The sign dependent expected utility functional representations

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Abstract: Starting from the needs of cumulative prospect theory this paper provides a discussion on difference representations of the asymmetric Choquet integral with respect to a signed fuzzy measure with bounded chain variation. There are given difference representations of the Choquet integral with respect to a signed fuzzy measure based on its representation as difference of two fuzzy measures. Further, there is obtained a representation of a comonotone symmetric maximum additive, weak symmetric minimum-homogeneous and monotone functional L (defined on the class of functions on finite set) as a symmetric maximum of two Sugeno integrals. There is considered general fuzzy rank and sign dependent functionals on functions defined on the infinitely countable basic set.

Key words and phrases: fuzzy measures, Sugeno integral, cumulative prospect theory.

1 Introduction

For the main field of application of Choquet integral, decision under uncertainty, an universal set X is a space, its elements are state of nature and functions from X to $\mathbb{R}$ are prospects. The preference relation $\preceq$ is defined on the set of prospects and we say that a utility functional L represents a preference relation if and only if $L(f) \leq L(g)$ for all pairs of prospects f, g such that $f \preceq g$. Schmeidler [19] showed that preference can be represented by Choquet integral model, so called Choquet expected utility model (cumulative utility). Choquet expected utility model is not an appropriate tool when the gain and loss must be considered in the same time.

In the field of decision theory the cumulative prospect theory (CPT), introduced by Tversky and Kahneman [18], see [4], combines cumulative utility and a generalization of expected utility, so called sign dependent expected utility. CPT holds if there exist two fuzzy measures, m^+ and m^-, which ensure that the utility functional L, model for preference representation, can be represented by the difference of two Choquet integrals, i.e.,

$$L(f) = (C)\int f^+\, dm^+ - (C)\int f^-\, dm^-, \tag{1}$$

where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. Narukawa et al. proved in [14] that comonotone-additive and monotone functional can be represented as a difference of two Choquet integrals and gave the conditions for which it can be represented by one Choquet integral, see [20].

Motivated by (1) in the first part of this paper we present by [11] some difference representations of asymmetric Choquet integral w.r.t a signed fuzzy measures. (4) is presented. We consider a difference representation (2) of Choquet integral w.r.t a signed fuzzy measure m with bounded chain variation. Introducing an interpreter and a frame for representation of the signed fuzzy measures we obtain that for every signed fuzzy measure $m \in BV$ there exists a representation of m and then applying this result, we present another difference representation (3) of Choquet integral w.r.t m.

In the second part of the paper we consider the analogous situation for the Sugeno integral, based on [17]. It is well known that the Sugeno integral is one of the non-linear functional on the class of measurable functions which is comonotone-maxitive, monotone and $\wedge$-homogeneous [2, 8, 15]. An extension of the Sugeno integral in the spirit of the symmetric extension of Choquet integral proposed by M. Grabisch in [6] is useful as a framework for cumulative prospect theory in an ordinal context. In this paper we consider representation by two Sugeno integrals of the functional L defined on the class of functions $f : X \to [-1, 1]$ on a finite set X. In the case of infinitely countable set X we obtain as a consequence of results on general fuzzy rank and sign dependent functionals that the symmetric Sugeno integral is comonotone-$\varovee$-additive functional on the class of functions with finite support. In all considerations we shall use two recently introduced operations in [7]: the symmetric maximum $\varovee$ and the symmetric minimum $\varowedge$, as extensions on the interval $[-1, 1]$ of the classical maximum $\vee$ and minimum $\wedge$, respectively.

2 Preliminaries

We assume that X is a non-empty universal set. Let $\mathcal{A}$ be a σ-algebra of subsets of X. A fuzzy measure m (see [8, 15]) is a non-negative real-valued set function $m : \mathcal{A} \to [0, \infty]$ with the following properties:
(FM1) $m(\emptyset) = 0$,
(FM2) $A \subset B \quad \Rightarrow \quad m(A) \leq m(B)$, for all $A, B \in \mathcal{A}$
(FM3) $A_n \in \mathcal{A}\ ,\ A_n \nearrow A \ \Rightarrow\ m(A_n) \nearrow m(A)$,
(FM4) $A_n \in \mathcal{A}\ ,\ A_n \searrow A$ and there exists n_0 such that $m(A_{n_0}) < \infty \ \Rightarrow$ $m(A_n) \searrow m(A)$.

In the following we assume that $m : \mathcal{A} \to [0,1]$ satisfies (FM1), (FM2) and $m(X) = 1$.

The chain variation of real-valued set functions, vanishing at the empty set, and the space BV will be introduced, see [1, 15].

Definition 1 *The chain variation of a real-valued set function m, $m(\emptyset) = 0$, for each $E \in \mathcal{A}$, is defined by*

$$|m|(E) = \sup\{\textstyle\sum_{i=1}^{n} |m(E_i) - m(E_{i-1})| :$$

$$\emptyset = E_0 \subset E_1 \subset \ldots \subset E_n = E,\, E_i \in \mathcal{A}, i = 1, \ldots, n\}.$$

In the previous definition, the supremum is taken over all chain between $\emptyset$ and E.

The chain variation $|m|$ of a set function m is positive, monotone set function, vanishing at the empty set, and the inequality $|m(E)| \leq |m|(E)$ is satisfied for each $E \in \mathcal{A}$. Consequently, if m is a fuzzy measure, then $|m|(E) = m(E)$, for all $E \in \mathcal{A}$.

Definition 2 *A real-valued set function m, $m(\emptyset) = 0$, is of bounded chain variation if $|m|(X) < \infty$.*

The family of all set functions of bounded chain variation, vanishing at the empty set, is denoted by BV. The functional $||m|| = |m|(X)$ is a norm on a Banach space (BV, $||\ ||$), see [1, 15].

Let $f : X \to [-1, 1]$ be a function on X with finite support. Consider the class of functions with finite support denoted by $\mathcal{K}_1(X)$:

$$\mathcal{K}_1(X) = \{\, f \mid f : X \to [-1, 1]\,,\, \mathrm{card}(\mathrm{supp}(f)) < \infty\,\},$$

where the support is given by $\mathrm{supp}(f) = \{\, x \,|\, f(x) \neq 0\}$. $\mathcal{K}_1^+(X)$ and $\mathcal{K}_1^-(X)$ denote the class of non-negative and non-positive functions with finite support, respectively.

Recall that two measurable functions f and g on X are called *comonotone*, see [15], if they are measurable with respect to the same chain $\mathcal{C}$ in $\mathcal{A}$ ($\mathcal{A}$ is a σ-algebra of subsets of X). Recall that equivalently, comonotonicity of the functions f and g can be expressed as follows: $f(x) < f(x_1) \;\Rightarrow\; g(x) \leq g(x_1)$ for all $x\,,\, x_1 \in X$.

3 Representation of the asymmetric Choquet integral with respect to signed fuzzy measure

An important characterization of space BV, given by following theorem, has been proven in [1, 15].

Theorem 1 *A set function m, $m(\emptyset) = 0$, belongs to BV if and only if it can be represented as difference of two monotone set functions m_1 and m_2 vanishing at the empty set.*

Using Theorem 1, another representation of Choquet integral with respect to a signed fuzzy measure can be obtained [15]. We have by [11] the following representation.

Theorem 2 *If m is a signed fuzzy measure such that $m \in BV$, then the asymmetric Choquet integral of $f \in \overline{\mathcal{M}}$ can be represented in the following manner*

$$C_m(f) = C_{m_1}(f) - C_{m_2}(f), \tag{2}$$

where m_1 and m_2 are two fuzzy measures such that $m = m_1 - m_2$ and $C_m(f)$ does not depend of the representation of m by means of Theorem 1.

As we have already mention it is not difficult to construct example to show that the representation of m given by Theorem 1 is not unique, but by Theorem 2 $C_m(f)$ does not depend of representation of m, see [11].

Now we shall consider a representation of a signed fuzzy measure $m : \mathcal{A} \to [-\infty, \infty]$ which belongs to the space BV. We will correspond to it a signed measure μ defined on a σ-algebra $\mathcal{B}$ of subsets of a set Y.

First, we will introduce an interpreter for measurable sets and a frame for representation [9], see [15].

Definition 3 *A mapping $H : \mathcal{A} \to \mathcal{B}$ is called an interpreter if H satisfies*

(i) $H(\emptyset) = \emptyset$ and $H(X) = Y$;

(ii) $H(E) \subset H(F)$, for all $E \subset F$.

A triple $(Y, \mathcal{B}, H)$ is called a frame of $(X, \mathcal{A})$, if H is an interpreter from $\mathcal{A}$ to $\mathcal{B}$.

Definition 4 *Let m be a signed fuzzy measure defined on $\mathcal{A}$. A quadruple $(Y, \mathcal{B}, \mu, H)$ is called a representation of m (or $(X, \mathcal{A}, m)$) if H is an interpreter from $\mathcal{A}$ to $\mathcal{B}$, μ is a signed measure on $(Y, \mathcal{B})$, and $m = \mu \circ H$.*

Theorem 3 *Every signed fuzzy measure m, $m \in BV$, has its representation.*

Remark 1 (i) As it is mentioned before, two fuzzy measures m_1 and m_2 are not unique, hence the representation of m given in Theorem 3 is *not unique*, too.

(ii) If m is a signed fuzzy measure such that $m \in BV$, and $\bar{m}$ is its conjugate set function, then a quadruple $(Y, \mathcal{B}, \mu, \bar{H})$ is a representation of $\bar{m}$, where the interpreter $\bar{H}$ is defined by $\bar{H}(E) = (-\bar{m}_2(E),\, \bar{m}_1(E))$ for all $E \in \mathcal{A}$, and $(Y, \mathcal{B}, \mu)$ is the same as in the proof of Theorem 3.

Now, we can apply Theorem 3 to obtain a representation of the asymmetric Choquet integral of a measurable function f with respect to a signed fuzzy measure m, see [11].

Theorem 4 *If m is a signed fuzzy measure, $m \in BV$ and $f \in \overline{\mathcal{M}}$, then there exist two functions $I_f^1 : Y \to [0, \infty]$ and $I_f^2 : Y \to [0, \infty]$ such that the asymmetric Choquet integral has the following difference representation*

$$C_m(f) = \int I^1_{f^+}\, d\lambda - \int I^2_{f^-}\, d\lambda, \tag{3}$$

where, $f^+ = f \vee 0$, $f^- = (-f) \vee 0$ and the integrals on the right-hand side are the Lebesgue integrals. $C_m(f)$ does not depend of the representation of m by means of Theorem 3.

4 The symmetric Sugeno integral

We shall need for the the considerations in the next section the following important notions. The *symmetric maximum* $\circledvee : [-1,1]^2 \to [-1,1]$, originally introduced in [7], is defined by

$$a \circledvee b = \begin{cases} -(|a| \vee |b|)\,, & b \neq -a \text{ and } |a| \vee |b| = -a \text{ or } \;= -b, \\ 0\,, & b = -a, \\ |a| \vee |b|\,, & \text{otherwise.} \end{cases}$$

The *symmetric minimum* $\circledwedge : [-1,1]^2 \to [-1,1]$, introduced in [7], is defined by

$$a \circledwedge b = \begin{cases} -(|a| \wedge |b|)\,, & \operatorname{sign} a \neq \operatorname{sign} b, \\ |a| \wedge |b|\,, & \text{otherwise.} \end{cases}$$

We have

$$a \circledvee b = (|a| \vee |b|) \operatorname{sign}(a + b),$$

$$a \circledwedge b = (|a| \wedge |b|) \operatorname{sign}(a \cdot b).$$

Let f and g be two functions defined on X with values in $[-1,1]$. Then, we define functions $f \varovee g$ and $f \varowedge g$ for any $x \in X$

$$(f \varovee g)(x) = f(x) \varovee g(x),$$

$$(f \varowedge g)(x) = f(x) \varowedge g(x),$$

and for any $a \in [0,1]$

$$(a \varowedge f)(x) = a \varowedge f(x).$$

Due to non-associativity of the operation $\varovee$ on $[-1,1]$, it cannot be used directly as n-ary operator. The expression $\bigovee_{i=1}^{n} a_i$ is unambiguously defined iff $\bigvee_{i=1}^{n} a_i \neq -\bigwedge_{i=1}^{n} a_i$. If equality occurs, several *rules of computation* can ensure uniqueness ([7]):

1. Put $\bigovee_{i=1}^{n} a_i = 0$. This rule is defined by

$$\left\lfloor \bigovee_{i=1}^{n} a_i \right\rfloor = \Big(\bigovee_{a_i \geq 0} a_i \Big) \varovee \Big(\bigovee_{a_i < 0} a_i \Big) = \Big(\bigvee_{a_i \geq 0} a_i \Big) \varovee \Big(\bigwedge_{a_i < 0} a_i \Big).$$

2. Discard all occurrences of $\bigvee_{i=1}^{n} a_i$ and $-\bigwedge_{i=1}^{n} a_i$ and continue with the reduced list of inputs, until the condition $\bigvee_{i=1}^{n} a_i \neq -\bigwedge_{i=1}^{n} a_i$ is satisfied. We denote this rule by $\langle \bigovee_{i=1}^{n} a_i \rangle$.

We refer the reader to [7] for a detailed study of the properties of the introduced rules.

Definition 5 *([6, 15]) Let μ be a fuzzy measure on the measurable space $(X, \mathcal{A})$.*

(i) The Sugeno integral *of $f \in \mathcal{K}_1^+(X)$ with respect to μ is defined by:*

$$(S)\int f\, d\mu = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \mu(\{x \mid f(x) \geq \alpha\}))$$

(ii) The symmetric Sugeno integral *of $f \in \mathcal{K}_1(X)$ with respect to μ is defined by:*

$$\circledS \int f\, d\mu = \Big((S)\int f^+\, d\mu \Big) \varovee \Big(-(S)\int f^-\, d\mu \Big),$$

where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0 = -(f \wedge 0)$.

Further, when X is a finite set, i.e., $X = \{x_1, \dots, x_n\}$, the Sugeno integral of a function $f \in \mathcal{K}_1^+(X)$ with respect to μ can be written as

$$(S) \int f \, d\mu = \bigvee_{i=1}^{n} f_{\alpha(i)} \wedge \mu(A_{\alpha(i)}),$$

where f has a comonotone maxitive representation $f = \bigvee_{i=1}^{n} f_{\alpha(i)} \wedge \mathbf{1}_{A_{\alpha(i)}}$ for $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(n))$ a permutation of index set $\{1, 2, \dots, n\}$ such that $0 \le f_{\alpha(1)} \le \dots \le f_{\alpha(n)} \le 1$ and $A_{\alpha(i)} = \{x_{\alpha(i)}, \dots, x_{\alpha(n)}\}$, $f_i = f(x_i)$ and $\mathbf{1}_A$ denotes characteristic function of the crisp subset A of X. The symmetric Sugeno integral of a function $f \in \mathcal{K}_1(X)$ can be considered as it is proposed in [6]

$$\mathbb{S}^1 \int f \, d\mu = \Big\langle \Big(\mathop{\otimes\!\!\!\!\vee}_{i=1}^{s} f_{\alpha(i)} \owedge \mu(\{x_{\alpha(1)}, \dots, x_{\alpha(i)}\}) \Big) \tag{4}$$

$$\ovee \Big(\mathop{\ovee}_{i=s+1}^{n} f_{\alpha(i)} \owedge \mu(\{x_{\alpha(i)}, \dots, x_{\alpha(n)}\}) \Big) \Big\rangle,$$

where α is a permutation of index set such that $-1 \le f_{\alpha(1)} \le \dots \le f_{\alpha(s)} < 0$ and $0 \le f_{\alpha(s+1)} \le \dots \le f_{\alpha(n)} \le 1$. More details about the symmetric Sugeno integral can be found in [6, 7].

Distributivity of the operation $\owedge$ w.r.t $\ovee$ does not hold in general. If we expect that distributivity is satisfied for $a, b \ge 0$ and $c \le 0$, we have to suppose some additional conditions as in the next result.

Proposition 1 *Let $a, b \ge 0$ and $c \le 0$. If $a \owedge b \ne a \owedge (-c)$ then*

$$a \owedge (b \ovee c) = (a \owedge b) \ovee (a \owedge c).$$

Note that the condition $a \owedge b \ne a \owedge (-c)$ is equivalent to the condition $(a \owedge b) \ovee (a \owedge c) \ne 0$. Further discussion of the distributivity can be found in [6].

5 Comonotone-$\ovee$-additive functional and its representation

The motivation for the paper [17] is based mainly on the axiomatic characterization of the preference relation $\preceq$ such that it is CPT, stated in [13], and our approach may be viewed as adequate base for an axiomatization for the preference representation in qualitative decision making. In order to examine the $\ovee$-additivity of the symmetric Sugeno integral, it is useful to consider the concept of comonotone functions. Note that any function $f : X \to [-1, 1]$ can be represented by symmetric maximum of two comonotone functions $f^+ \ge 0$ and $-f^- \le 0$, i.e.,

$$f = f^+ \ovee (-f^-). \tag{5}$$

Namely, for $x \in X$ the value $f(x) \neq 0$ is equal either to $f^+(x)$ or $-f^-(x)$. Hence for any $a \in [0,1]$ we have

$$a \odot f = a \odot (f^+ \oplus (-f^-)) = (a \odot f^+) \oplus (a \odot (-f^-)), \tag{6}$$

where $a \odot f^+$ and $a \odot (-f^-)$ are comonotone functions.

For infinitely countable set X we have the following compatibility relation between the operation $\oplus$ pointwise extended for functions and the positive and negative parts of a function.

Proposition 2 *Let X be an infinitely countable set. For any comonotone functions $f,\, g \in \mathcal{K}_1(X)$ we have*
(i) $(f \oplus g)^+ = f^+ \oplus g^+$;
(ii) $(f \oplus g)^- = f^- \oplus g^-$.

We remark that Proposition 2 does not hold in general for X finite set, see example in [17].

For $a \in [0,1]$ and $A \subseteq X$, the following functions, defined by

$$s(x) = (a \odot \mathbf{1}_A)(x) = \begin{cases} a & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

$$s(x) = (a \odot (-\mathbf{1}_A))(x) = \begin{cases} -a & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

are called basic functions. We shall denote by $\mathcal{B}_1(X)$, the class of all basic functions from X into $[-1,1]$.

Note that for any $s_i \geq 0$ and $s_i' \leq 0$, $s_i,\ s_i' \in \mathcal{B}_1(X)$, $i = 1, \ldots, n$, and for any $a \in [0,1]$ we have

$$a \odot \Big(\bigoplus_{i=1}^{n} s_i\Big) = \bigoplus_{i=1}^{n} (a \odot s_i), \tag{7}$$

$$a \odot \Big(\bigoplus_{i=1}^{n} s_i'\Big) = \bigoplus_{i=1}^{n} (a \odot s_i'). \tag{8}$$

Both formulas are unambiguous with respect to possible associativity problem, since the first one contains only positive terms, and second one only negative terms. Distributivity is satisfied, too.

Let X be an arbitrary set and $f \in \mathcal{K}_1(X)$ with $\operatorname{supp}(f) = \{f_1, f_2, \ldots, f_n\}$. Let $f \in \mathcal{K}_1^+(X)$. The function f admits a comonotone $\oplus$-additive representation:

$$f = \bigoplus_{i=1}^{n} s_i, \qquad \text{where } s_i = a_i \odot \mathbf{1}_{A_i}\,, \tag{9}$$

$0 \leq a_1 \leq \ldots \leq a_n \leq 1$ and $a_i = f_{\alpha(i)},\ A_i = A_{\alpha(i)}$.
Let $f \in \mathcal{K}_1^-(X)$. It admits a comonotone $\oslash$-additive representation:

$$f \quad = \quad \bigovee_{i=1}^{n} s_i', \qquad \text{where } s_i' = a_i \odot (-\mathbf{1}_{A_i})\,, \tag{10}$$

$0 \leq a_1 \leq \ldots \leq a_n \leq 1$ and $a_i = -f_{\alpha(i)},\ A_i = A_{\alpha(i)}$.

Obviously, the function $f \in \mathcal{K}_1^+(X)$ admits many comonotone $\oslash$-additive representations. The minimal one, uniquely determined, can be obtained by omitting the possible null term, and by assuming $0 < a_1 < \ldots < a_n \leq 1$. The same is truth for $f \in \mathcal{K}_1^-(X)$. By means of the equations (5), (9) and (10) we obtain a comonotone $\oslash$-additive representation of $f \in \mathcal{K}_1(X)$. Again, the representation is not unique. It is well known fact that the Sugeno integral of a non-negative function f is independent with respect to its comonotone maxitive representation, see [2]. This fact together with Definition 5 (ii) ensures that the symmetric Sugeno integral of function $f \in \mathcal{K}_1(X)$ is independent with respect to its comonotone $\oslash$-additive representation.

Now we extend the notion of the symmetric Sugeno integral.

Definition 6 *A functional $L : \mathcal{K}_1(X) \to [-1,1]$ is a* fuzzy rank and sign dependent functional (f.r.s.d.) *on $\mathcal{K}_1(X)$ if there exist two fuzzy measures μ^+ and μ^- such that for all $f \in \mathcal{K}_1(X)$*

$$L(f) = \Big((S)\int f^+\, d\mu^+\Big) \oslash \Big(-(S)\int f^-\, d\mu^-\Big).$$

Note that in the case when $\mu^+ = \mu^-$ the fuzzy rank and sign dependent functional (f.r.s.d. functional for short) is exactly the symmetric Sugeno integral. If a f.r.s.d. functional L is the symmetric Sugeno integral then we have

$$L(-f) = -L(f).$$

Definition 7 *Let $L : \mathcal{K}_1(X) \to [-1,1]$, be a functional on $\mathcal{K}_1(X)$.*
(i) L is comonotone-$\oslash$-additive *iff*

$$L(f \oslash g) \quad = \quad L(f) \oslash L(g) \tag{11}$$

for all comonotone functions f, $g \in \mathcal{K}_1(X)$.
(ii) L is monotone *iff*

$$f \leq g \quad \Rightarrow \quad L(f) \leq L(g) \tag{12}$$

for all functions f, $g \in \mathcal{K}_1(X)$.
(iii) L is positive $\odot$-homogeneous *iff*

$$L(a \odot f) \quad = \quad a \odot L(f) \tag{13}$$

for all $f \in \mathcal{K}_1(X)$ *and* $a \in [0,1]$.
(iv) L *is weak* $\circledtriangle$-homogeneous *iff*

$$L(a \circledtriangle \mathbf{1}_A) = a \circledtriangle L(\mathbf{1}_A) \quad \text{and} \quad L(a \circledtriangle (-\mathbf{1}_A)) = a \circledtriangle L(-\mathbf{1}_A) \tag{14}$$

for all $a \in [0,1]$ *and* $A \subseteq X$.

Weak $\circledtriangle$-homogeneity does not imply positive $\circledtriangle$-homogeneity in general.

Example 1 Let $X = \{1,2\}$ and $f : X \to [-1,1]$. Let L be a functional on $\mathcal{K}_1(X)$ defined by

$$L(f) = f(1) \circledvee f(2).$$

For all $a \in [0,1]$, and $\emptyset \neq A \subseteq X$ we have $L(a \circledtriangle (\mathbf{1}_A)) = a = a \circledtriangle L(\mathbf{1}_A)$ and $L(a \circledtriangle (-\mathbf{1}_A)) = -a = a \circledtriangle L(-\mathbf{1}_A)$. Therefore L is weak $\circledtriangle$-homogeneous functional on $\mathcal{K}_1(X)$. It is not positive $\circledtriangle$-homogeneous, e.g., for f defined by $f(1) = 0.5$ and $f(2) = -0.8$ and $a = 0.3$ we have $L(0.3 \circledtriangle f) = 0.3 \circledvee (-0.3) = 0$ and $0.3 \circledtriangle L(f) = 0.3 \circledtriangle (0.5 \circledvee (-0.8)) = -0.3$.

Remark 2 Note that the Sugeno integral with respect to a fuzzy measure μ is a comonotone-$\circledvee$-additive functional which maps $\mathcal{K}_1^+(X)$ into $[0,1]$. This implies that for all comonotone functions f, $g \in \mathcal{K}_1(X)$ we have

$$(S) \int (f^+ \circledvee g^+)\, d\mu = \left((S) \int f^+ \, d\mu\right) \circledvee \left((S) \int g^+ \, d\mu\right),$$

and an analogous equality holds for f^- and g^-.

In the case of finite set X and $\mathcal{K}_1(X)$ class of functions $f : X \to [-1,1]$ we have the next result.

Theorem 5 *Let* X *be a finite set. If* $L : \mathcal{K}_1(X) \to [-1,1]$ *is a comonotone-*$\circledvee$*-additive, weak* $\circledtriangle$*-homogeneous and monotone functional on* $\mathcal{K}_1(X)$*, then* L *is a f.r.s.d functional, i.e., there exist two fuzzy measures* μ_L^+ *and* μ_L^- *such that*

$$L(f) = \left((S) \int f^+ \, d\mu_L^+\right) \circledvee \left(-(S) \int f^- \, d\mu_L^-\right).$$

Remark 3 Note that the condition of the monotonicity of L in Theorem 5 can be replaced with the weaker one:

$$f \text{ and } g \text{ comonotone and } \; f \leq g \;\Rightarrow\; \; L(f) \leq L(g).$$

Theorem 6 *Let* X *be an infinitely countable set. If* $L : \mathcal{K}_1(X) \to [-1,1]$ *is a f.r.s.d functional such that* $L(f) \neq 0$*, for all* $f \in \mathcal{K}_1(X)$*,* $f \neq 0$*, then it is a comonotone-*$\circledvee$*-additive functional on the set* $\mathcal{K}_1(X)$*.*

A f.r.s.d. functional on $\mathcal{K}_1(X)$, where X is a finite set, is not always comonotone-$\circledS$-additive.

Corollary 1 *Let X be an infinitely countable set. The symmetric Sugeno integral is comonotone-$\circledS$-additive functional on the class of functions*

$$\{f \in \mathcal{K}_1(X) \mid \circledS \int f\, d\mu \neq 0\}.$$

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