Adaptive Vibration Damping Based on Causal Time-invariant Green-Functions and Fractional Order Derivatives

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Abstract – **In this paper a simple nonlinear, adaptive control using causal time-invariant Green-functions and fractional order derivatives is applied for damping the vibration of a car forced during passing along a bumpy road. Its key idea is the replacement of the integer order derivatives in a Green-functions-based nonlinear controller with a time-shift invariant, causal approximation of the Riemann-Liouville fractional derivative that also behaves like a Green-function. Since its physical essence is rather frequency filtering than providing integer order derivatives in limit cases, the approximation applied numerically is quite convenient. In this way simple kinematic design of the desired damping becomes possible. The adaptive part of the controller guarantees the realization of this kinematic design without making it necessary for the designer to have accurate and complete dynamic model of the system to be controlled or to design a sophisticated linear "*CRONE*" controller that has to take the responsibility for the unknown dynamics of the system. The applicability of the approach is illustrated via simulations for a paradigm that is a rough model of a car. It was found that both adaptivity and the use of fractional order derivatives in the control are essential parts of the success of the method.**

I. INTRODUCTION

Externally excited vibration normally is undesired phenomenon that occurs in various physical systems therefore its efficient damping is of great practical significance. From the point of control technology the task has the „delicate" nature that the controller cannot be provided with the "exact" system-model, and/or with complete information on the system's actual physical state. A novel branch of soft computing simultaneously using the Modified Renormalization Transformation and simple ancillary methods [1] has recently been developed that is flexible enough to incorporate various algebraic blocks like e.g. special Symplectic Transformations [2]. It was shown that in the case of a wide class of physical systems with the aid of this method quite robust adaptive controllers could be developed for the control of very inaccurately and partially modeled physical systems that can have even unmodeled internal degrees of freedom [3]. As an input the method requires the desired trajectory of the generalized coordinates of the system that has to directly be controlled.

This approach should be useful if the desired trajectory of the generalized coordinates could be prescribed with respect to an inertial system of reference. However, in the most cases the system considered is a part of a moving object that does not serve as a basis of an inertial frame, as e.g. a car proceeding along a bumpy road crossing hills and valleys. In this case some slow motion along certain average distance between the chassis and the wheel can be prescribed for the controller because it is locally measurable quantity. A feasible kinematic formulation of this goal is the application of some „forgetting integral" that does not allow abrupt changes with respect to the former values of this distance but allows its slow variation. The desired behavior of this distance can practically be prescribed to some extent by the terms used in the traditional linear controllers as frequency filters etc. The most plausible means would be the application of a simple PD type controller to keep a finite error at bay. However, the integrating term of this controller does not „forget" the past, and for an even small but constant error it generates infinite signal for feedback. As generalizations of the concept of the derivative the concept of fractional order derivatives found more and more physical applications to describe the „longer term memory" of various physical systems like in the case of visco-elastic phenomena [4] etc. The problem of designing fractional order control systems within the frames of linear control obtained considerable attention recently, e.g. [5]. The French expression invented by Oustaloup *„CRONE: Commande Robuste d'Ordre Non Entier"* [6] almost became a „trademark" hallmarking a well-elaborated design methodology that obtained application in vibration control, too [7]. Understanding and application of this method requires deep engineering knowledge in the realm of linear systems, frequency spectrum analysis, the use of Laplace transforms and complex integrals, various typical diagrams as e.g. the Nichols plot, etc. The aim of the present paper is to propose an alternative approach not strictly restricted to the traditional way of thinking in the case of linear systems. Tackling the problem from a more general nonlinear basis requires less amount of profound and specific engineering knowledge, the application of which can be evaded by the controller's adaptive nature or learning abilities. For this purpose the „long term memory" or slowly forgetting nature of the fractional order systems is considered in a more general view.

II. FRACTIONAL ORDER DERIVATIVES AND GREEN-FUNCTIONS

In the case of a normal PID-type controller the desired trajectory reproduction can be prescribed in a purely kinematics based manner. For the second time-derivative of the actual coordinate errors the desired relation can be formulated as:

$$\ddot{h}^d = -Ph - D\dot{h} - I\int_{-\infty}^{t} h(t')dt' \qquad (1)$$

The use of the integrating term in (1) is expedient whenever very accurate tracing of the nominal trajectory is needed, because even for very small permanent error sooner or later it yields quite considerable feedback. Since this typically is not the goal of a vibration-damping task in which some uncertainty may be desired in order to avoid the rigid transmission of high-frequency vibrations we propose an alternative solution in the form

$$\ddot{h}^d(t) = -\mu\int_0^t h(\tau)G(\tau - t)d\tau + S(t) \qquad (2)$$
$$\text{for } t \geq 0, h(0) = h_0, \dot{h}(0) = \dot{h}_0$$

in which $G(\xi)$ has the following properties:

$$G(\xi) = 0 \; if \; \xi \geq 0, \; G(\xi) \geq 0 \; if \; \xi \leq 0, \; \int_{-\infty}^{0} G(\xi)d\xi = 1 \;, \qquad (3)$$

and let $G(\xi)$ be *continuously differentiable* with the exception of certain discrete points of finite number. The main idea behind (2) and (3) is to replace a non-forgetting integral with a forgetting one. Regarding the replacement of the other terms in (1) it is expedient to introduce a "supplementary" term $S(t)$ that is expected to be necessary for maintaining the decreasing nature of the quadratic error integrated in the "moving window" defined as

$$V(t) := \int_0^t h^2(\tau)G(\tau - t)d\tau \qquad (4)$$

It is not difficult to show that due to the properties of $G(\xi)$ V satisfies the following differential equations:

$$\dot{V}(t) = -h^2(0)G(-t) + \int_0^t 2h(\tau)\dot{h}(\tau)G(\tau - t)d\tau \qquad (5)$$

$$\ddot{V}(t) = -\mu F^2(t) + h_0^2\dot{G}(-t) + 2\int_0^t \left[\dot{h}^2(\tau) + S(\tau)h(\tau)\right]G(\tau - t)d\tau \qquad (6)$$

in which

$$F(t) := \int_0^t h(\tau)G(\tau - t)d\tau \;. \qquad (7)$$

Via calculating the integral below we obtain that

$$0 \leq \int_0^t \left[h(\tau) - F(t)\right]^2 G(\tau - t)d\tau \Rightarrow V(t) \geq F(t)^2 \geq 0 \qquad (8)$$

that indicates that if $S(t)$ is chosen as

$$S(\tau)h(\tau) := -\dot{h}(\tau)^2 - \rho\dot{h}(\tau)h(\tau) \qquad (9)$$

with a constant $\rho>0$ then

$$\ddot{V}(t) = -\rho\dot{V}(t) - \mu F(t)^2 + h_0^2\left[\dot{G}(-t) + \rho G(-t)\right] \qquad (10)$$

that has a simple and lucid interpretation. Since G can be so constructed that both its value and its derivative converges to zero as $t\rightarrow\infty$ the last term in it converges to zero and can be omitted following some transient phase. For the terms remained, in the phase space determined by μ and dV/dt the location of the points in which the 2[nd] derivative of V can take the value 0 must be in the V>0 half plane approximately along the straight line of the equation $\dot{V} = -\mu V/\rho$. The 2[nd] derivative of V also changes sign along this line, that therefore attracts the phase trajectories of V, and asymptotically leads them to the V=0, dV/dt=0 point, so the moving average of the square of the error $h^2(t)$ approaches zero. To avoid division by zero in calculating S, instead of (9) in the practice the approximation

$$S(\tau) := -sign(h(\tau))\frac{\dot{h}(\tau)^2}{\varepsilon + |h(\tau)|} - \rho\dot{h}(\tau) \qquad (11)$$

can be used with a small positive ε value.

In the present paper the following Green-function was chosen:

$$G(\xi) := \begin{cases} \beta e^{\beta D} e^{\beta\xi} \; if \; \xi \leq -D \\ 0 \; otherwise \end{cases} \qquad (12)$$

with D=5 ms and $\beta\cong$0.99 ms^{-1}.

If the dynamics of the system to be controlled is not very well known the feedback force calculated on the basis of (2) cannot accurately be estimated, therefore normally very big gains --i.e. P, and D in (1) or μ in (2) and ρ in (11)-- has to be applied, normally resulting in a noisy control, that is instead damping the vibration the control can introduce even more noise into the system. This situation can be improved by using fractional order derivative in S and in the 2[nd] time-derivative of the nominal trajectory as

$$\ddot{x}^d(t) = \frac{d^\beta}{dt^\beta}\dot{x}^N(t) - \mu\int_0^t h(\tau)G(\tau - t)d\tau - sign(h(\tau))\frac{\left[\frac{d^\beta}{dt^\beta}h(\tau)\right]^2}{\varepsilon + |h(\tau)|} - \rho\frac{d^\beta}{dt^\beta}h(\tau) \qquad (13)$$

in which the symbol d^β/dt^β denotes some fractional order derivative. In the literature, starting from the 19[th] century various definitions has been elaborated for this concept [8, 9], and comprehensive works were also issued recently, e.g. [10]. For our present purposes the form invented by Caputo –it can be derived from the Riemann-Liouville fractional derivative (e.g. in [11] and [12]) under simple smoothness conditions– seems to be the most expedient as

$$\frac{d^\beta}{dt^\beta}u(t) := \frac{1}{\Gamma(1-\beta)}\int_0^t \left[\frac{du(\tau)}{d\tau}\right](t-\tau)^{-\beta}d\tau \,, \beta\in(0,1) \qquad (14)$$

For t>0 (14) physically has the following simple meaning: the full 1[st] order derivative in the integrand removes the constant component from the signal, and this derivative is "causally reintegrated" by the use of a Green-function that has a slowly forgetting nature (the contribution of the far past becomes more and more negligible in it), while its singularity in τ=0 enhances the relative weight of the contribution of the $\tau\cong<t$ instants. Furthermore, the relatively slowly decreasing "tail" of this function also acts as a frequency filter that rejects the high-frequency components of the traditional 1[st] derivative. Due to the singularity of the Green-function in (14) a common finite-element numerical integration cannot accurately be done. Instead of that, we can suppose that at least $u'(\tau)$ is a

relatively slowly varying function of time, therefore it can be considered as constant during the integration over a small time-interval, while the variation of the Green function can be taken into account accurately. Furthermore, to introduce symmetry against the translation of the signal in time we can omit the very long tail of the Green-function and we can go back in time only to some time t-T instead of 0. The proposed approximation of (14) in this paper was taken as

$$\frac{d^{\beta}}{dt^{\beta}}u(t) \cong \frac{u'(t)\delta^{-\beta+1}}{\Gamma(2-\beta)} + \sum_{0<s \text{ while } s\delta<T} \frac{\delta^{-\beta+1}\left[(s+1)^{-\beta+1}-s^{-\beta+1}\right]}{\Gamma(2-\beta)}u'(t-s\delta) \quad (15)$$

(It is not our purpose to obtain exact integer order derivatives from this approximation when $\beta\rightarrow 1$ or $\beta\rightarrow 0$.)

In the numerical simulations in this paper $\delta=1$ *ms*, $T=50$ *ms* values were chosen, and the appropriate value for β was determined experimentally via simulations. The components of the Green function were stored in an array variable, the u' values were stored in a shift-register.

III. THE MODEL OF THE CAR AND ITS CONTROL

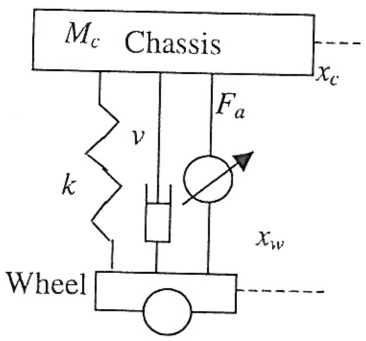

Fig. 1. The rough model of the suspension system

The model of the system considered is described in Fig. 1. The mass of the wheel is supposed to be negligible with respect to that of the chassis of mass $M_c=100$ *kg* the model value of which was supposed to be 150 *kg*. (This value cannot exactly be known *a priori* since one or more than one travelers of even 100 *kg* weight each can sit in the car.) The passive suspension system consisted of a spring of stiffness $k=2\times10^4$ *N/m* and viscosity of $v=1\times10^4$ *N/(m/s)*. The force of the active suspension F^a was supposed to be generated according to the control law. The coordinates x_c and x_w in *m* units describe the height of the chassis and the wheel, respectively, with respect to an inertial frame, i.e. with respect to the sea level, so they are not available as direct data for the controller. The nominal height of the chassis while rigidly following the wheel was prescribed to be $x_{cnom}=x_w+L_0$ with $L_0=0.5$ *m*. In the case of loose trajectory tracking little humps and dips need not be traced by the chassis, but climbing a higher hill or deeper valley must be traced. However, the error of this trajectory tracking is available via local measurements within the car as

$$h=x_c-x_{cnom}=(x_c-x_w)-L_0 \quad (16)$$

The 1[st] order time-derivative of the error can be numerically estimated by finite element methods. Because x_w and x_c are measured with respect to an inertial frame their 2[nd] traditional time-derivatives also are measurable even by the use of micro-sensors developed on a chip. These error values can be used in (2), with the numerical approximation of the integrals given in (15).

The so obtained equation of motion represents a tracing requirement expressed by the use of purely kinematic terms. The main expectation behind it is the supposition that for small feedback gains loose tracking can be achieved the accuracy of which is increased by the "filtered" integrals at low frequency (that is for hill climbing), while for the higher frequency components occurring when small dips are passed it remains loose. By the use of the approximate dynamic model of the system the appropriate active force can be estimated. Due to the approximate nature of the dynamic model exertion of this force will not result in the desired acceleration of the chassis. For the realization of (2) adaptive control is needed. Its main principles are given in the following part.

IV. PRINCIPLES OF THE ADAPTIVE CONTROL

For the adaptive control there is given an imperfect system model as a starting point. On the basis of that some excitation is calculated to obtain a desired system response $\mathbf{i}^d$ as $\mathbf{e}=\varphi(\mathbf{i}^d)$. This model is step by step refined in the following manner. If we apply the above approximate excitation, according to the actual system's inverse dynamics described by the unknown function a realized response $\mathbf{i}^r=\psi(\varphi(\mathbf{i}^d))=f(\mathbf{i}^d)$ is obtained instead of the desired one, $\mathbf{i}^d$. Normally one can obtain information via observation only on the function $f()$ considerably varying in time, and no any possibility exists to directly "manipulate" the nature of this function: only $\mathbf{i}^d$ as the input of $f()$ can be "deformed" to $\mathbf{i}^{d*}$ to achieve and maintain the $\mathbf{i}^d=f(\mathbf{i}^{d*})$ state. [Only the *model function* $\varphi()$ can directly be manipulated.] On the basis of the modification of the method of renormalization widely applied in Physics the following "scaling iteration" was suggested for finding the proper deformation:

$$\mathbf{i}_0;\mathbf{S}_1\mathbf{f}(\mathbf{i}_0)=\mathbf{i}_0;\mathbf{i}_1=\mathbf{S}_1\mathbf{i}_0;\ldots;\mathbf{S}_n\mathbf{f}(\mathbf{i}_{n-1})=\mathbf{i}_0; \\ \mathbf{i}_{n+1}=\mathbf{S}_{n+1}\mathbf{i}_n;\mathbf{S}_n\xrightarrow[n\to\infty]{}\mathbf{I} \quad (17)$$

in which the $\mathbf{S}_n$ matrices denote some linear transformations that map the observed response to the desired one, and the construction of each matrix corresponds to a step in the adaptive control. It is evident that if this series converges to the identity operator just the proper deformation is approached, and the controller „learns" the behavior of the observed system by step-by-step amendment and maintenance of the initial model. Since (17) does not unambiguously determine the possible applicable quadratic matrices, we have additional freedom in choosing appropriate ones. In this paper the Special Symplectic Transformations described in details e.g. in [2] were chosen as algebraic means. The main idea is the completion of the dimensions of the $\mathbf{f}$ and $\mathbf{i}_0$ vectors with physically not interpreted "dummy" dimensions to avoid $\mathbf{0}\rightarrow\mathbf{0}$ or $\mathbf{0}\rightarrow$**finite** or **finite**$\rightarrow\mathbf{0}$ mappings. Following that by properly placing further blocks around these arrays easily invertible special matrices belonging to some special

Lie Group, in this case to the Symplectic Group can be obtained, so unique proposition can be given for the S_n matrices via simple computation. These matrices automatically approach the unit matrix as $f \to i_0$, and since they are the elements of a group, their matrix multiplication also belongs to this group, that is we remain within the mathematical frameworks of a simple Lie group the elements of which arbitrarily can approach the unit matrix. In the lack of enough free space, regarding the details we refer to [2].

V. SIMULATION RESULTS

In the simulation examples a hill of parabolic shape and height of 0.5 m was climbed by the use of a bumpy road containing bumps of the size from 0.1 to 2 *m* diameters and maximal height of about 1 *cm*. The road was modeled by a Fourier series-like construction as

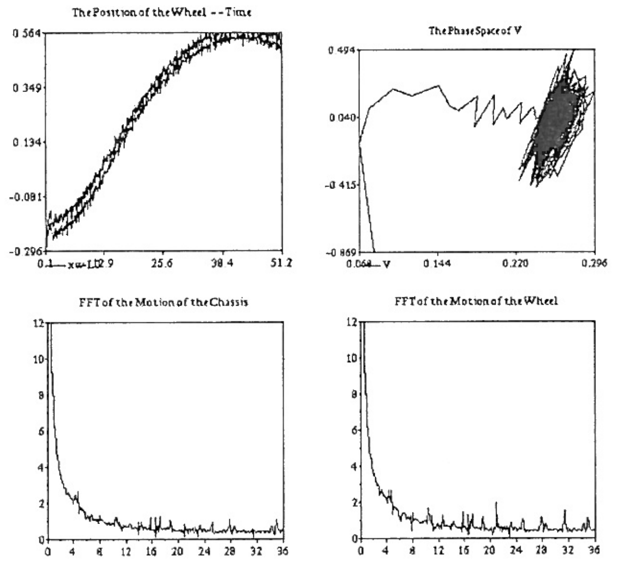

Fig. 2. Simulation results for the passive suspension only: the wheel's height (black –upper- line) and the height of the chassis minus L_0 (blue – lower- line) in *m* units vs. time in *s* units, the phase-space of the quadratic error-integral (dV/dt vs. V) (1st row), and the Fast Fourier Transform's absolute value vs. frequency in *Hz* units for the height of the wheel and of the chassis (2nd row)

$$x_w(s) = x_{w0}(s) + \sum_{d=1}^{20} A_d \left(1 - \cos \frac{2\pi s}{0.1 \times d}\right) \qquad (18)$$

This model of bumpy road was inspired by the state of the old Thököly Street in Budapest. It is covered by basalt cubes of the approximate size of 10 cm. Its surface was many times corrupted in smaller-larger details due to amendment of the pipelines under it. Following such reconstructions the surface of the street could not remain 'even', this explains the dips/bumps of about 1 cm depth/height and length between 0.1 to 2 m. In (18) *s* measures the horizontal displacement of the car, which is obtained by constant 1 m/s^2 acceleration till achieving the 14 *m/s* velocity. Following that, this velocity component remains constant. According to (18) this means a random-like excitation for the suspension system. In Fig. 2 typical results are given for the passive suspension when no active force is applied. It is evident that the passive suspension in this case results in a poor vibration damping quality. The vibration of the wheel is transmitted to the chassis.

In order to improve noise-suppression a non-adaptive active, and for further improvement adaptive active version of the above control for strongly decreased proportional feedback coefficient μ=5 s^{-2} and the same derivative coefficients ρ=100 s^{-1} but with non-integer order derivatives with β=0.08 order of derivation can be applied (Fig. 3). The noise suppression was considerably improved due to the high-frequency active forces but the low frequency tracking is poor due to the very approximate dynamic model used in the case of the non-adaptive control. Switching on adaptivity improves tracking for low frequency while the vibrations of high-frequency components are suppressed.

In Fig. 4 further details of the adaptive active control are given. The quadratic error-integral V now really well approaches zero. While the high-frequency terms are well suppressed due to the filtered goal of the control, adaptivity makes it possible to really implement the prescribed loose tracking for the low frequency components, therefore the chassis now climbs the hill without taking the higher frequency vibrations caused by the bumpy road.

VI. CONCLUSIONS

In this paper the combination of the concept of fractional order derivatives and a novel branch of soft computing was applied for vibration suppression purposes in the case of a simple car model.

The main contribution of the non-integer order derivatives lies in providing the controller with appropriate causal goal functions of significantly filtered high-frequency components.

The adaptive law gives help in implementing the result of this essentially purely kinematic design without requiring *a priori* accurate information on the dynamics of

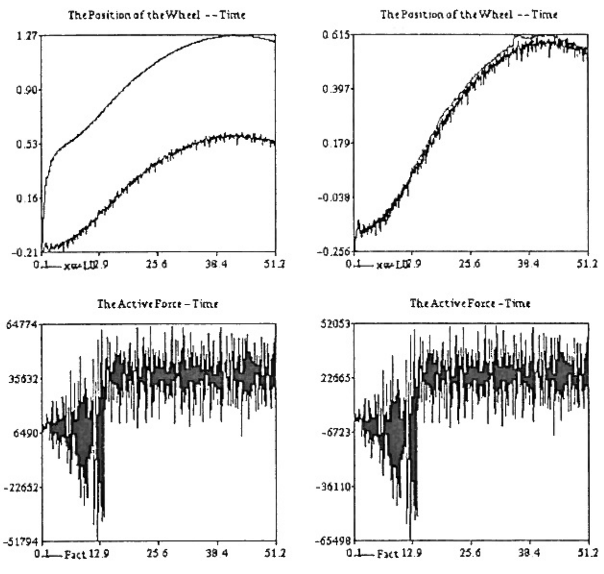

Fig. 3.Simulation results for the active suspension for soft tracing using b=0.08 fractional order derivatives: the wheel's height (black -lower- line) and the height of the chassis minus L0 (blue -upper- line) in m units vs. time in s units (1st row, left picture: non-adaptive, right picture: adaptive), and the active force in N vs. time in s units (2nd row, left picture: non-adaptive, right picture: adaptive)

the system.

In contrast to the rather traditional "*CRONE*" design methods in this case little profound engineering knowledge is needed. Adaptivity takes away the burden of dealing with dynamic effects from the linear fractional order controller. Furthermore, in this case no particular suppositions are needed for the nature of the vibration that assumptions used to be typical in the traditional control literature, e.g. that vibration can be treated with low order Taylor series expansion around some equilibrium position, or that the suspension system and the external excitation have characteristic eigenfrequencies or peaks in their Fourier spectra for the absorption of which the eigenfrequency of the damping medium has to be properly tuned. It needs the possibility only for fast feedback signals.

For further research the active adaptive vibration control of physical systems having unmodeled internal degrees of freedom can be considered for which the adaptive approach was also found to be applicable.

VII. ACKNOWLEDGMENT

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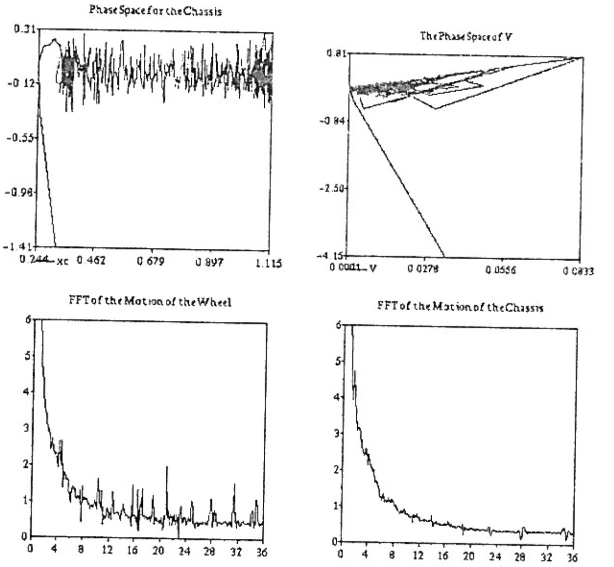

Fig. 4. Simulation results for the active adaptive suspension for soft tracing using β=0.08 fractional order derivatives: the phase space of the motion of the chassis in *m/s* and *m* units, the phase-space of the quadratic error-integral (d*V*/*dt* vs. *V*) (1st row), and the Fast Fourier Transform's absolute value vs. frequency in *Hz* units for the wheel and for the height of the chassis (2nd row)

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