The Cross Product of Fuzzy Numbers and its Applications in Geology

Dedicated to the 80th Birthday of Professor György Bárdossy

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Abstract: In fuzzy arithmetic, the multiplication operation based on Zadeh's extension principle owns several unnatural properties both from theoretical and practical points of view. To overcome some of these shortcomings, a new operation called cross product has been introduced recently. We show the main properties of the cross product. We also present a comparative study of the traditional multiplication and the cross product in geological applications, especially for estimating of resources of solid mineral deposits.

Keywords: Fuzzy number, cross product, solid mineral deposit estimation

1 Introduction

Uncertainties in different scientific areas arise mainly from the lack of human knowledge. In many practical problems the uncertainties are not of statistical type. This situation occurs mainly in the case of modeling the linguistic expressions appeared in different scientific areas because of dependence on human judgment. Also, as it is shown in several recent works (see e.g. [1]), the uncertainty on different measurements due to finite resolution of measuring instruments is in many cases more possibilistic than probabilistic, since, in many applications the measurements cannot be repeated. Mainly in Geosciences, we cannot have two holes in the same place in order to repeat the measurement, so, every experiment can be considered as unique. This shows us that uncertainties on the measurements in geological data are more of possibilistic type than of probabilistic type, since, in order to obtain the statistical distribution of a variable we need several experiments.

Fuzzy numbers allow us to model in an easy way these non-probabilistic uncertainties. This justifies the increasing interest on theoretical and practical aspects of fuzzy arithmetic in the last years, especially directed to: operations over fuzzy numbers and properties, ranking and canonical representation of fuzzy numbers.

Usually, the definition of the addition and multiplication of fuzzy numbers is based on the extension principle ([16]). A main disadvantage of the multiplicative operation in this case is that by multiplication, the shape of $L - R$ type fuzzy numbers (so triangular or trapezoidal numbers) is not preserved. In many situations this problem is solved by approximating the result of the extension principle-based multiplication by a triangular or trapezoidal number. This can lead to unexpected results in the case of iterative application of these approximations (i.e. can increase or decrease a defuzzified value in a considerable way).

Nevertheless, there exist other directions of the development of fuzzy arithmetic. For example, in [12] new operations between fuzzy numbers are defined starting from a representation of a fuzzy number by a location index number and two fuzziness index functions.

Recently, in [2] a new multiplicative operation of product type is introduced, the so-called cross product of fuzzy numbers and its algebraic and analytic properties were studied. The main point is that this product preserves the shape of $L - R$ fuzzy numbers under multiplication, is consistent to the classical error theory and has good algebraic and metric properties. The idea to define the cross product started from the approximation formulas ([5], p. 55) of the multiplication (obtained by Zadeh extension principle) of two $L - R$ type fuzzy numbers by an $L - R$ type fuzzy number if the spreads are small compared with the means of the numbers. The consistency of the cross product with the classical error theory is also proved in [2].

The above mentioned properties motivate the use of cross product in geological applications as a possible alternative of the product obtained by Zadeh's extension principle, mainly in the case of iterative calculations using product at every step.

In Section 2 we recall some concepts from fuzzy arithmetic then in Section 3, we recall the definition and some properties of the cross product. In Section 4, we consider applications in geology. The application presented in this paper concerns the solid mineral deposit estimation.

2 Basic Concepts

Let us recall the following well-known definition of a fuzzy number. The addition of fuzzy numbers and multiplication of a fuzzy number by a crisp number are provided by Zadeh's extension principle.

Definition 1 A fuzzy number is a function $u : \mathbf{R} \to [0,1]$ with the following properties:

(i) u is normal, i.e., there exists $x_0 \in \mathbf{R}$ such that $u(x_0) = 1$;

(ii) $u(\lambda x + (1-\lambda) y) \geq \min\{u(x), u(y)\}, \forall x, y \in \mathbf{R}, \forall \lambda \in [0,1]$;

(iii) u is upper semicontinuous on $\mathbf{R}$, i.e., $\forall x_0 \in \mathbf{R}$ and $\forall \varepsilon > 0$ there exists a neighborhood $V(x_0)$ such that $u(x) \leq u(x_0) + \varepsilon, \forall x \in V(x_0)$;

(iv) The set $\overline{\operatorname{supp}(u)}$ is compact in $\mathbf{R}$, where $\operatorname{supp}(u) = \{x \in \mathbf{R}; u(x) > 0\}$.

We denote by R_F the set of all fuzzy numbers.

Let $a, b, c \in \mathbf{R}, a < b < c$. The fuzzy number $u : \mathbf{R} \to [0,1]$ denoted by (a,b,c) and defined by $u(x) = 0$ if $x \leq a$ or $x \geq c, u(x) = \frac{x-a}{b-a}$ if $x \in [a,b]$ and $u(x) = \frac{c-x}{c-b}$ if $x \in [b,c]$ is called a triangular fuzzy number.

For $0 < r \leq 1$ and $u \in R_F$ we denote $[u]^r = \{x \in \mathbf{R}; u(x) \geq r\}$ and $[u]^0 = \overline{\{x \in \mathbf{R}; u(x) > 0\}}$. It is well-known that for each $r \in [0,1], [u]^r$ is a bounded closed interval, $[u]^r = \left[\underline{u}^r, \overline{u}^r\right]$. Let $u, v \in R_F$ and $\lambda \in \mathbf{R}$. We define the sum $u \oplus v$ and the scalar multiplication λu by

$[u \oplus v]^r = [u]^r + [v]^r = \left[\underline{u}^r + \underline{v}^r, \overline{u}^r + \overline{v}^r\right]$ and

$$[\lambda u]^r = \lambda [u]^r = \begin{cases} \left[\lambda \underline{u}^r, \lambda \overline{u}^r\right], & \text{if } \lambda \geq 0, \\ \left[\lambda \overline{u}^r, \lambda \underline{u}^r\right], & \text{if } \lambda < 0, \end{cases}$$

respectively, for every $r \in [0,1]$.

We denote by $-u = (-1)u \in R_F$ the symmetric of $u \in R_F$.

The product $u \cdot v$ of fuzzy numbers u and v, based on Zadeh's extension principle, is defined by

$$\underline{(u\cdot v)}^r = \min\{\underline{u}^r\underline{v}^r, \underline{u}^r\overline{v}^r, \overline{u}^r\underline{v}^r, \overline{u}^r\overline{v}^r\}$$

$$\overline{(u\cdot v)}^r = \max\{\underline{u}^r\underline{v}^r, \underline{u}^r\overline{v}^r, \overline{u}^r\underline{v}^r, \overline{u}^r\overline{v}^r\}.$$

Definition 2 *A fuzzy number* $u \in R_F$ *is said to be positive if* $\underline{u}^1 \geq 0$, *strict positive if* $\underline{u}^1 > 0$, *negative if* $\overline{u}^1 \leq 0$ *and strict negative if* $\overline{u}^1 < 0$.

Let $u, v \in R_F$. *We say that* $u \prec v$ *if* $\underline{u}^r \leq \underline{v}^r$ *and* $\overline{u}^r \leq \overline{v}^r$ *for all* $r \in [0,1]$. *We say that* u *and* v *are on the same side of* 0 *if* $u \prec 0$ *and* $v \prec 0$ *or* $0 \prec u$ *and* $0 \prec v$.

Remark 1 If u is positive (negative) then $-u$ is negative (positive).

Definition 3 *For arbitrary fuzzy numbers* u *and* v *the quantity*

$$D(u,v) = \sup_{0\leq r\leq 1}\{\max\{|\underline{u}^r - \underline{v}^r|, |\overline{u}^r - \overline{v}^r|\}\}$$

is called the (Hausdorff) distance between u *and* v.

It is well-known (see e.g. [co-mi]) that (R_F, D) *is a complete metric space and* D *verifies* $D(ku,kv) = |k|D(u,v),\ \forall u,v \in R_F,\ \forall k \in \mathbf{R}$.

3 Definition of the Cross Product

In this section we study the theoretical properties of the cross product of fuzzy numbers. Let $R_F^* = \{u \in R_F : \ u \text{ is positive or negative}\}$. Firstly we begin with a theorem which was obtained by using the stacking theorem ([13]).

Theorem 1 *If* u *and* v *are positive fuzzy numbers then* $w = u \odot v$ *defined by* $[w]^r = \left[\underline{w}^r, \overline{w}^r\right]$, *where* $\underline{w}^r = \underline{u}^r\underline{v}^1 + \underline{u}^1\underline{v}^r - \underline{u}^1\underline{v}^1$ *and* $\overline{w}^r = \overline{u}^r\overline{v}^1 + \overline{u}^1\overline{v}^r - \overline{u}^1\overline{v}^1$, *for every* $r \in [0,1]$, *is a positive fuzzy number.*

Corollary 1 *Let* u *and* v *be two fuzzy numbers.*

(i) *If* u *is positive and* v *is negative then* $u \odot v = -(u \odot (-v))$ *is a negative fuzzy number;*

(ii) *If* u *is negative and* v *is positive then* $u \odot v = -((-u) \odot v)$ *is a negative fuzzy number;*

(iii) *If* u *and* v *are negative then* $u \odot v = (-u) \odot (-v)$ *is a positive fuzzy number.*

Definition 4 *The binary operation* $\odot$ *on* R_F^* *introduced by Theorem 1 and Corollary 1 is called cross product of fuzzy numbers.*

Remark 1) *The cross product is defined for any fuzzy numbers in*

$R_F^{\wedge} = \{u \in R_F^*;$ there exists an unique $x_0 \in \mathbf{R}$ such that $u(x_0) = 1\}$, *so implicitly for any triangular fuzzy number. In fact, the cross product is defined for any fuzzy number in the sense proposed in [6] (see also [14]).*

2) *The below formulas of calculus can be easily proved (* $r \in [0,1]$ *):*

$$\underline{(u \odot v)}^r = \overline{u}^r \underline{v}^1 + \overline{u}^1 \underline{v}^r - \overline{u}^1 \underline{v}^1,$$
$$\overline{(u \odot v)}^r = \underline{u}^r \overline{v}^1 + \underline{u}^1 \overline{v}^r - \underline{u}^1 \overline{v}^1$$

if u *is positive and* v *is negative,*

$$\underline{(u \odot v)}^r = \underline{u}^r \overline{v}^1 + \underline{u}^1 \overline{v}^r - \underline{u}^1 \overline{v}^1,$$
$$\overline{(u \odot v)}^r = \overline{u}^r \underline{v}^1 + \overline{u}^1 \underline{v}^r - \overline{u}^1 \underline{v}^1$$

if u *is negative and* v *is positive. In the last possibility, if* u *and* v *are negative then*

$$\underline{(u \odot v)}^r = \overline{u}^r \overline{v}^1 + \overline{u}^1 \overline{v}^r - \overline{u}^1 \overline{v}^1,$$
$$\overline{(u \odot v)}^r = \underline{u}^r \underline{v}^1 + \underline{u}^1 \underline{v}^r - \underline{u}^1 \underline{v}^1.$$

3) *The cross product extends the scalar multiplication of fuzzy numbers. Indeed, if one of operands is the real number* k *identified with its characteristic function then* $\underline{k}^r = \overline{k}^r = k, \forall r \in [0,1]$ *and following the above formulas of calculus we get the result.*

The main algebraic properties of the cross product are the following.

Theorem 2 *If* $u, v, w \in R_F^*$ *then*

(i) $(-u) \odot v = u \odot (-v) = -(u \odot v)$;

(ii) $u \odot v = v \odot u$;

(iii) $(u \odot v) \odot w = u \odot (v \odot w)$;

(iv) *If* u *and* v *have the same sign then* $(u \oplus v) \odot w = (u \odot w) \oplus (v \odot w)$;

(v) $(u \odot v)^{\odot n} = u^{\odot n} \odot v^{\odot n}, \forall n \in N^*$, *where* $a^{\odot n} = \underbrace{a \odot \ldots \odot a}_{n \text{ times}}$ *for any* $a \in R_F^*$.

Remark *1) If u is positive and v negative (or u is negative and v positive) then the property of distributivity in (iv) is not verified even if u and v are real numbers.*

2) The above properties (i)-(iii) hold for the usual product " $\cdot$ " based on the extension principle. The property (iv) holds in a weaker form: If u and v are on the same side of 0 then for any $w, w \prec 0$ or $0 \prec w$ we have $(u \oplus v) \cdot w = (u \cdot w) \oplus (v \cdot w)$.

The so-called L - R fuzzy numbers are considered important in fuzzy arithmetic. These and their particular cases triangular and trapezoidal fuzzy numbers are used almost exclusively in applications.

Definition 5. *([5], p. 54, [14]) Let $L, R : [0, +\infty) \to [0,1]$ be two continuous, decreasing functions fulfilling $L(0) = R(0) = 1, L(1) = R(1) = 0$, invertible on $[0,1]$. Moreover, let a^1 be any real number and suppose $\underline{a}, \overline{a}$ be positive numbers. The fuzzy set $u : \mathbf{R} \to [0,1]$ is an L - R fuzzy number if*

$$u(t) = \begin{cases} L\left(\frac{a^1 - t}{\underline{a}}\right), & \text{for } t \le a^1 \\ R\left(\frac{t - a^1}{\overline{a}}\right), & \text{for } t > a^1. \end{cases}$$

Symbolically, we write $u = \left(a^1, \underline{a}, \overline{a}\right)_{L,R}$, where a^1 is called the mean value of u, $\underline{a}, \overline{a}$ are called the left and the right spread. If u is an L - R fuzzy number then (see e. g. [wa-ha-sc])

$$[u]^r = \left[a^1 - L^{-1}(r)\underline{a}, a^1 + R^{-1}(r)\overline{a}\right].$$

Theorem 3 *If u and v are strict positive L - R fuzzy numbers then $u \odot v$ is a strict positive L - R fuzzy number.*

Since we are interested mainly in the applications of the cross product we may restrict our attention to positive fuzzy numbers, however in other cases some similar properties can be obtained (see [3]).

The cross product verifies the following metric property.

Theorem 4 *If u, v have the same sign and $w \in R_F^*$ then*

$$D(w \odot u, w \odot v) \le K_w D(u, v), \text{ where } K_w = \max\{|\overline{w}^1|, |\underline{w}^1|\} + \overline{w}^0 - \underline{w}^0.$$

By using the previous metric property, several properties can be obtained with respect to continuity, differentiability (using the H-differential) and integrability of the product of fuzzy-number-valued functions (see [2]).

The following interpretation related to error theory is a further theoretical motivation of the use of the cross product of fuzzy numbers. Indeed, the consistency of the cross product with the classical error theory motivates its use in the case of modeling uncertain data (uncertainty being due to errors of measurement).

We introduce two kinds of errors of fuzzy numbers corresponding to absolute error and relative error in classical error theory and we study these with respect to sum and cross product.

Definition 6 *Let u be a fuzzy number. The crisp number $\Delta_L^r(u) = \underline{u}^1 - \underline{u}^r$ is called r - error to left of u and the crisp number $\Delta_R^r(u) = \overline{u}^r - \overline{u}^1$ is called r -error to right of u, where $r \in [0,1]$. The sum $\Delta^r(u) = \Delta_L^r(u) + \Delta_R^r(u)$ is called r -error of u.*

If u expresses the fuzzy concept A then $\Delta_L^r(u)$ and $\Delta_R^r(u)$ can be interpreted as the values of tolerance of level r from the concept A to left and to right, respectively. For example, if the triangular fuzzy number $u = (5,7,9)$ expresses "early morning" then $\Delta_L^{\frac{1}{2}}(u) = 1$ (one hour) is the tolerance of level $\frac{1}{2}$ of u towards night from the concept of "early morning" and $\Delta_R^{\frac{1}{4}}(u) = 0.5$ (30 minutes) is the tolerance of level $\frac{1}{4}$ of u towards moon from the concept of "early morning".

A new argument in the use of addition of fuzzy numbers as extension (by Zadeh's principle) of real addition is the validity of the formula

$$\Delta^r\left(u \oplus v\right) = \Delta^r\left(u\right) + \Delta^r(v)$$

which is consistent to the classical error theory. It is an immediate consequence of the obvious formulas

$$\Delta_L^r\left(u \oplus v\right) = \Delta_L^r\left(u\right) + \Delta_L^r\left(v\right)$$

and

$$\Delta_R^r\left(u \oplus v\right) = \Delta_R^r(u) + \Delta_R^r\left(v\right).$$

Now, let us study the relative error of the cross product.

Definition 7 *Let u be a fuzzy number such that $\underline{u}^1 \neq 0$ and $\overline{u}^1 \neq 0$. The crisp numbers $\delta_L^r\left(u\right) = \frac{\Delta_L^r(u)}{\left|\underline{u}^1\right|}$ and $\delta_R^r\left(u\right) = \frac{\Delta_R^r(u)}{\left|\overline{u}^1\right|}$ are called relative r -errors of u to left and to right. The quantity $\delta^r(u) = \delta_L^r(u) + \delta_R^r(u)$ is called relative r - error of u.*

Theorem 5 *If* u *and* v *are strict positive or strict negative fuzzy numbers then* $\delta^r(u \odot v) = \delta^r(u) + \delta^r(v)$.

Corollary 2 *If* u *is a strict positive fuzzy number then* $\delta_L^r(u^{\odot n}) = n\delta_L^r(u)$, $\delta_R^r(u^{\odot n}) = n\delta_R^r(u)$ *and* $\delta^r(u^{\odot n}) = n\delta^r(u)$.

The above theorems show us that the cross product is consistent with the classical error theory (the propagation of errors is governed by a similar law as in the classical case).

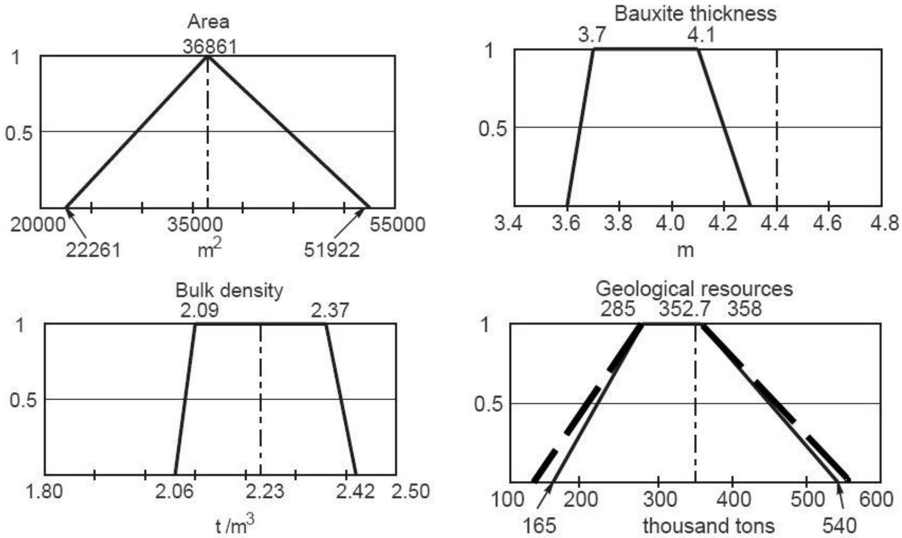

Figure 1

Fuzzy numbers of the tonnage calculation, Szőc-Szárhegy I and I/A. Thick dashed lines denote the result obtained with the cross product. Thin dashed line: results of traditional resource estimation (method of the geological blocks)

4 Applications of the Cross Product in Geology

Recently, fuzzy arithmetic has found several applications in geology (see [1]). In the above cited work the usual (Zadeh's extension principle based) product is used for estimation of resources of solid mineral deposits. In this section we propose an alternative study of the same problem, by using the cross product. The reasons of the possible usefulness of the cross product are the following.

Firstly, in this case the shape of the result of the product is conserved, i.e. the product of triangular numbers is triangular and the product of trapezoidal numbers is trapezoidal. Secondly, the 1-level sets are better taken into account by the use of cross product. Also, the consistency of the cross product with the classical error theory motivates this study.

As in [1], we perform resource estimation on several bauxite deposits in Hungary. In the same way as with the traditional methods, the tonnage of the resources is obtained by the product of the deposit area, the average thickness and the average bulk-density of the studied ore or mineral commodity. Large deposits can be split into blocks, preferably along natural boundaries, such as tectonic lines. We present the results obtained by the usual multiplication and the results obtained by using the cross product.

Furthermore, if we defuzzify the two results obtained by the two different product-type operations we conclude that the results are different. Also, we observe that after defuzzification (by centroid method) the result of the cross product in the study of the Óbarok deposit is smaller than that of the usual product i.e. the cross product leads to a more pessimistic result than the usual multiplication in this case. So, the risks of an investment at this site can be more realistically evaluated.

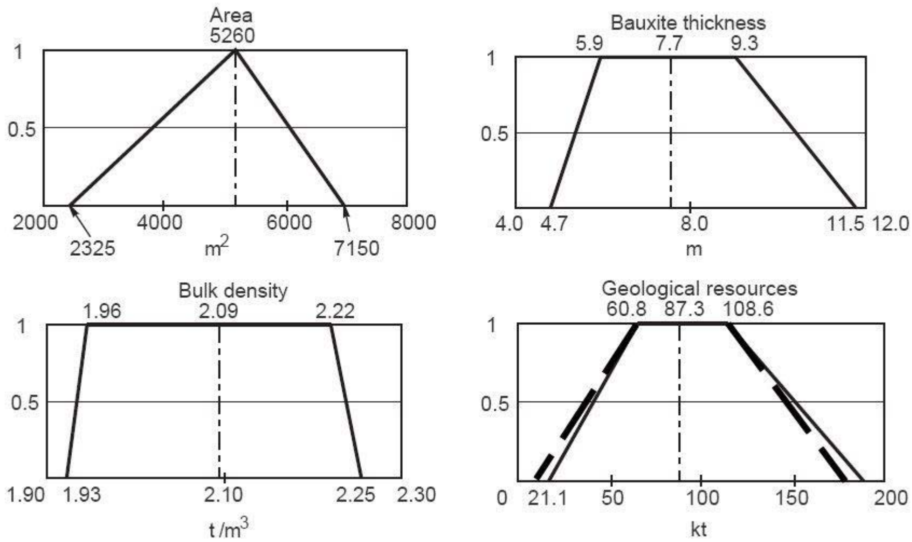

Figure 2

Fuzzy numbers of the tonnage calculation, Óbarok IX. Thick dashed lines: denote the result obtained with the cross product. Thin dashed line: results of traditional resource estimation (method of the geological blocks)

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