Uninorms and Absorbing Norms in Image Processing

Imre J. Rudas, János Fodor

Department of Intelligent Engineering Systems
Budapest Tech Polytechnical Institution
Bécsi út 96/B, H-1034 Budapest, Hungary
e-mail: rudas@bmf.hu, fodor@bmf.hu

Barnabás Bede [1]

Department of Mechanical and System Engineering
Budapest Tech Polytechnical Institution,
Népszínház u. 8, H-1081 Budapest, Hungary
e-mail: bede.barna@bgk.bmf.hu

Abstract: Recently it has been shown that sum and product are not the only operations that can be used in order to define concrete approximation operators. So, in Image Processing the sum and multiplication are not the only operations that can be used, but also, several other operations provided by fuzzy logic. In this sense, in the present paper, we will investigate how a pair consisting of a uninorm and an absorbing norm can be used for approximation purposes. We propose an approximation operator based on a continuous strictly increasing additive generator for the uninorm and on the same function used as a multiplicative generator for the absorbing norm. The proposed operator is defined based on the classical Shepard operator. We show that in Image Processing, the result of the proposed method outperforms in several cases the classical Shepard approximation operators based on sum and product operations.

1 Introduction

In classical Functional Analysis and classical Approximation Theory, the underlying algebraic structure is the linear space structure. The mathematical analysis using nonlinear mathematical structures is called idempotent analysis (see

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[10]) or pseudo-analysis (see [13], [12]) and it is shown to be a powerful tool in several applications.

Recently we proposed the same problem in Approximation Theory i.e., is the linear structure the only one that can be used in the classical Approximation Theory? Moreover, are the addition and multiplication of the reals the only operations that can be used for defining approximation operators? All the approximation operators need to be linear? The answer to this question is negative, and in this sense in [3] max-product Shepard approximation operators are studied. Also, in [4] Pseudo-Linear Approximation Operators are studied from the theoretical point of view and it is shown that even a parametric family of operators can be used for approximation purposes instead of the sum and multiplication of the reals. The idea of the possible usefulness of these operators is coming from Fuzzy Logic. For example in [5], normal forms are regarded as approximations, and this lead us to the idea that sum and addition are not the only operations which can be used in approximation theory.

We will continue in the present paper this line of research by studying Shepard-type approximation operators based on a pair consisting of a uninorm and an absorbing norm. Uninorms were introduced by Yager and Rybalov [15] as a generalization of t-norms and t-conorms. For uninorms, the neutral element is not forced to be either 0 or 1, but can be any value in the unit interval. Absorbing norms are a generalization of the well-known median studied in [9]. For a more general treatment of this operator, we refer to [7]. For the absorbing norm (see [2]), a given element is absorbing, i.e. the absorbing element if it is composed with any other element the result is the absorbing element itself.

We use in the present paper a pair of operations which are distributive one with respect to the other. As a consequence we will use a uninorm based on an additive generator and an absorbing norm based on the same (but now multiplicative) generator. Such a pair of operations ensures that the distributivity property holds.

Surely any approximation operator can be used in Image Processing (for example in zooming, compression, etc). In the present paper we propose also to investigate whether a pair consisting of generated uninorm and absorbing norm can be used in Image Processing.

After a preliminary section, we define and study in Section 3 the approximation method using them as operations instead of sum and product. Surely any approximation method can be used immediately in Image Processing. In Section 4 we present some numerical results which show the effectiveness of the proposed method compared with sum-product based classical Shepard approximation operators. At the end of the paper some conclusions and topics for further research are given.

2 Preliminaries

2.1 Uninorms and Absorbing Norms

Firstly let us recall the definitions and some properties of uninorms and absorbing norms.

Definition 1 ([15]) A uninorm $\oplus$ is a commutative, associative and increasing binary operator with a neutral element $e \in [0,1]$, i.e., for all $x \in [0,1]$ we have $x \oplus e = x.$

Definition 2 ([2]) An absorbing norm $\odot$ is a commutative, associative and increasing binary operator with an absorbing element $a \in [0,1]$, i.e. $(\forall x \in [0,1])(x \odot a) = a)$.

In the present paper we will use the so-called representable uninorms. These are defined as follows. Given $e \in (0,1)$ and a strictly increasing continuous $[0,1] \to \mathbf{R}$ mapping h with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$. The binary operator $\oplus$ defined by

$$x \oplus y = h^{-1}(h(x) + h(y))$$

for any $(x, y) \in [0,1]^2 - \{(0,1),(1,0)\}$, and either $0 \oplus 1 = 1 \oplus 0 = 0$ or $0 \oplus 1 = 1 \oplus 0 = 1$, is a uninorm with neutral element e. The class of uninorms that can be constructed in this way has been characterized [8].

Let us consider now the operation

$$x \odot y = h^{-1}(h(x)h(y)).$$

Then is is easy to check that the distributivity of $\odot$ w.r.t. $\oplus$ holds.

Theorem 1 Given e and h as above, then the operation $\odot$ has the following properties:

(i) The operation $\odot$ is an absorbing norm, having e as absorbing element.

(ii) The element $e' = h^{-1}(1)$ is neutral element w.r.t. $\odot$.

(iii) $\odot$ is distributive with respect to $\oplus$.

Proof (i) It is easy to check that $\odot$ is commutative, associative and increasing. Also, by direct computation it follows that $x \odot e = h^{-1}(0) = e, \forall x \in (0,1),$ that is, e is absorbing element.

(ii) It is easy to check that $x \odot e' = h^{-1}(h(x) \cdot 1) = x.$

Remark 1 Nullnorms are absorbing norms fulfilling some boundary conditions. In our case the boundary conditions are not satisfied. Instead, on the boundaries, if we pass to limit, we get

$$x \odot 0 = \begin{cases} 1 \text{ if } x < e \\ e \text{ if } x = e \\ 0 \text{ if } x > e \end{cases}$$

and

$$x \odot 1 = \begin{cases} 0 \text{ if } x < e \\ e \text{ if } x = e \\ 1 \text{ if } x > e \end{cases}$$

Let us also observe here the strong relationship between pseudo-analysis and our proposed pairs of operations. The algebraic structure induced by the above described pair of operations is a semiring as in the case of pseudo-analysis.

2.2 Approximation Operators

Let us recall that the main problem solved by classical approximation theory is to approximate a function $f : [a,b] \to \mathbf{R}$, where $[a,b]$ is a real interval, by some more simple function, e.g. (trigonometric) polynomial, rational function or wavelet. Crisp approximation theory provides many different approximation operators: Bernstein polynomials, Shepard-type rational approximation operators, Jackson-type trigonometric polynomials, wavelets (see e.g. [6]), to mention only a few. These operators are using exclusively sum and product of the reals as operations, and so, the linear structure over $\mathbf{R}$ as underlying algebraic structure. Usually, the form of such an operator is

$$L_n(f,x) = \sum_{i=0}^{n} K_n(x,x_i) \cdot f(x_i),$$

where $x_i \in [a,b]$, $i = 0,...,n$ are the knots and $K_n(x,x_i)$ are functions having relatively simple expression (polynomials, trigonometric polynomials, rational functions, wavelets).

Let us observe that all these operators are linear with respect to the target function to which they are associated i.e.,

$$L_n(f+g,x) = L_n(f,x) + L_n(g,x), \forall f,g : [a,b] \to \mathbf{R} .$$

Usually, the error estimates in crisp approximation theory are provided in terms of the modulus of continuity. So, let us recall it's definition and main properties adapted to our case (for the general definition see [6]).

Definition 3 Let $f:[a,b]\to\mathbf{R}$ be a function. Then $\omega(f,\cdot):[0,\infty)\to[0,\infty)$, defined by

$$\omega(f,\delta)=\sup\{|f(x)-f(y)|;x,y\in X,\ |x-y|\le\delta\}$$

is called the modulus of continuity of f.

3 Approximation Operators Based on a Uninorm and an Absorbing Norm

Let us consider a continuous target function $f:[a,b]\to[0,1]$. Let also, $x_i\in[a,b]$ be sampled data (i.e. we suppose that the values $f(x_i)$ are known). The idea of defining nonlinear approximation operators is very simple. We change the addition to a uninorm and multiplication to absorbing norm, in the approximation operators, i.e. the general discrete form of such an operator is

$$P_n(f,x)=\bigoplus_{i=0}^{n}B_{n,i}(x)\odot f(x_i), \tag{1}$$

where $B_{n,i}(x):\ X^2\to[0,1]$ are some given continuous functions, $i=1,...,n$.

Let us denote by $C([a,b])$ the space of continuous functions $f:[a,b]\to\mathbf{R}$.

The operator $P_n:C([a,b])\to C([a,b])$, $P_n(f,x)=\bigoplus_{i=0}^{n}B_{n,i}(x)\odot f(x_i)$, is continuous and satisfies the property

$$P_n(\alpha\odot f\oplus\beta\odot g,x)=\alpha\odot P_n(f,x)\oplus\beta\odot P_n(g,x).$$

In the present paper we use Shepard kernels:

$$A_{n,i}(x)=\frac{\frac{1}{|x-x_i|^\lambda}}{\sum_{j=0}^{n}\frac{1}{|x-x_j|^\lambda}}.$$

The classical Shepard approximation operator is

$$Sh_n(f,x)=\sum_{i=0}^{n}A_{n,i}(x)f(x_i).$$

Let us transform the Shepard kernel by h^{-1}, i.e., let

$$B_{n,i}=h^{-1}\circ A_i$$

and let $P_n(f,x)$ given as in (1).

Theorem 2 ([14]) For the error in approximation by the classical Shepard operator given above we have the following Jackson-type estimate

$$|Sh_n(f,x) - f(x)| \leq C\omega\left(f, \frac{1}{n}\right),$$

where C is some absolute constant.

Analogously in the case of using the proposed operators we get the same error estimate.

Theorem 3 Let $P_n(f,x)$ be the operator given above and the operations $\oplus$ and $\odot$ generated by a Lipschitz-type generator. Then the following Jackson-type estimate holds true

$$|P_n(f,x) - f(x)| \leq C\omega\left(f, \frac{1}{n}\right),$$

where C is some absolute constant.

Proof It is easy to check that

$$P_n(f,x) = \bigoplus_{i=0}^{n} B_{n,i}(x) \odot f(x_i) = h^{-1}\left(\sum_{i=1}^{n} A_{i,n}(x) \cdot hf(x_i)\right).$$

Then, by using the Lipschitz property for h^{-1}, the error can be estimated as follows:

$$|P_n(f,x) - f(x)| = \left| h^{-1}\left(\sum_{i=1}^{n} A_{i,n}(x) \cdot hf(x_i)\right) - h^{-1}hf(x) \right|$$

$$\leq L_1 \left| \sum_{i=1}^{n} A_{i,n}(x) \cdot hf(x_i) - hf(x) \right|,$$

L_1 being the Lipschitz constant of h^{-1}. If we apply now the classical error estimate for the Shepard approximation operator, we get

$$|P_n(f,x) - f(x)| \leq L_1 C\omega\left(hf, \frac{1}{n}\right) \leq L_1 L_2 C\omega\left(f, \frac{1}{n}\right),$$

where L_2 is the Lipschitz constant of h, and the theorem is proved.

It is now straightforward the following corollary which is a Weierstrass-type theorem for the proposed Shepard-type approximation operators.

Corollary 1 Any continuous function $f : [0,1] \to [0,1]$ can be arbitrarily closely approximated by Shepard-type approximation operators based on uninorm absorbing norm pair.

However all the properties are straightfoward and however the method is based on the classical approximation result, it worth study, because firstly, the result obtained for approximation of the target function is different and also they are different seen as approximation operators, one being linear and the other is not linear.

4 Image Processing Experiments

We perform in this section some preliminary image processing experiments by using Shepard-type approximation operators based on uninorms and absorbing norms. Let us regard the following results as competition between addition, multiplication pair and other pairs of operations in the framework of image processing.

The compression method used in this comparison is to simply select one point from each image block of $n \times n$ pixels (upper left corner) and deleting the rest.

The decompression step is as follows. Having the interpolation points selected as in the previous step we compute the values in the missing points by using the proposed approximation operators with the sliding neighborhood method.

As generators of the uninorm and absorbing norm operations we have used

$$h(x) = \ln \frac{x^a}{1 - x^a}, \tag{2}$$

which generates a parametric family of approximation operators. The neutral element of the uninorm in this case is $\frac{1}{2^{\frac{1}{\alpha}}}$. This parametric family of operations, for $\alpha = 1$ contains the famous 3 PI oeration.

We present three experiments. In the first and second experiment, original images Lenna and Text are compressed with compression rates $\frac{1}{4}$, $\frac{1}{9}$ respectively. The decompression results are presented in Fig. 1, and Fig. 2 respectively.

Figure 1
Decompression result for Image Lenna, compression rate ¼, operations generated by (2), parameter a=1.25

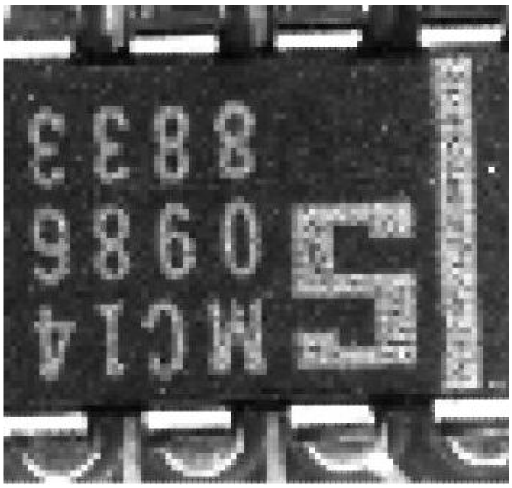

Figure 2
Decompression result for Image Text, compression rate 1/9, operations generated by (2), parameter a=1

The results show that the pair consisting of a uninorm and absorbing norm can be used in Image processing as alternatives of sum and product operations.

In the third experiment image Text is compressed with the compression rate $\frac{1}{16}$. by using sum and multiplication and then using uninorm absorbing norm, for different values of the parameter α.

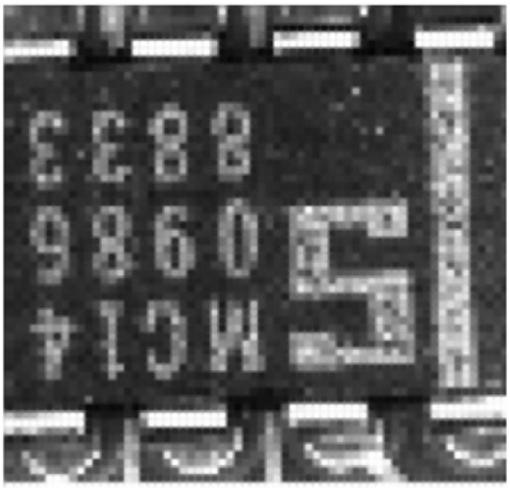

Figure 3
Decompression result for Image Text, compression rate 1/16, operations sum and product

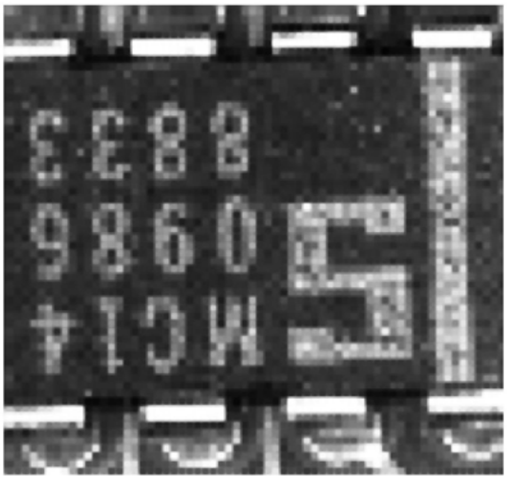

Figure 4
Decompression result for Image Text, compression rate 1/16, operations generated by (2), parameter a=2.25

In Fig. 3 and Fig. 4 the result of the decompression can be compared. Visually the quality is approximatively the same. For a more accurate comparison we compare the decompression quality by using the RMSE. The dependence of the RMSE w.r.t. α is shown in Fig. 5.

Let us remark here that the RMSE for image Text, when using classical Shepard approximation operator was 28.5981.

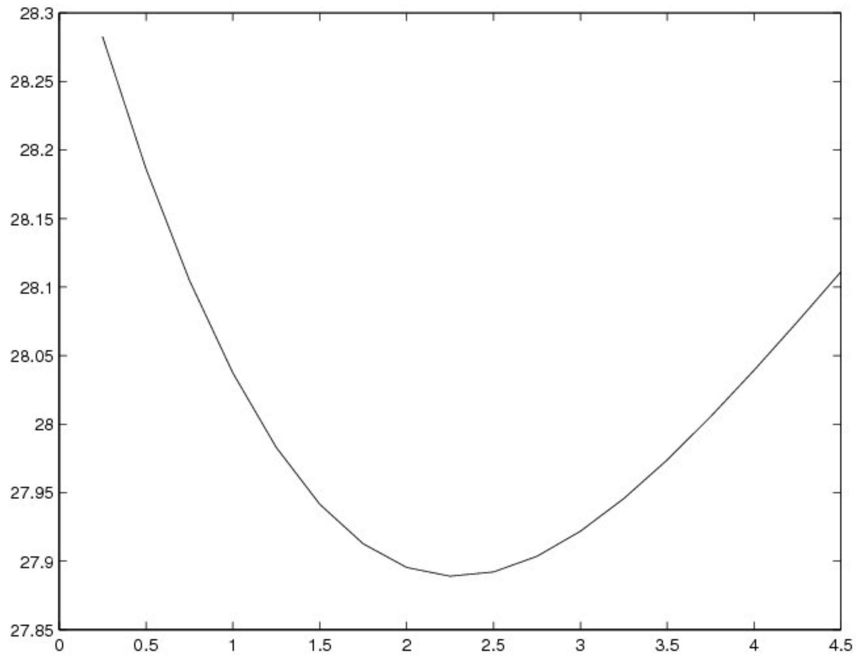

Figure 5

Dependence of the rmse of the decompression results for Image Text, compression rate 1/16 with respect to the parameter *a*

If we regard the results presented above as a competition between the classical and other operations we can observe that the addition and product are outperformed in this experiment by pairs of uninorm absorbing norm.

Conclusions and Further Research

We have proposed the use of a pair of generated uninorm and absorbing norm in Approximation Theory and in Image Processing. The theoretical study shows that the main properties are conserved.

As a conclusion of the experiments proposed in the previous section it is easy to see that the sum-product based approximation is sometimes outperformed by our proposed method. Another promising research topic is the use of the proposed approximation operators for noise reduction in images. Surely more practical image compression methods can be imagined.

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