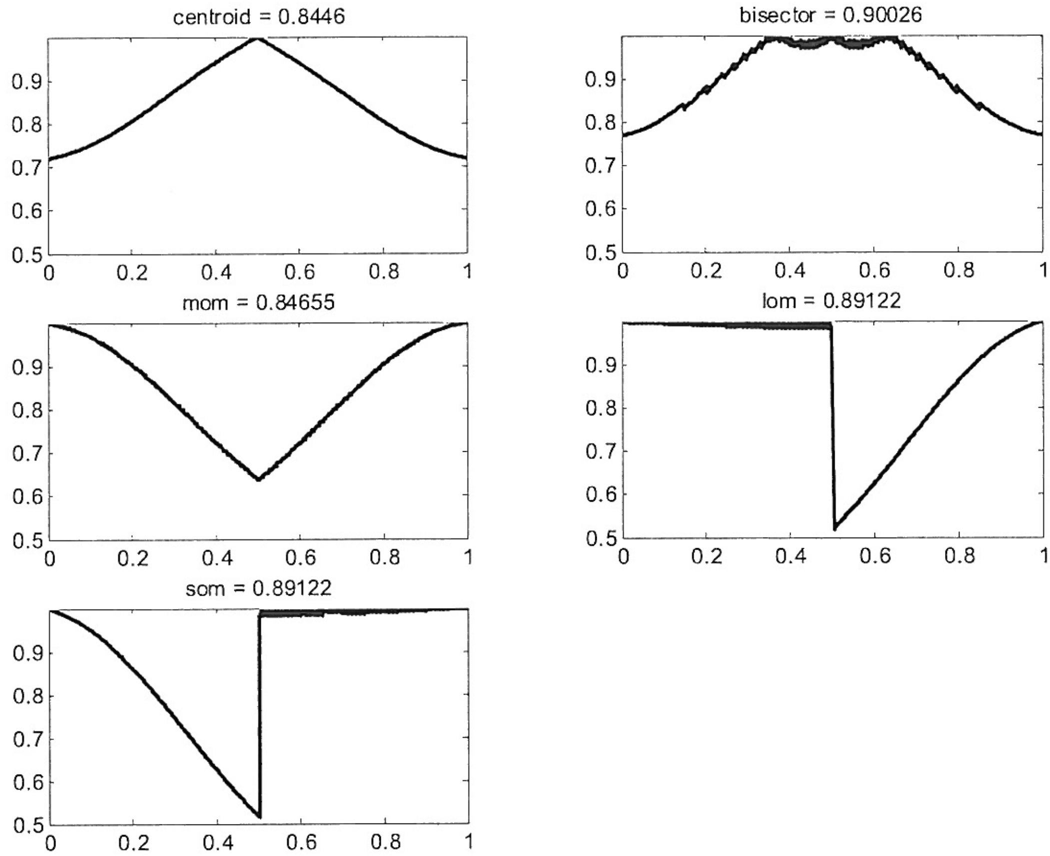

Fig. 2. Plot of the optimality degrees vs. the input values for the synthetic FIS, with different choices of defuzzification procedures.

optimality degree highlights the variation of quality of fuzzy inference systems in regions of the input domain, while its global definition provides a quantitative estimate of the quality of the system, which may be affected by different design choices. Results of an illustrative simulation have shown that different defuzzification methods can be compared in terms of optimality degrees and a high value of such measure is a necessary condition for transparent fuzzy inference systems. Therefore, the optimality degree can be considered as a mathematical tool to analyze and compare different defuzzification methods or different output interfaces in general.

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