A Cascade Fuzzy Controller Design Based on Fuzzy Lyapunov Stability

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1. Introduction

Although in some way, a rigorous mathematical framework of a control systems stability theory is in opposition with vagueness of fuzzy controller properties, stability remains the key issue in the fuzzy controller design. The main criticism to fuzzy control is related to the lack of precise stability analysis. That is why huge efforts are put in investigation of various techniques that have a potential to solve the stability issue in a fuzzy controlled system [1]. The problem is that an operator, whose experience is a base for a fuzzy control algorithm, can guide a system into a desired state according to some criterion without knowing why taken actions cause a stable behavior of the system. In the same time the operator is aware that there exists a set of maneuvers, which could be a source of instability. From the operator's point of view, the stability region is not strictly defined since actions that lead to the unstable controlled system are described by linguistic values. Being in the position of a fuzzy controller designer, we should take these descriptive justifications of forbidden (unstable) actions into account in the final structure of the controller

Some of techniques have roots in the stability analysis of nonlinear control systems described with their nonlinear mathematical models. Some of methods being developed are applicable only to special problem cases or strictly determined structures of fuzzy controllers [2]. For example, in [3] it is shown that for a class of fuzzy controllers, which can be described as a multi-level nonlinear relay element, a Nyquist stability criterion can be used for determination of a stability region for a fuzzy controlled system.

Stability can be assessed through analysis of a so-called sliding-mode operation of a fuzzy controller. So, in [4] a fuzzy sliding mode controller is proposed and a proof of stability of a so controlled system is made.

In [5] a phase plane has been partitioned into nine regions and a linear function in the consequent part of fuzzy rules depends on the region a phase trajectory is in.

In [6] a fuzzy sensitivity concept has been used to solve a problem of stability assessment enabling a designer to find out a range of parameter variations for which a given fuzzy controller will maintain a stable system performance.

Since a fuzzy logic can also be applied for determination of fuzzy system models, in [7, 8, 9] methods are given that prove a stability of fuzzy process models. A Takagi-Sugeno controller is used and it is shown that by using a Lyapunov function, a fuzzy model will be stable only if linear models in the consequent parts of the rules are stable.

2. Fuzzy Lyapunov stability

In the Introduction we have mentioned that operator can define by linguistic values not only stabilizing (allowed) actions, but also destabilizing (forbidden) control actions. The question is: if we replace a crisp mathematical definition of Lyapunov stability conditions with linguistic terms, can we still treat these conditions as a valid test for stability? The answer to this query was proposed in [10]. Instead of using numbers in calculation of the derivative of a Lyapunov function, *computing with words (CW)* is proposed [11]. It has been shown that only partial knowledge of the system was enough for design of a simple fuzzy controller that stabilizes a process; an inverted pendulum was used as an example. The other approach, presented in [12], goes further and defines *gradual stability* that embeds fuzzy reasoning in forms of *tolerance space* and *feasible state space*.

Let us recall the Lyapunov stability criterion for the second order system

$$V = \frac{1}{2}\left(x_1^2 + x_2^2\right) \Rightarrow \dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 . \qquad (1)$$

As already mentioned, an inverted pendulum was used as an example in [10]. If a *pendulum angle* and a *change in pendulum angle* are chosen as the process variables x_1 and x_2, then from the inverted pendulum model we know that $\dot{x}_2$ is proportional to the controller output u_{FC}, which stands for a force applied to the cart. Authors in [10] examined signs of x_1 and x_2 and proposed a fuzzy controller with only four rules (Table 1) that fulfills inequality (1).

Linguistic values for state variables are *negative* and *positive* while controller output is partitioned in *negative big*, *zero* and *positive big*. Since the proposed fuzzy controller does not take into account magnitudes of linguistic variables, it cannot provide fine changes in the controller output. Furthermore, by using only the sign, number of rules is less than it could be if all available information about the process were used. As domains of state variables and controller output are known in advance there is no reason why we should not partition these domains in more linguistic values, thus providing more rules and a finer controller output. An extension of a CW design in this direction was proposed in [13]. Instead of using only signs of state variables, authors integrated their magnitudes in a form of *fuzzy numbers*.

Table 1 Control rules according to [10].

x_1	x_2	$u_{FC}\ (\sim \dot{x}_2)$	$\dot{V}$
positive	*positive*	*negative big*	*negative*
positive	*negative*	*zero*	*negative*
negative	*positive*	*zero*	*negative*
negative	*negative*	*positive big*	*negative*

3. Fuzzy numbers and fuzzy arithmetics

A fuzzy number, which is a special case of a fuzzy set, represents a concept of a set of "numbers close to ?" where ? is the number being fuzzified. We denote a fuzzy number as $\tilde{\varsigma}$.

Definition 1 (a fuzzy number) A fuzzy number $\tilde{\varsigma}$ is a fuzzy set that has a bounded support and a convex and normal membership function $\mu_?(x)$, i.e.

$$\mu_{\varsigma}\left[\lambda x_1 + (1-\lambda x_2)\right] \geq \min\left[\mu_{\varsigma}(x_1), \mu_{\varsigma}(x_2)\right],$$
$$\forall x_1, x_2 \in X, \lambda \in [0,1], \quad \sup\left[\mu_{\varsigma}(x)\right] = 1 .$$

A most commonly used form of a fuzzy number is a *triangular fuzzy number (L-R fuzzy number)*. As the name says, the L-R fuzzy number has a triangular membership function and is written as $\tilde{\varsigma} = \langle L, c, R \rangle$, where L is a left margin, c is a center and R is a right margin of the number. A general procedure that provides extension of crisp mathematical expressions to fuzzy domains is called the *extension principle* [14]. It states that having a function $y=f(x)$ and a fuzzy number $\tilde{a} = \{(\mu_a(x), x) : x \in X\}$, then

$$\tilde{b} = f(\tilde{a}) = \{(\mu_a(y), y) : y \in X\}. \quad \textbf{(2)}$$

In other words, an outcome of a mathematical expression (2) is a fuzzy number that is obtained by computation of the image of the interval while a membership function is carried through. In case $f(x)$ is a many-to-one mapping, $\mu_a(y)$ is calculated as a *maximum* of multiple entries. Implementation of the extension principle to arithmetic operations gives the following definition.

Definition 2 (arithmetic of fuzzy numbers) Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers and let "?" denote any of four basic arithmetic operations (+, - , •, /). Then

$$\mu_{a \circ b}(c) = \sup_{c = a \circ b} \min\{\mu_a(x), \mu_b(y)\} \quad \textbf{(3)}$$

For 0 as an element of the fuzzy set $\tilde{b}$ division is undefined. If L-R fuzzy numbers are used in operations, the results of addition and subtraction are also L-R numbers. Moreover, in that case if $\tilde{a} = \langle L_a, c_a, R_a \rangle$ and $\tilde{b} = \langle L_b, c_b, R_b \rangle$ then

$$\tilde{c} = \tilde{a} + \tilde{b} = \langle L_a + L_b, c_a + c_b, R_a + R_b \rangle,$$
$$\tilde{c} = \tilde{a} - \tilde{b} = \langle L_a - R_b, c_a - c_b, R_a - L_b \rangle. \quad \textbf{(4)}$$

Multiplication and division of L-R fuzzy numbers result in a fuzzy number that is not an L-R number. However, for the engineering purpose multiplication and division can be approximated [15].

Having arithmetics defined, the other subject to be addressed is the *ordering* of fuzzy numbers. Since a Lyapunov condition for system stability is represented by an inequality, in order to be able to determine whether the system is stable or not, we have to define how to compare two fuzzy numbers.

Due to the nature of a fuzzy number it is clear that the order of fuzzy numbers can be ascertained in various ways. Generally, we discern two classes of ordering methods. Methods in the first class are based on an ordering relation proposed in [16].

Definition 3 (ordering of fuzzy numbers) Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers and let $\tilde{\succ}$ denote an ordering function (*greater than or equal to*). Then $\tilde{a} \tilde{\succ} \tilde{b}$ if and only if $a_\alpha \geq b_\alpha$, $\forall \alpha \in (0,1]$; $a_\alpha \geq b_\alpha$ if and only if $\overline{a}_\alpha \geq \overline{b}_\alpha$ and $\underline{a}_\alpha \geq \underline{b}_\alpha$.

A problem is that above definition may be inconsistent, i.e. for two overlapping fuzzy numbers we may get different orderings for different values of

a. Nevertheless, in case of fuzzy numbers that are usually used in control applications Definition 3 gives exclusive ordering.

The ordering methods that belong to the second class are able to overcome the inconsistency problem and generally make ordering consistent. They are based on a crisp representation of a fuzzy number [17]. In the text that follows we use ordering according to Definition 3 since it does not require calculation of a fuzzy number index.

4. Fuzzy controller design

Let us now return to fuzzy controller stability described in [13]. Instead of using only signs of state variables, input domains are partitioned in 5 linearly distributed fuzzy sets: NM, NS, ZE, PS, PM. Inclusion of these linguistic values in the Lyapunov stability condition (1) and by using fuzzy arithmetic (4) and (5), the rules shown in Table 2 are obtained.

In case $\tilde{x}_1 = NS$ and $\tilde{x}_2 = NM$, then $\tilde{x}_2 \cdot (\tilde{x}_1 + \tilde{u}_{FC}) = NM \cdot (NS + \tilde{u}_{FC}) \prec \tilde{0}$ (a set $\tilde{0}$ is a fuzzy singleton having 0 as its only element). It is clear that fulfillment of this inequality, i.e. stability, depends on domains of involved fuzzy numbers. Since $\underline{0} = \overline{0} = 0$, i.e. both, infimum and supremum, of fuzzy singleton $\tilde{0}$ are equal to 0, domain of $\tilde{x}_2 \cdot (\tilde{x}_1 + \tilde{u}_{FC})$ should be (-8, 0). In [13] this fact is stated in a form of the theorem, which says that a fuzzy control system is asymptotically stable if domain of $\tilde{\dot{V}}$ is (-8, 0), where $\tilde{\dot{V}}$ is a linguistic value of the Lyapunov function derivative.

It should be noted that the theorem expresses only a sufficient condition for stability, which can be easily checked on the rules from Table 2.

Table 2 Control rules according to [13].

x_2 \ x_1	NM	NS	ZE	PS	PM
NM	PL	PB	PM	PS	ZE
NS	PB	PM	PS	ZE	NS
ZE	PM	PS	ZE	NS	NM
PS	PS	ZE	NS	NM	NB
PM	ZE	NS	NM	NB	NL

If inputs and output are described by linearly distributed triangular fuzzy numbers, then for example, a fizzy Lyapunov criterion in case $\tilde{x}_2 \cdot (\tilde{x}_1 + \tilde{u}_{FC}) = PS \cdot (NS + ZE)$ is not satisfied. However, the proposed controller in Table 2 is stable. This situation is caused by the fact that fuzzy arithmetic does not utilize all available information, thus imprecision of obtained results is greater than or equal to imprecision of used fuzzy numbers. In the standard fuzzy arithmetic, for example, $\tilde{a} - \tilde{a} \neq 0$ or $\tilde{a} / \tilde{a} \neq 1$, which contradicts with intuition (new approaches to fuzzy arithmetic try to resolve this issue by redefinition of basic fuzzy arithmetic operations [18]).

In order to overcome problems caused by imprecision, output of the controller in Table 2 may be represented by singletons instead of triangular fuzzy numbers.

In that case, our example $\tilde{x}_2 \cdot (\tilde{x}_1 + \tilde{u}_{FC}) = PS \cdot (NS + ZE)$ becomes $\tilde{x}_2 \cdot (\tilde{x}_1 + \tilde{u}_{FC}) = PS \cdot (NS + \tilde{0})$ which is less than $\tilde{0}$ and the Lyapunov stability condition is satisfied.

5. Cascade fuzzy controller design

The system selected to demonstrate fuzzy arithmetic in the fuzzy controller stability analysis is a well known ball and beam control problem. The system, shown in Figure 1, consists of a ball that is free to roll on a beam.

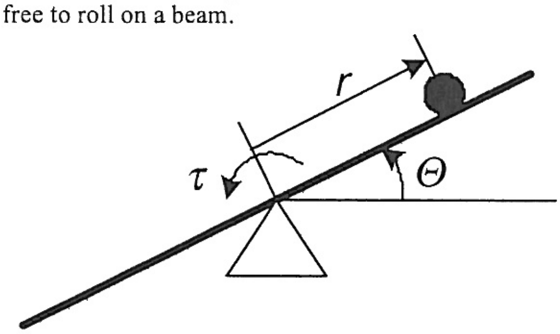

Figure 1. The ball and beam system.

The system is challenging from the control point of view as it is unstable and highly nonlinear. Our goal is to obtain a fuzzy controller that stabilizes the system around a set point r_{ref}. There exist many solutions to the problem, from standard algorithms to neural networks and fuzzy controllers [19, 20]. Here we solve the problem assuming that only basic knowledge about the system is available in a form of linguistic statements.

By using the Lagrange equation we may obtain mathematical description of the system. Although the mathematical model is not used in the stability analysis and controller design, it is given here:

$$\left(\frac{J_B}{R^2} + m\right)\ddot{r} = mr\dot{\theta}^2 - mg\sin\theta,$$

$$\left(mr^2 + J_b + J_B\right)\ddot{\theta} = \tau - 2mr\dot{r}\dot{\theta} - mgr\cos\theta.$$

where: m is the ball mass, R is the ball radius, J_b is the ball moment of inertia, J_B is the beam moment of inertia around the center, g is a gravitational constant, t is the torque applied to the beam center, r is the ball position and ? is the beam angle.

We know the following facts about the system:

- the range of the beam angle ? is ±p/4,
- the ball should be held within ±0.1 m from the center of the beam,
- the ball position and the beam angle are measured.

Even though we assume that an exact physical law of motion is unknown, from the common experience we distinguish that the ball acceleration increases as the beam angle increases, $\ddot{r} \sim \theta$ (note that according to Figure 1 a positive angle causes movement in the negative direction). Also, we are acquainted with the aspect that angular acceleration of the beam is somehow proportional to the applied torque, $\ddot{\theta} \sim \tau$. Since the ball position and the beam angle are measured we choose r and ? as process variables.

Now, let us define L-R fuzzy numbers that will represent linguistic values of deviations of process variables from the set points, $e_r = r_{ref} - r$ and $e_\theta = \theta_{ref} - \theta$. We define 3 linguistic values; *negative*, *zero* and *positive*. The ranges of r and θ are known, thus

$$\tilde{e}_{rN} = \langle -0.2,-0.1,0\rangle,\ \tilde{e}_{rZ} = \langle -0.1,0,0.1\rangle,$$
$$\tilde{e}_{rP} = \langle 0,0.1,0.2\rangle, \tilde{e}_{\theta N} = \langle -\pi/2,-\pi/4,0\rangle,$$
$$\tilde{e}_{\theta Z} = \langle -\pi/4,0,\pi/4\rangle,\ \tilde{e}_{\theta P} = \langle 0,\pi/4,\pi/2\rangle.$$

Their derivatives, $\dot{e}_r$ and $\dot{e}_\theta$, are approximated with a difference between two consecutive measurements (sampling time T_s=10 [ms]), $\dot{e}_r \approx e_{dr}(k) = [e_r(k) - e_r(k-1)]/T_s$ and $\dot{e}_{r\theta} \approx e_{d\theta}(k) = [e_\theta(k) - e_\theta(k-1)]/T_s$. Since we assume that the system dynamics is unknown, one way how we can determine fuzzy numbers for these two variables is to impose requirements that the ball velocity and the beam angular velocity should remain inside predefined values. We bound $|\dot{r}| \le 0.03[m/s]$ and $|\dot{\theta}| \le \pi/2[rad/s]$, which gives

$$\tilde{e}_{drN} = \langle -1,-0.03,0\rangle,\ \tilde{e}_{drZ} = \langle -0.03,0,0.03\rangle,$$
$$\tilde{e}_{drP} = \langle 0,0.03,1\rangle,\ \tilde{e}_{d\theta N} = \langle -\pi,-\pi/2,0\rangle,$$
$$\tilde{e}_{d\theta Z} = \langle -\pi/2,0,\pi/2\rangle,\ \tilde{e}_{d\theta P} = \langle 0,\pi/2,\pi\rangle.$$

It should be noted that although centers of proposed fuzzy numbers correspond with predefined boundaries we leave wide margins since actual values of velocities are unknown. Having defined the deviations of process variables and their linguistic values we may proceed with a fuzzy Lyapunov stability test. We consider the Lyapunov function of the following form:

$$V = \frac{1}{2}\left(e_r^2 + \dot{e}_r^2 + e_\theta^2 + \dot{e}_\theta^2\right).$$

Its derivative gives (recall that $\ddot{r} \sim \theta$, and $\ddot{\theta} \sim \tau$)

$$\begin{aligned}\dot{V} &= e_r\dot{e}_r + \dot{e}_r\ddot{e}_r + e_\theta\dot{e}_\theta + \dot{e}_\theta\ddot{e}_\theta \\ &= e_r\dot{e}_r + \dot{e}_r\theta + e_\theta\dot{e}_\theta - \dot{e}_\theta\tau\end{aligned}.$$

For the system to be asymptotically stable we require $\dot{V} < 0$. By using the extension principle, the inclusion of linguistic values of the variables in the form of fuzzy numbers in above equation, will give a fuzzy Lyapunov stability criterion that eventually defines rules of a fuzzy controller. Since each variable has 3 linguistic values, there are 81 possible combinations that should be tested. Hence, a final fuzzy controller will have 81 rules. In order to reduce the number of rules we apply a different approach.

Let us study each of two terms in the derivative of the Lyapunov function, separately. First we determine stability conditions for $e_r\dot{e}_r + \dot{e}_r\theta < 0$ and then for $e_\theta\dot{e}_\theta - \dot{e}_\theta\tau < 0$. In that way the fuzzy controller is split into two parts; the first part, created by the first term, should generate the set point (the commanded beam angle θ_{ref}) for the second part, whose design is based on the second term. An output from the second part of the fuzzy controller is a torque t applied to the beam. A so obtained fuzzy controller forms a cascade control scheme shown in Figure 2 [21]. The advantage of this approach is a significant reduction of a number of rules. While a standard controller contains 81 rules, a cascade fuzzy controller has only 9+9=18 rules.

Insertion of fuzzy numbers that represent linguistic values of variables involved in the first term $e_r\dot{e}_r + \dot{e}_r\theta$, results in the following:

$$\tilde{e}_{drN}(\tilde{e}_{rN} + \tilde{\theta}_{ref}) \prec \tilde{0},\ \tilde{e}_{drN}(\tilde{e}_{rZ} + \tilde{\theta}_{ref}) \prec \tilde{0},$$
$$\tilde{e}_{drN}(\tilde{e}_{rP} + \tilde{\theta}_{ref}) \prec \tilde{0},\ \tilde{e}_{drZ}(\tilde{e}_{rN} + \tilde{\theta}_{ref}) \prec \tilde{0},$$
$$\tilde{e}_{drZ}(\tilde{e}_{rZ} + \tilde{\theta}_{ref}) \prec \tilde{0},\ \tilde{e}_{drZ}(\tilde{e}_{rP} + \tilde{\theta}_{ref}) \prec \tilde{0},$$
$$\tilde{e}_{drP}(\tilde{e}_{rN} + \tilde{\theta}_{ref}) \prec \tilde{0},\ \tilde{e}_{drP}(\tilde{e}_{rZ} + \tilde{\theta}_{ref}) \prec \tilde{0},$$
$$\tilde{e}_{drP}(\tilde{e}_{rP} + \tilde{\theta}_{ref}) \prec \tilde{0}.$$

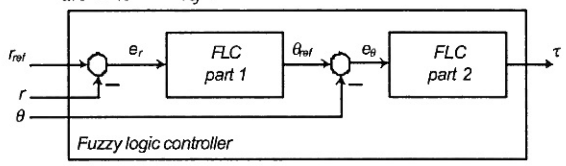

Figure 2. The cascade fuzzy controller used for ball and beam system stabilization.

The range of θ_{ref} should be the same as the range of θ. In order to get a smooth control surface, the domain of θ_{ref} is represented by 5 fuzzy numbers, *negative large*, *negative, zero, positive* and *positive large* defined as:

$\tilde{\theta}_{refNL} = \langle -1.2, -0.8, -0.4 \rangle$, $\tilde{\theta}_{refN} = \langle -0.8, -0.4, 0 \rangle$,

$\tilde{\theta}_{refZ} = \langle -0.4, 0, 0.4 \rangle$,

$\tilde{\theta}_{refP} = \langle 0, 0.4, 0.8 \rangle$, $\tilde{\theta}_{refPL} = \langle 0.4, 0.8, 1.2 \rangle$.

For each of inequalities we have to determine a linguistic value of the beam angle set point, θ_{ref}, such that all of them are fulfilled. In case we choose $\tilde{\theta}_{refPL}$ for the first inequality we get

$$\tilde{e}_{drN}(\tilde{e}_{rN} + \tilde{\theta}_{refPL}) = \langle -1, -0.03, 0 \rangle \begin{pmatrix} \langle -0.2, -0.1, 0 \rangle \\ + \langle 0.4, 0.8, 1.2 \rangle \end{pmatrix}$$
$$= \langle -1, -0.03, 0 \rangle \langle 0.2, 0.7, 1.2 \rangle$$
$$= \langle -1.2, -0.0105, 0 \rangle \precsim \tilde{0},$$

thus, the inequality is satisfied and the first rule has a form "IF e_r is *negative* AND e_{dr} is *negative* THEN θ_{ref} is *positive large* ". Other rules can be obtained in the same manner. A final fuzzy rule table determined according to the first part of the Lyapunov function derivative is shown in Table 3 a).

Table 3 A fuzzy rule table for the a) first part, and b) second part of a cascade fuzzy controller.

a)

e_{dr} \ e_r	N	Z	P
N	PL	P	Z
Z	P	Z	N
P	Z	N	NL

b)

$e_{d\theta}$ \ e_θ	N	Z	P
N	NL	N	Z
Z	N	Z	P
P	Z	P	PL

Let us now analyze the second part of the Lyapunov function derivative, $e_\theta \dot{e}_\theta - \dot{e}_\theta \tau < 0$. Same as in the previous case we attain 9 inequalities that have to be fulfilled in order to get a stable behavior of the closed loop system,

$$\tilde{e}_{d\theta N}(\tilde{e}_{\theta N} - \tilde{\tau}) \prec \tilde{0}, \ \tilde{e}_{d\theta N}(\tilde{e}_{\theta Z} - \tilde{\tau}) \prec \tilde{0},$$
$$\tilde{e}_{d\theta N}(\tilde{e}_{\theta P} - \tilde{\tau}) \prec \tilde{0}, \ \tilde{e}_{d\theta Z}(\tilde{e}_{\theta N} - \tilde{\tau}) \prec \tilde{0},$$
$$\tilde{e}_{d\theta Z}(\tilde{e}_{\theta Z} - \tilde{\tau}) \prec \tilde{0}, \ \tilde{e}_{d\theta Z}(\tilde{e}_{\theta P} - \tilde{\tau}) \prec \tilde{0},$$
$$\tilde{e}_{d\theta P}(\tilde{e}_{\theta N} - \tilde{\tau}) \prec \tilde{0}, \ \tilde{e}_{d\theta P}(\tilde{e}_{\theta Z} - \tilde{\tau}) \prec \tilde{0},$$
$$\tilde{e}_{d\theta P}(\tilde{e}_{\theta P} - \tilde{\tau}) \prec \tilde{0}.$$

The first inequality gives

$$\tilde{e}_{d\theta N}(\tilde{e}_{\theta N} - \tilde{\tau}) = \langle -\pi, -\pi/2, 0 \rangle \begin{pmatrix} \langle -\pi/2, -\pi/4, 0 \rangle \\ -\tilde{\tau} \end{pmatrix} \precsim \tilde{0},$$

which yields

$$\tilde{\tau} \precsim \langle -\pi/2, -\pi/4, 0 \rangle .$$

Inspection of other inequalities makes clear that the applied torque value for the first inequality should be the most negative one, i.e., we assign a linguistic value *negative large* with $\tilde{\tau}_{NL} = \langle -3\pi/4, -\pi/2, -\pi/4 \rangle$. A corresponding fuzzy rule is "IF *e?* is *negative* AND *ed?* is *negative* THEN *t* is *negative large* ". The obtained fuzzy rule table for the second part of the fuzzy controller is shown in Table 3 b).

A problem with calculated values of the torque is that they are fully based on elementary knowledge of the system. It is clear that torque $\tilde{\tau}_{NL} = \langle -3\pi/4, -\pi/2, -\pi/4 \rangle$ may not be enough to move the beam in the right direction if, for example, the ball mass is significant. Nevertheless, the obtained fuzzy controller is a solid first step in the stability analysis and design.

The response of the autonomous system (r_{ref}=0) controlled with a cascade fuzzy controller with initial conditions *r*=0.1 [m] and *?*=-0.3 [rad] are shown in Figure 3 (dotted lines).

One may see that the system is stable, but rather slow. Since determination of the fuzzy numbers representing changes in errors was based on the assessments without knowledge about actual boundaries, we can readjust these values in order to make the system dynamics faster. Division by factor 2 gives the results shown in Figure 3 (solid line). The system remains stable with faster response containing a slight overshoot.

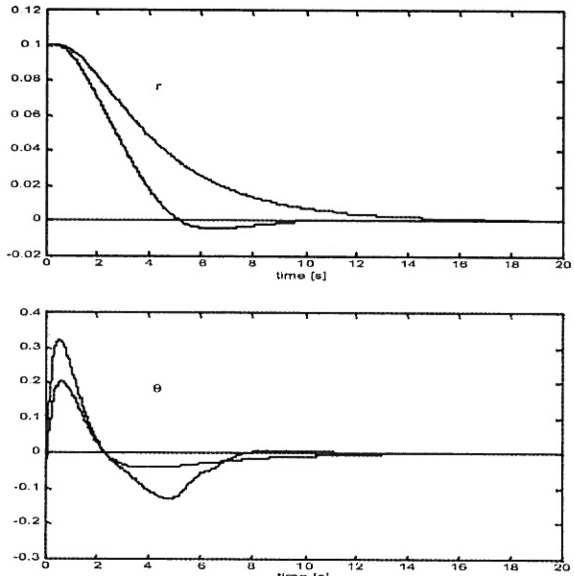

Figure 3. The response of the ball and beam system controlled with a cascade fuzzy controller (initial conditions r=0.1 [m] and ?=-0.3 [rad]).

A tracking performance of the system is tested with a signal r_{ref}=0.05sin(0.94*t*) (Figure 4), indicating a very good and stable system behavior.

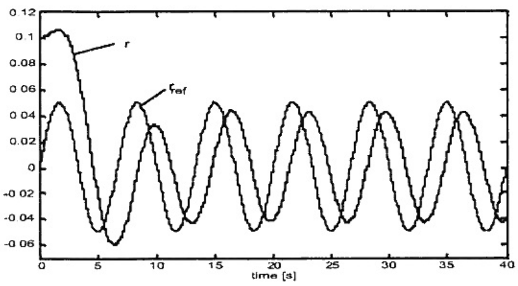

Figure 4. A tracking performance of the ball and beam system controlled with cascade fuzzy controller; initial conditions r=0.1 [m] and ?=-0.3 [rad], $r_{ref}=0.05\sin(0.94t)$.

6. Conclusions

In general, a fuzzy controller design is heuristic; it leans on the descriptive knowledge of the operater who knows how the target system should be controlled. While in conventional control the synthesis of controller structures and parameters is carried out according to various stability criteria, the same philosophy in fuzzy control has not been so successful yet. Representing one of most promising approaches, the Lyapunov stability approach has been successfully used for stability assessment of some fuzzy controller structures and corresponding fuzzy control systems.

Fuzzy Lyapunov stability, based on fuzzy numbers and fuzzy arithmetic presented herein, is still under development, which implies that many issues are not resolved yet. However, due to its simplicity, a fuzzy Lyapunov stability approach may be exploited as the first, elementary step in the fuzzy controller design and fuzzy controller stability analysis, especially when only rudimentary information regarding a controlled process is available.

In the paper we have shown that complex nonlinear control problems such as a ball and beam control problem can be successfully solved by employment of a cascade fuzzy control scheme obtained from the fuzzy Lyapunov stability criteria for the outer and inner control loop. In this way, a stable system performance has been achieved with only 18 rules, i.e., with 9 rules for each control loop.

7. Literature

1. Y. L. Kandel, Y.-Q. Zhang, "Stability analysis of fuzzy control systems", Fuzzy Sets and Systems 105 (1999), 33-48.
2. C.-S. Chen, "Design of stable fuzzy control systems using Lyapunov's method in fuzzy hypercubes" Fuzzy Sets and Systems 139 (2003), 95–110.
3. K.S. Ray, D. Majumder; "Application of circle criteria for stability analysis of linear SISO and MIMO systems associated with fuzzy logic controller", IEEE Transaction on systems, man and cybernetics, Vol. 14, No. 2, 1984, 345-349.
4. G.C. Hwang, S.C. Lin; "A stability approach to fuzzy control design for nonlinear systems", Fuzzy sets and systems, 48, 1992, 279-287.
5. J.C. Wu, T.S. Liu; "Fuzzy Control Stabilization with Applications to Motorcycle Control", IEEE Transaction on SMC, Vol. 26, No. 6, 1996, 836-847.
6. S.S. Farinwata, G. Vachtsevanos; "Robust stability of fuzzy logic control systems", Proceedings of the American Control Conference, Seattle, 1995, 2267-2271.
7. M.A.L. Thathachar, P. Viswanath; "On the stability of fuzzy systems", IEEE Transaction on Fuzzy Systems, Vol. 5, No. 1, 1997, 145-151.
8. K. Tanaka, M. Sugeno; "Stability analysis and design of fuzzy control systems", Fuzzy sets and systems, 45, 1992, 135-156.
9. L. Luoh, " New stability analysis of T–S fuzzy system with robust approach", Mathematics and Computers in Simulation 59 (2002), 335–340.
10. M. Margaliot, G. Langholz, "Fuzzy Lyapunov based approach to the design of fuzzy controllers", Fuzzy Sets and Systems 106 (1999), 49–59.
11. L.A. Zadeh, "From computing with numbers to computing with words— from manipulation of measurements to manipulation of perceptions", Int. J. Appl. Math. Comput. Sci., 2002, Vol.12, No.3, 307–324.
12. H. Bandemer, S. Hartmann, "A fuzzy approach to stability of fuzzy controllers", Fuzzy Sets and Systems 96 (1998), 161-172.
13. C. Zhou, "Fuzzy–arithmetic–based Lyapunov synthesis in the design of stable fuzzy controllers: a computing–with–words approach", Int. J. Appl. Math. Comput. Sci., 2002, Vol.12, 411–421.
14. D. Dubois, H. Prade, "Fuzzy numbers: an overview", in: J.C. Bezdek (Ed.), Analysis of Fuzzy Information, CRC-Press, Boca Raton 1988, vol. 2, 3-39
15. M. Oussalah, J. De Schutter, "Approximated fuzzy LR computation", Information Sciences 153 (2003), 155–175.
16. G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall, Upper Saddle River, N J, 1995.
17. Yager, R.R., A procedure for ordering fuzzy subsets of the unit interval, *Information Sciences*, 24 , 143-161, 1981.
18. G.J. Klir, "Fuzzy arithmetic with requisite constraints", Fuzzy Sets and Systems 91 (1997), 165-175.
19. P.H. Eaton, D.V. Prokhorov, and D. C. Wunsch II,"Neurocontroller Alternatives for "Fuzzy" Ball-and-Beam Systems with Nonuniform Nonlinear Friction", IEEE Transactions On Neural Networks, VOL. 11, NO. 2, MARCH 2000.
20. J. Hauser, S. Sastry, and P. Kokotovic, "Nonlinear control via approximate input-output linearization: The ball and beam example", IEEE Transactions on Automatic Control, 37:392-398, 1992.
21. W. Guanghui, T. Yantao, H. Wei, J. Huimin, "Stabilization and Equilibrium Control of Super Articulated Ball and Beam System", Proceedings of the 3rd World Conference on Intelligent Control and Automation, 2000.