Rule of Distance Based Uninorm Group in Approximate Reasoning

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Abstract **– In this paper an overview of approximate reasoning methods is given, based on distance based operators group. An overview of approximate reasoning based on similarity measures of fuzzy sets is given. Furthermore, based on the definitions and theorems for lattice ordered monoids and left continuous uninorms and t-norms, certain distance-based operators are focused on, with the help of which the uninorm-residuum based approximate reasoning system becomes possible in fuzzy approximate reasoning and Fuzzy Logic Control systems. It will be shown, that those types of the reasoning partially satisfy the expected conditions for approximate reasoning and inference mechanism for FLC systems.**

I. INTRODUCTION

The concept of approximate reasoning in the known framework of the linguistic information was introduced by Zadeh [15]. The system state is described by a fuzzy rule base system, and the relationship between fuzzy rule base system, system output and system input is modeled by compositional rule of inference. The fuzzy logic control (FLC) systems are based on linguistic variables [15], and fuzzy approximate reasoning, and therefore they are successful and effective [16]. A practical application of fuzzy sets was realized by means of the Mamdani inference [8] but since it has been introduced it is disputable how it fits into the theory of general approximate reasoning, and especially in the residuum type reasoning.

Knowledge representation in a rule-based system is done by means of IF...THEN rules. Furthermore, approximate reasoning systems allow fuzzy inputs, fuzzy antecedents, fuzzy consequents and fuzzy logic rules. In contrast with multi-valued logics, "fuzzy logic differs from conventional logical systems in that it aims at providing a model for approximate rather than precise reasoning", [1].

Although application-oriented fuzzy systems seek to be simple and comprehensible, it is obvious that they are heavily related to the fields of classical multi-valued logic, operation research and functional analysis. There are numerous models that are yet to gain exact mathematical description, but have already proven their applicability in practice.

Taking all this into consideration, and using the possibilities offered by uninorm operators group, the Mamdani-based approach can be generalized further. The advantage of the uninorms in approximate reasoning is that in certain part of the universe of related fuzzy numbers we have t-norms, and in another part there are t-conorms. In approximate reasoning environment it means, that for some situations the operation between rule base input and rule premise can be treated as pessimistic (calculated by t-norm,) and in another part optimistic (calculated by t-conorm).

In fact the uninorms offer new possibilities in fuzzy approximate reasoning [18]. On the one hand, because the low level of covering over of rule premise and rule input has measurable influence on rule output, and on the other hand, because of the above described pessimistic and optimistic calculations.

Engineering applications are satisfied with the minimum operator, but from a mathematical point of view it is interesting to study the behavior of other t-norms in inference mechanism. The using of distance based operators in fuzzy logic control (FLC) was described in [13] and [14].

It is also to be considered, that the Mamdani type controller is based on Generalized Modus Ponens (GMP) inference rule (but not in all aspects), and the rule output is given with a fuzzy set, which is derived from rule consequence, as a *cut* of them. This cut is the generalized *degree of firing level* of the rule, considering actual rule base input, and usually it is the supreme of the minimum of the rule premise and rule input (calculating with their membership functions, of course). The firing level depends on the covering over of the rule base input and rule premise, but first of all it depends on the *height* of those covered membership functions.

For uninorms, and especially for distance based operators, the height of the firing level is not a real measure of the covering over of the rule premise and rule input any more, we get a normal fuzzy set as uninorm of them. We need a new height-independent measure, and it opens new possibilities and also new problems in approximate reasoning. In this paper rational similarity measures of the rule premise and rule input are reviewed, and it is investigated whether the conditions are satisfied by approximate reasoning with these measures.

An other question raised previously was: whether there are uninorm operation groups which satisfy the residuum-based approximate reasoning, but at the same time are easily comprehensible and acceptable to application-oriented experts. How come the residuum-based approximate reasoning is not as wide-spread in mathematical logic as the Mamdani-type?

The axiom system reviewed in this paper declare the expectations of the approximate reasoning systems build in distance-based operators environment, and it becomes clear to what degree they satisfy or violate this system.

The theoretical basis for fuzzy operators used in this paper provided by [7]. The other basic background of this research are the distance-based operators introduced by Rudas in his work [10],[11]. The paper heavily relies on approximate reasoning and fuzzy logic theory (focusing on implications). The works from [6] were mainly used in this investigation.

Concerning the structure of the work is the next: the first section contents an overview of uninorm and distance based operators, with its important properties and role in residuum-based approximate reasoning. It is emphasized which operators from distance based operator group are corresponding to the uninorm-based residuum known so far. The next section gives an overview of the novel reasoning systems in a uninorm environment using similarity measures for fuzzy sets, and using uninorm-residuum. Furthermore, it becomes clear to what degree those type of approximate reasoning satisfy or violate the axiom system expected in fuzzy control systems.

II UNINORMS AND DISTANCE BASED OPERATORS

Both the neutral element 1 of a t-norm and the neutral element 0 of a t-conorm are boundary points of the unit interval. However, there are many important operations whose neutral element is an interior point of the underlying set. The fact that the first three axioms (commutativity, associativity, monotonicity) coincide for t-norms and for t-conorms, i.e., the only axiomatic difference lies in the location of the neutral element, has led to the introduction of a new class of binary operations closely related to t-norms and t-conorms.

A *uninorm* is a binary operation U on the unit interval, i.e., a function $U:[0,1]^2 \to [0,1]$ which is commutative, associative, monotone in the first component, and has a neutral element exists, which is $e \in]0,1[$.

The distance-based operators [11] can be expressed by means of the min and max operators as follows:

the *maximum distance minimum operator with respect* to $e \in]0,1[$ is defined as

$$max_e^{min} = \begin{cases} max(x,y) & \text{if } y > 2e - x \\ min(x,y) & \text{if } y < 2e - x \\ min(x,y) & \text{if } y = 2e\text{-}x \end{cases}$$

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the *maximum distance maximum operator with respect to* $e \in]0,1[$ is defined as

$$max_e^{max} = \begin{cases} max(x,y) & \text{if } y > 2e - x \\ min(x,y) & \text{if } y < 2e - x \\ max(x,y) & \text{if } y = 2e\text{-}x \end{cases}$$

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The distance-based operators have the following properties max_e^{min} *and* max_e^{max} are uninorms,

the dual operator of the uninorm max_e^{min} *is* max_{1-e}^{max} *, and*

the dual operator of the uninorm max_e^{max} *is* max_{1-e}^{min}.

For modified distance-based operators (the only modification on distance based operators described in [9] is the boundary condition for neutral element e): the *maximum distance minimum operator with respect* to $e \in [0,1]$ and the *minimum distance minimum operator* with respect to $e \in]0,1[$ the following can be proofed.

Based on results from [2] and [3] we conclude:

Operator $max_{0.5}^{min}$ is a conjunctive left-continuous idempotent uninorm with neutral element $e \in]0,1[$ with the super-involutive decreasing unary operator $g(x) = 2e - x = 2 \cdot 0.5 - x \Rightarrow g(x) = 1 - x$.

Operator $min_{0.5}^{max}$ is a disjunctive right-continuous idempotent uninorm with neutral element $e \in]0,1[$ with the sub-involutive decreasing unary operator $g(x) = 2e - x = 2 \cdot 0.5 - x \Rightarrow g(x) = 1 - x$.

According to Theorem 8. in [2] we introduce implicator of distance based operator $max_{0.5}^{min}$.

Consider the conjunctive left-continuous idempotent uninorm $max_{0.5}^{min}$ with the unary operator $g(x) = 1 - x$, then its residual implicator $Imp_{max_{0.5}^{min}}$ is given by

$$Imp_{max_{0.5}^{min}} = \begin{cases} max(1-x,y) & \textit{if} \quad x \le y \\ min(1-x,y) & \textit{elsewhere} \end{cases} \qquad (2.1.).$$

III APPROXIMATE REASONING

In theory of the approximate reasoning introduced by Zadeh in 1979, [16] much of the knowledge of system behaviour and system control can be stated in the form of if-then rules. In most sources it was suggested to represent an

if x is A then x is B

rule in the form of fuzzy implication (shortly $Imp(A,B)$), relation (shortly $R(A,B)$), or simply as a connection (for example as a t-norm, $T(A,B)$) between the so called rule premise: x is A and rule consequence: y is B. Let x be from universe X, y from universe Y, and let x and y be linguistic variables. Fuzzy set A in X is characterised by its membership function μ_A: $x \to [0,1]$. The most significant differences between the models of approximate reasoning lie in the definition of this connection, relation or implication.

The other important part of the reasoning is the inference mechanism. One of the widely used methods is the Generalised Modus Ponens (GMP), in which the main point is, that the inference y is B' is obtained when the propositions are:

- the i^{th} rule from the rule system of n rules: if x is A_i then y is B_i
- and the system input x is A'.

Let RB a fuzzy rule system of n rules, with rule premises x is A_i and rule consequences: y is B_i ($i = 1,2,...n$). Let x is A' be the system input, where A' is characterized by its membership function $\mu_{A'}$: $x \to [0,1]$.

Applying the generalized compositional rule of inference to given components, the *i-th* rule output, with respect to the given RB and given system input A', is y is B_i' given by the expression ([6],[16])

$$B_i'(y) = \sup_{x \in X} T(A'(x), Imp(A_i(x), B_i(y))).$$

The Mamdani inference rule says, that the membership function of the consequence in the i-th rule B_i' is defined by

$$Bi'(y) = \sup_{x \in X}(T(A'(x), T(Ai(x), Bi(y)))$$

where T is a t-norm.

Using the t-norm properties, from the above expression $Bi'(y) = T(\sup_{x \in X}(T(A'(x), Ai(x))), Bi(y))$. Generally speaking, the consequence (rule output) is given with a fuzzy set $B'(y)$, which is derived from rule consequence $B(y)$, as a cut of the $B(y)$. This cut, $\sup_{x \in X}(T(A'(x), A(x)))$, is the generalized degree of firing level of the rule, considering actual rule base input $A'(x)$, and usually depends on the covering over $A(x)$ and $A'(x)$. But first of all it depends on the *sup* of the membership function of $T(A'(x), A(x))$.

The fuzzy rule base output is constructed as a crisp value calculated with a defuzzyfication model, from rule base output. Rule base output is an aggregation of all rule consequences $B_i'(y)$ from the rule base. As aggregation operator, t-conorm is usually used.

$$B'_{out}(y) = S(B_n', S(B_{n-1}', S(...., S(B_2', B_1'))))..$$

This conclusion is defined as the union of individual conclusions, i.e., it is a pessimistic concluding [3]

3.1. The axioms of inference mechanism

On a general level, Imp is the relationship between rule base premise and rule base consequence, satisfying the following conditions:

(out1) If the input coincides with one of the premises, then the resulting output coincides with the corresponding consequence, i.e.,
$(\exists i \in \{1,2,...n\})(A' = A_i)$ then $B_i' = B_i$.

(out2) For each normal input A' the output is not contained in all consequences, i.e.,
$(\exists i \in \{1,2,...n\})(B_i' < B_i)$.

(out3) The rule output belongs to the convex hull of B_i, ($i \in I$), where $I = \{i | 1 \le i \le n, Supp(A') \cap Supp(A_i) \ne 0\}$.

In [9] we can find an axiom system on the same principle.

IV APPROXIMATE REASONING WITH DISTANCE-BASED UNINORMS

4.1. Approximate reasoning with distance-based Uninorms based on similarity measures of fuzzy sets

In system control one would intuitively expect to make the powerful coincidence between fuzzy sets stronger, and the weak coincidence even weaker. The distance-based operators group satisfy these properties, but the covering over $A(x)$ and $A'(x)$ are not really reflected by the *sup* of the membership function of the $min_e^{max}(A_i(x), A'(x))$.

Hence, and because of the non-continuous property of distance-based operators, it was unreasonable to use the classical degree of firing, to give expression to the coincidence of the rule premise (fuzzy set A), and system input (fuzzy set A'), therefore a Degree of Coincidence (*Doc*) for those fuzzy sets has been initiated.

Based on definition of similarity measures from [4] and [5] from the class of rational cardinality-based similarity measures, with some modifications, the next are acceptable for expressing of degree of coincidence between the rule premise (fuzzy set A), and system input (fuzzy set A').

The proportion of area under membership function of the distance-based intersection of rule premise and system input, and the area under membership function of their union (using *max* as the fuzzy union)

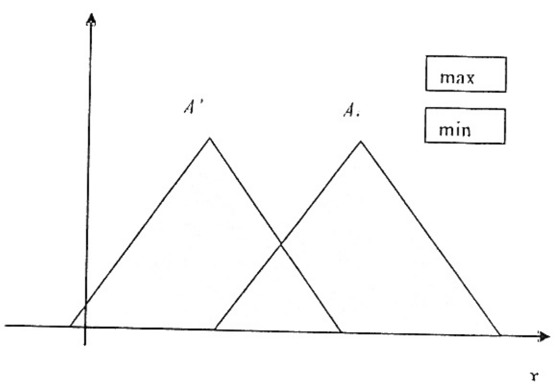

Figure 1.

The areas figured in the Doc_i

$$Doc_i = \frac{\int_X max_e^{min}(A_i(x), A'(x))dx}{\int_X max(A_i(x), A'(x))dx}$$

is the first possibility (Figure 1.).

It supports the a pessimistic concluding in fuzzy approximate reasoning, because the conclusion is defined as the union of individual conclusions from individual rules.

$$B_i'(y) = min(Doc_i, B_i(y)),$$

$$B'_{out}(y) = S(B_n', S(B_{n-1}', S(\ldots, S(B_2', B_1')))).$$

The proportion of area under membership function of the distance-based intersection of rule premise and system input, and the area under membership function of the union of all rule premises (using *max* as the fuzzy union)

$$Doc_{iopt} = \frac{\int_X max_e^{min}(A_i(x), A'(x))dx}{\int_X \max_{forallrules}(A_i(x))dx}$$

is the second possibility (Figure 2.).

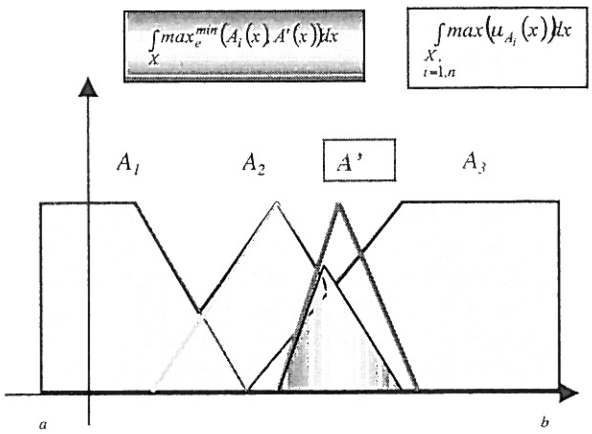

Figure 2.
The areas figured in Doc_{iopt}

It supports partially the optimistic logical concluding model in fuzzy approximate reasoning, because the conclusion is defined using the union of all rule premises for every individual conclusion.

$$B_i'(y) = min(Doc_{iopt}, B_i(y))$$

$$B'_{out}(y) = S(B_n', S(B_{n-1}', S(\ldots, S(B_2', B_1')))). .$$

The question arises to what extent and under what conditions the approximate reasoning based on similarity measures satisfies the axioms (out1)-(out3).

The sufficient condition is that the rule base input A' is a fuzzy set, and not a crisp value. Furthermore, let the rule premises cover the input space over, i.e.

$$\bigcup_{forallrules} Supp(A_i) = X.$$

For rule base system described in section 3. with $\bigcup_{forallrules} Supp(A_i) = X$, and for input fuzzy set $A'(x)$, the infrerence mechanism described in above satisfies the axiom system (out1)-(out3) if $A'(x) \le A_i(x)$ for all rule premisses $A_i(x)$.

$A'(x) \le A_i(x)$ implies $Doc^i_{similarity} \le 1$ and $B_i'(y) = min(Doc^i_{similarity}, B_i(y)) \le 1$.

(out1) If $A'(x) = A_i(x)$ than $Doc^i_{similarity} = 1$ and $B_i'(y) = min(1, B_i(y)) = B_i(y)$.

(out2) The proposition $A'(x) \le A_i(x)$ insures that this axiom is satisfied.

(out3) Because the rule outputs are cuts of rule consequences, rule output is obviously subset of the convex hull of B_i, ($i \in I$).

Interpolation methods can be achieved also, if it is supposed by new methods that the rule input does not coincide with, but covers over some rule premisses, i.e., $A'(x) \ge A_i(x)$, because in this case it is not convenient to use similarity measures (because in this case $Doc^i_{similarity} \ge 1$)

4.2. Residuum-based approximate reasoning with distance based operator

Although the minimum plays an exceptional role in fuzzy control theory, there are situations requiring new models. In system control one would intuitively expect: to make the powerful coincidence between fuzzy sets stronger, and the weak coincidence even weaker. The distance-based operators group satisfy these properties. Let we consider a mathematical approach: residuum-based approximate reasoning and inference mechanism. Hence, and because of the results from sections of this paper we can n consider the general rule consequence for i-th rule from a rule system as

$$B_i'(y) = \sup_{x \in X}\left(max_{0.5}^{min}\left(A'(x), Imp_{max_{0.5}^{min}}(A_i(x), B_i(y))\right)\right)$$

or, using formula (2.*1*.)

$$B_i'(y) = \sup_{x \in X} \begin{cases} max_{0.5}^{min}(A'(x), max(1 - A_i(x), B_i(y))) & if\ A_i(x) \le B_i(y) \\ max_{0.5}^{min}(A'(x), min(1 - A_i(x), B_i(y))) & elsewhere \end{cases}$$

The rule base output is constructed as a crisp value calculated with a defuzzification model, from rule base output. Rule base output is an aggregation of all rule consequences $B_i'(y)$, from the rule base. As aggregation operator, in this case, dual operator $max_{0.5}^{max}$ of $max_{0.5}^{min}$ can be used.

$$B'_{out}(y) = max_{0.5}^{max}(B_n'(y), max_{0.5}^{max}(B_{n-1}'(y), \ldots, max_{0.5}^{max}(B_2'(y), B_1'(y)))).$$

.

Taken into account Proposition 13. from [2], it can be conclude, that conjunctive left-continuous idempotent uninorm $max_{0.5}^{min}$ and its implicator $Imp_{max_{0.5}^{min}}$ satisfy the inequality

$$B'_i(y) = max_{0.5}^{min}\left(A'(x), Imp_{max_{0.5}^{min}}(A_i(x), B_i(y))\right) \leq B_i(y)$$

for i-th rule in rule base system, if $A'(x) = A_i(x)$ for all $x \in X$. It means, that this type of reasoning partially satisfies the conditions for approximate reasoning (the axiom (out1)), hence $B'_i(y) = B_i(y)$ or $B'_i(y) < B_i(y)$ if $A'(x) = A_i(x)$ for all $x \in X$.

Axiom (out2)). In most of cases uninorm-residuum based approximate reasoning violates this axioms of inference mechanism, because for normal input A' the output is contained in all consequences, if we have not "fired" rule.

Axiom (out3): If $A \neq A'$, the rule output belongs not to the convex hull of B_i, $(i=1,n)$.

In [12] it was proved in simulations, that in this case if all the rules, where the rule does not have real influence on output, have been real time eliminated, the results are acceptable.

V CONCLUSION

Building the distance based uninorms in the fuzzy approximate reasoning applications new offers are raised. On the one hand a modified Mamdani's model was investigated with novel degree of coincidence for fuzzy rule premises and fuzzy input, on the other hand uninorm residuum-based approximate reasoning is defined and investigated based on distance-based uninorms. The modified Mamdani's approach does not rely on the compositional rule inference any more, but still satisfies the basic conditions supposed for the approximate reasoning for a fuzzy rule base system. Those types of fuzzy approximate reasoning partially satisfy the conditions for the fuzzy inference mechanism and for FLC systems.

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