Preference Relations in Decision Models

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Abstract: In this paper we summarize some results on constructing preference relations from criterion values, in the framework of multiple criteria decision making. First we recall two types of methods to build a comprehensive preference model. Then we show how to construct crisp and fuzzy preferences from criterion evaluations.

Keywords: Preference relations, multiple criteria decision making, criterion value, fuzzy preferences.

1 Introduction

Most of the real-world decision problems take place in a complex environment where conflicting systems of logic, uncertain and imprecise knowledge have to be considered. To face such complexity, preference modelling requires the use of specific tools, techniques and concepts which allow to reflect the available information with the appropriate granularity.

2 Multiple Criteria Decision Making

In the framework of multiple criteria decision making (MCDM), decision problems can often be formulated as comparing and/or discriminating between m potential alternatives (i.e., variants, projects, candidates) of a set $A = \{a_1, \ldots, a_m\}$, on the basis of one or several criteria. Usually, the set of criteria $G = \{g_1, \ldots, g_n\}$ is considered as a set of real-valued functions defined on A, where a_{ij} denotes the score $g_j(a_i)$ of alternative a_i according to the criterion function g_j, for any $a_i \in A$. For simplicity, the usual assumption is the higher the score, the better the alternative.

One can distinguish two main types of methods to build a comprehensive preference model:

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1. *Scoring methods based on the aggregation of fuzzy evaluations*: The preference analysis is conducted on the set A. An overall evaluation is given to each alternative on the basis of its partial scores on each criterion. Alternatives are then ordered on the basis of these overall evaluations. The most typical example of this approach is to found the discrimination between alternatives on the basis of the weighted average of their scores (but other aggregative operators may be considered as well). The usual implicit assumption when using such a model is the *complete and transitive* comparability of alternatives. Considering an overall score indeed, a is preferred to b as soon as the aggregated score of a is strictly greater than the aggregated score of b.
2. *Pairwise comparisons methods based on the aggregation of fuzzy preference relations*: The preference analysis is conducted on the Cartesian product $A \times A$. This approach is mainly used in social choice theory (Sen, 1986) and outranking methods (e.g. Roy and Bouyssou, 1993). Transitive preference relations are built on each dimension j from scores of alternatives on this dimension. The resulting relations are then aggregated into a comprehensive model which reflects a kind of majoritarian preference among the set of criteria. This approach allows a fine and flexible description of preferences without forcing arbitrarily alternatives to be comparable. The main difficulty with this approach is that intransitivities or cycles can occur in the overall preference model resulting from the aggregation of preferences. This crucial point was proved by Arrow (1959) in the case of nonfuzzy relations. Then, Leclerc (1984) and Perny (1992) have provided extensions of these results in the case of fuzzy relations. This shows the necessity of resorting to an additional exploitation procedure to derive a transitive information when a ranking is required.

3 Construction of Preference Relations from Criterion Values

A *criterion* is a real-valued function g defined on the set of alternatives A, allowing these alternatives to be compared as follows:

$$g(a) \geq g(b) \quad \Longrightarrow \quad aRb, \tag{1}$$

where R is a weak preference (or outranking) relation describing the preference of the decision maker (DM) from a particular point of view, represented by g.

When we have several criteria, such as $g_1, \ldots, g_n$ then the corresponding m weak preferences are $(R_1, \ldots, R_m)$, in accordance with (1).

Quantities $g_j(a)$ are called *criterion values*, representing *partial evaluations* or *scores* of a according to criterion j.

Consider now a single criterion denoted by g. In what follows, we study the problem of building up crisp as well as fuzzy preferences from g.

3.1 Constructing Crisp Preferences

Conventional way of building a preference model: g is interpreted as a *true-criterion*: a is strictly preferred to b when $g(a) > g(b)$, not matter of how large the difference is (*total preorder*). In real-life problems however a small positive difference of scores is not always a justification for a preference.

A classical attitude is to assess *discrimination thresholds* to distinguish between *significant* and *not significant* differences of scores. A typical example consists of the *semi-criterion* and the associated *semi-order* structure.

A *standard semi-criterion* is defined by a criterion function g and a threshold function q such that:

$$g(a) > g(b) \Rightarrow g(a) + q(g(a)) \geq g(b) + q(g(b)),$$

$$aPb \Leftrightarrow g(a) - g(b) > q(g(b)),$$

$$aIb \Leftrightarrow |g(a) - g(b)| \leq q(g(a)) \wedge q(g(b)).$$

The first condition is a *local consistency* condition, making it impossible to consider the difference $g(c) - g(a)$ as significant when a greater difference $g(c) - g(b)$ is not significant (assuming $g(a) > g(b)$).

Then the preference structure $(P, I, \emptyset)$ defines a *semiorder* R, by $R = P \cup I$. Without the local consistency (called nonstandard semi-criterion), *interval orders* are rediscovered.

Some drawbacks of this modelling follows now.

Suppose two candidates a and b are such that

$$g(a) - g(b) = q(g(b)) - \varepsilon/2,$$

where ε is a positive quantity very small compared to $q(g(b))$.

If a slightly superior score $(+\varepsilon)$ was attached to a, we would obtain

$$g(a) - g(b) = q(g(b)) + \varepsilon/2,$$

transforming the previous indifference aIb into strict preference aPb.

One can overcome these difficulties as follows. Separate the preference area from the indifference area by inserting an intermediate zone called weak-preference area (Roy and Vincke, 1984). A possible interpretation is a hesitation between strict preference and indifference.

Formally, consider two discrimination threshold functions

- the indifference threshold q,
- the preference threshold p,

to define a *pseudo-criterion*.

This leads to define crisp binary relations called *strict preference* P, *indifference* I, and *weak preference* Q. They form together a *pseudo-order structure*, and are defined as follows:

$$aPb \Leftrightarrow g(a) - g(b) > p(g(b)),$$

$$aQb \Leftrightarrow p(g(b)) \geq g(a) - g(b) > q(g(b)),$$

$$aIb \Leftrightarrow |g(a) - g(b)| \leq q(g(a)) \wedge q(g(b)),$$

where q and p satisfy the local consistency condition, and

$$p(a) > q(a) \quad (a \in A).$$

This last model offers new possibilities, but does not really solve the problem of sensitivity. All these difficulties will remain as long as we will try to make discrete a continuum of preference situations. The real solution is provided by using fuzzy relations.

3.2 Constructing Fuzzy Preferences

We introduce now two ways (which may be mixed) of implementing the fuzzy approach to model and process preference information. The first way leads to a model which can be seen as an extension of the previous crisp model obtained by replacing pseudo-orders (I_j, Q_j, P_j) by fuzzy semi-orders. Let us detail the typical features of these fuzzy preference structures and their link with the generalized outranking relations introduced in Perny and Roy, 1992.

For any criterion j whose scale is X_j, the fuzzy binary relation S_j is said to be a *monocriterion outranking relation* if there is a real valued function t_j, defined on X_j^2, verifying $S_j(a_i, a_k) = t_j(a_{ij}, a_{kj})$, for all a_i, a_k in A, such that:

$$\begin{array}{ll} \forall y \in X_j, & t_j(x,y) \ is\ a\ nondecreasing\ function\ of\ x, \\ \forall x \in X_j, & t_j(x,y) \ is\ a\ nonincreasing\ function\ of\ y, \\ \forall z \in X_j, & t_j(z,z) = 1. \end{array}$$

As shown by Perny (1992), the relation S_j is a reflexive, complete, semi-transitive and Ferrers fuzzy relation, and thus it is a fuzzy semi-order. Every α-cut of S_j is a crisp semi-order, and these semi-orders form together a homogeneous family compatible in the sense of Roberts (1971) with the classical weak order relation $\geq$ on scores (for more details see Perny and Roy, 1992).

Thus, a natural extension of indifference and preference relations in the fuzzy case is given by setting:

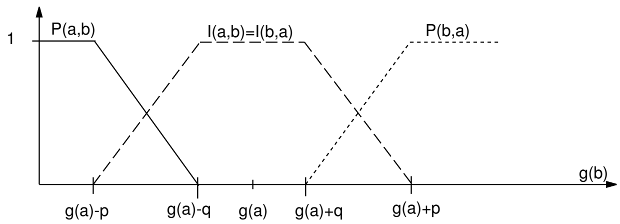
$$\forall (a_i, a_k) \in A \times A, \quad \begin{array}{l} I_j(a_i, a_k) = \min\{S_j(a_i, a_k), S_j(a_k, a_i)\} \\ P_j(a_i, a_k) = 1 - S_j(a_k, a_i) \end{array} \tag{2}$$

As it can be proved, these relations are the symmetric and asymmetric part of the outranking relations S_j where $S_j(a_i, a_k)$ represents the degree to which a_i is not worse than a_k. Such indices are widely involved in Electre Methods (see Roy, 1978) and Promethee Methods (see Brans and Vincke (1985)).

In the Electre III method proposed by Roy (1978), the relations S_j are characterized by the following function.

$$t_j(x, y) = \frac{p_j(x) - \min\{y - x, p_j(x)\}}{p_j(x) - \min\{y - x, q_j(x)\}}$$

This amounts to define fuzzy indifference and preference relations I_j and P_j, $(j = 1, \ldots, n)$, as shown in the next figure.



A second way of including fuzziness in pairwise comparisons methods is to consider fuzzy scores. In this case, we associate to each alternative a_i, for each criterion function g_j, a fuzzy interval defined as follows:

$$\tilde{a}_{ij} = \{(x, \mu_{\tilde{a}_{ij}}(x)) | x \in X_j\}.$$

By definition, $\tilde{a}_{ij}$ is a normalized convex fuzzy subset of the real line, characterized by its membership function $\mu_{\tilde{a}_{ij}}$ such that:

$$\begin{array}{l} \inf_{x \in X_j} \mu_{\tilde{a}_{ij}}(x) = 0, \quad \sup_{x \in X_j} \mu_{\tilde{a}_{ij}}(x) = 1 \\ \forall x_1, x_2 \in X_j, \forall x_3 \in [x_1, x_2], \ \mu_{\tilde{a}_{ij}}(x_3) \ \geq \ \min\{\mu_{\tilde{a}_{ij}}(x_1), \mu_{\tilde{a}_{ij}}(x_2)\} \end{array}$$

Then equation (2) allow to define fuzzy binary relations I_j and P_j for any dimension j by setting:

$$S_j(a_i, a_k) = \max_{x \geq y} \min\{\mu_{\tilde{a}_{ij}}(x), \mu_{\tilde{a}_{kj}}(y)\}. \tag{3}$$

In this case $S_j(a_i, a_k)$ measures the possibility for a_i to be given a score as least as good as the score of a_k. In other words: the possibility of the event a_k *is outranked by* a_i. Moreover, $I_j(a_i, a_k)$ measures the possibility for a_i and a_k to be equivalent whereas $P_j(a_i, a_k)$ measures the necessity of the event a_k *is outranked by* a_i.

Thus, in both cases, we get a fuzzy representation of preferences which seems to be well fitted to the imprecise nature of information. In particular, this kind of representation allows to be sure that small variations of input data will modify in a continuous way the resulting model of preferences.

In case of having all the marginal outranking relations $S_j(a_i, a_k)$, we must determine a global outranking relation $S(a_i, a_k)$ with respect to all the objectives. Once more, the subjectivity of the decision maker must be taken into account and we are looking for an operator h such that :

$$S = h(S_1, \dots, S_n)$$

for any pair (a_i, a_k).

Fodor and Roubens (1994) have suggested to consider monotonic and idempotent operators corresponding to the following compromise attitudes :

- a_i globally outranks a_k if a_i outranks a_k for all significant objectives :

  $$h_{I^{\rightarrow}}(S_1, \dots, S_n) = \min_{j=1,\dots,n} I^{\rightarrow}(\omega_j, S_j(a_i, a_k))$$

  represents the degree of truth of the statement "for all objectives, if it is significant, then a_i is as good as a_k".
- a_i globally outranks a_k if a_i outranks a_k for at least one significant objective :

  $$h_{I_c^{\rightarrow}}(S_1, \dots, S_n) = \max_{j=1,\dots,n} I_c^{\rightarrow}(1 - \omega_j, S_j)$$

  represents the degree of truth of the statement "there exists at least one objective for which it is not true that if it is significant, then a_i is not as good as a_k".

$I^{\rightarrow}$ represents a fuzzy implication and such that $I^{\rightarrow}(1, x) = x$, and $I_c^{\rightarrow}(0, x) = x$. Moreover, $I_c^{\rightarrow}$ is a coimplication defined as $I_c^{\rightarrow}(x, y) = 1 - I^{\rightarrow}(1-x, 1-y)$.

It is interesting to notice that

$$\min_{j=1,\dots,n} S_j \leq h_{I^{\rightarrow}}(S_1, \dots, S_n) \leq h_{I_c^{\rightarrow}}(S_1, \dots, S_n) \leq \max_{j=1,\dots,n} S_j.$$

$h_{I^{\rightarrow}}$ corresponds to the pessimistic attitude and $h_{I_c^{\rightarrow}}$ to the optimistic attitude. If the Kleene-Dienes implication-coimplication is used, one obtains the weighted minimum-maximum operators studied by Dubois and Prade (1986) :

$$h_{\min}(S_1, \dots, S_n) = \min_{j=1,\dots,n} \max\{1 - \omega_j, S_j\}$$
$$h_{\max}(S_1, \dots, S_n) = \max_{j=1,\dots,n} \min\{\omega_j, S_j\}$$

which correspond to weighted medians.

If the Gödel residual implication-coimplication is considered, then

$$h_{I^{\rightarrow}}(S_1, \ldots, S_n) = \min_{\substack{j \\ S_j < \omega_j}} S_j \quad \text{(or 1 if all } j \text{ give } S_j \geq \omega_j\text{)}$$

$$h_{I_c^{\rightarrow}}(S_1, \ldots, S_n) = \max_{\substack{j \\ S_j > 1-\omega_j}} S_j \quad \text{(or 0 if all } j \text{ give } S_j \leq 1 - \omega_j\text{)}$$

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