The Geometry of Associativity and its Application

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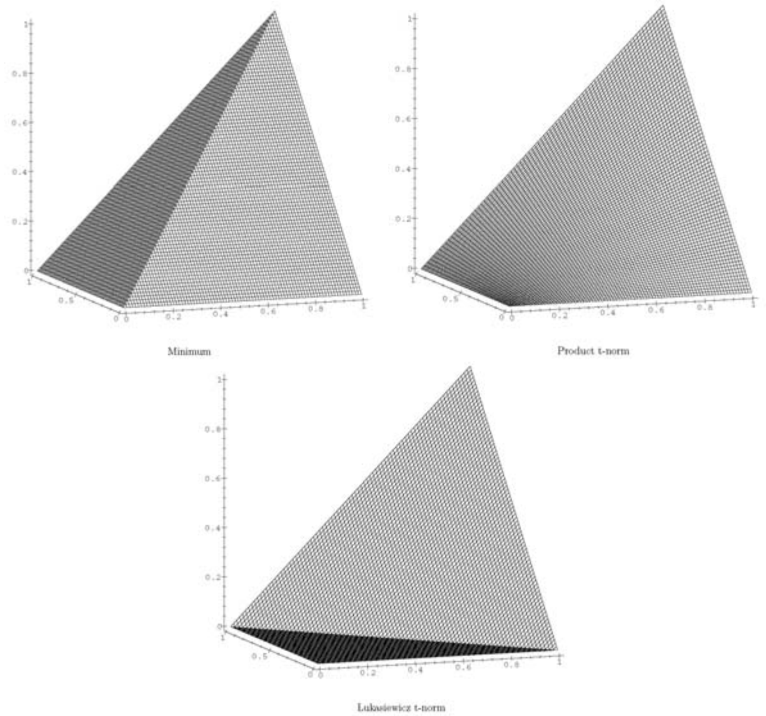
*Abstract: Commutativity of (binary) operations, that is the interchangeability of their arguments (x*y=y*x) is easily seen from the graph of the operations. The meaning of commutativity is just the invariance of the graph with respect to a reflection to the plane defined by x=y. Similar geometrical description for associativity is not known. That is, associativity of binary operations can not be seen simply by "looking at" their graphs. The following three operations are commutative and associative, their commutativity is readily seen from their graphs, but their associativity is not (see the figure below).*



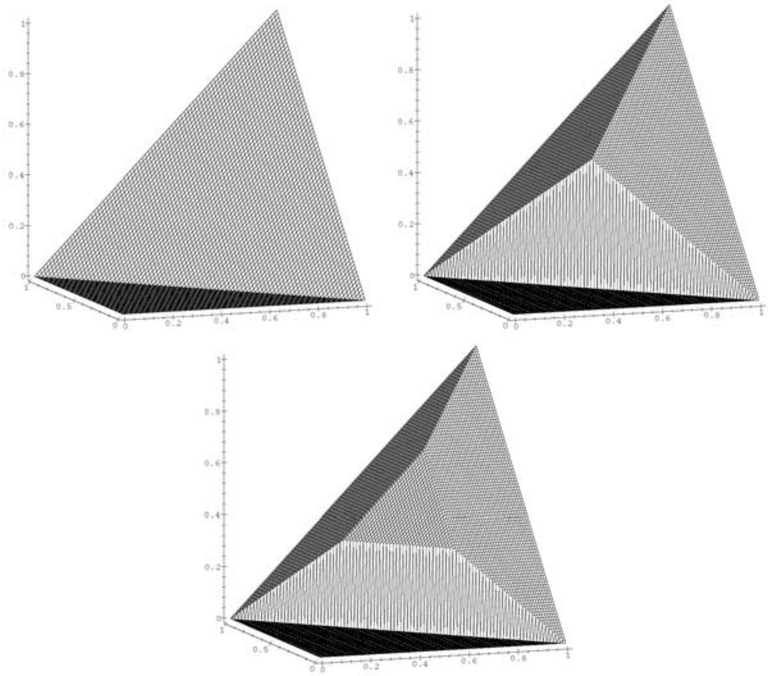
Investigation of associativity is one of the major problems in algebra. For example, semigroups, groups, rings and fields are all associative structures. In my opinion the reason of the difficulty of investigation of associativity is that we are able to “see things” in three dimensions only. In three dimensions the graph of an operation is defined as follows: There are two independent variables x and y, and the value $x*y$ is taken in the third axle. The meaning of associativity together with commutativity is that we can freely interchange the operands of the operation, that is, any two operands are interchangeable. We have seen above that interchangeability is just the invariance of the graph with respect to a reflection to a plane. Consider now the graph of an associative and commutative operation in four dimensions: There are three independent variables x, y, and z, and the value $x*y*z$ is taken in the fourth axle. It follows from the previous arguments that associativity and commutativity together are equivalent to the invariance of the four-dimensional graph with respect to three reflections to the “spaces” $x=y$, $x=z$, and $y=z$, respectively. That is, if we were able to “see things” in four dimension, then associativity together with commutativity were easily seen from the graph of the operation “for the first sight”.

Similar geometrical description of associativity is not known as of today.

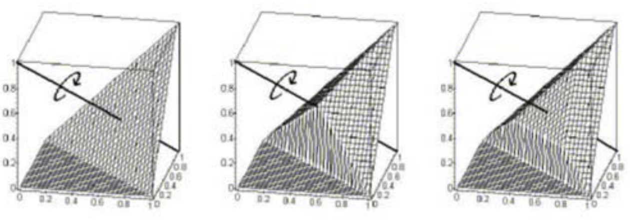
I have reported on a surprising geometrical property of a *special* class of associative functions in

> S. Jenei, *Geometry of left-continuous triangular norms with strong induced negations,*
> Belg. J. Oper. Res. Statist. Comput. Sci. **38** (1998), 5-16

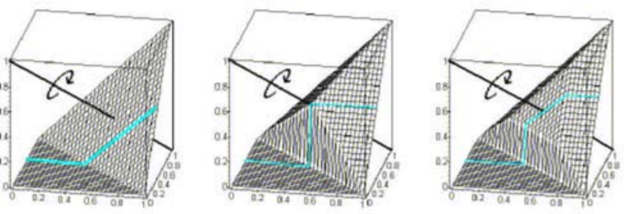
Namely, if we, in addition to commutativity and associativity, assume that the “border line” in between the 0 and the positive part of the graph is the function $y=1-x$, (three examples are plotted below)



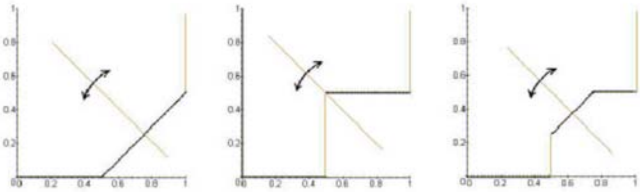
then the corresponding graphs are rotation-invariant with respect to a rotation with 120 degree (an illustration is in the following figure).



Moreover, vertical sections of graphs of such operations (see the yellow lines)



show as well a kind of symmetry.



The mentioned geometrical property does not characterize associativity. That is, there exist rotation-invariant functions which are not associative. The question suggests itself:

— Does there exist a geometrical characterization which does not assume the “border line” property, and which do characterize associativity.

This question will be answered in this talk. That is, associativity can be “seen” from the 3-dimensional graph. We will point out how the geometric understanding of associativity can be applied in algebra (partially-ordered semigroups), in functional equations (associative functions), and in logics (non-classical logics).

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